Cost, Supply, and Farm Structure: A Pedagogical Note

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Abstract. *Starting with an individual firm and its quadratic production function, this paper derives all related functions marginal and average cost, supply, profit, and input-demand Since derivatives in other functions correspond to parameters of the quadratic, the results generalize Explicit aggregation from firm to market shows that properly specified aggregate functions depend on firm numbers To illustrate the results, marginal and average cost functions for several dairy farms are drawn to scale, noting that large farms get more output per cow than small farms Juxtaposing the cost curves with trends in dairy farms by size shows the link between firm-level profit and structural change*

Keywords. *Dairy farms, production, cost, duality, aggregation, technical change, structural change, quadratic forms*

The relationship between cost and supply curves is well-known among economists, but not always well illustrated Consequently, perceptions of the relationship may be distorted This note is offered in the spirit of Jacob Viner's (1931) instructions to his draftsman

My basic criticism[1] is that textbook cost curves for the "representative" firms are drawn too close together The impression conveyed is that alternative technologies are differentiably close to one another Within any industry, or subsector of agriculture, there coexist firms that employ widely different technologies For example, the technology on farms with 500 milk cows is very different from the technology on farms with 5 or 50 cows The large farm is not a small farm that "grow'd up" Rather, it discarded the old technology and put on the new Not many 500-cow dairies have a 50-cow barn alongside a 450-cow facility

The curves are not always drawn to scale, and only infrequently are they related to a particular mathematical function, Beattie and Taylor's (1985) text is an exception to this generalization Without a cardinal sense of distance, distortions can arise With a sense of distance in the graph, relationships among firms of different size can be better visualized Relations between supply elasticity and cost surve location (and the implied envelopes) become evident

To illustrate the relationships with a concrete example, consider the circumstances in the U S dairy sector (fig 1) The marginal and average variable cost curves for three sizes of farms are plotted in the top panel The lower panels illustrate the trends in the number of dairy farms by size Small farms have decreased in number exponentially Large farms have increased in number over the last half century The cost curves help to understand the trends in farm numbers For any given milk price, the profit on the large farm substantially exceeds that on smaller farms On the graph, profit is the area of a rectangle formed by the quantity at which marginal cost equals the price and the difference between marginal and average cost at that point The rectangle formed from MC1 and AC1 is clearly smaller than the rectangle formed from MC3 and AC3 As prices change, the effect on per-firm profits can easily be mapped out

Where did those cost curves come from? They are somewhat hypothetical, but not altogether arbitrary The curves are those implied by a quadratic production function They are each parametrized by a supply elasticity and a point on the marginal cost (supply) curve In each case, the functions were evaluated where the marginal cost (price) is \$12 per hundredweight (cwt) of milk The quantity and elasticity pairs are 50 cows and an elasticity of 0 45 in case 1, 200 cows and an elasticity of 0 3 in case 2, and 500 cows and an elasticity of 0 15 in case 3 More specifically, the quantity dimension is the "cow equivalent" unit of milk production—about 150 cwt per cow The quantities chosen represent class boundaries in the size distribution of farms All the small farms have cost curves to the left of MC1 (unless their price response is extremely elastic) Most of the large farms have marginal cost curves between MC1 and MC3 The overhead (or fixed) costs of the farm are not recognized in figure 1 Matulich (1978) estimated that construction, equipment, labor, insurance, and tax costs on large (375-1200 cow) dairy farms were asymptotic to about \$150 per cow in 1978, about \$1 per cwt of milk

What is the mathematical source of those curves? The quadratic production function is the basis of

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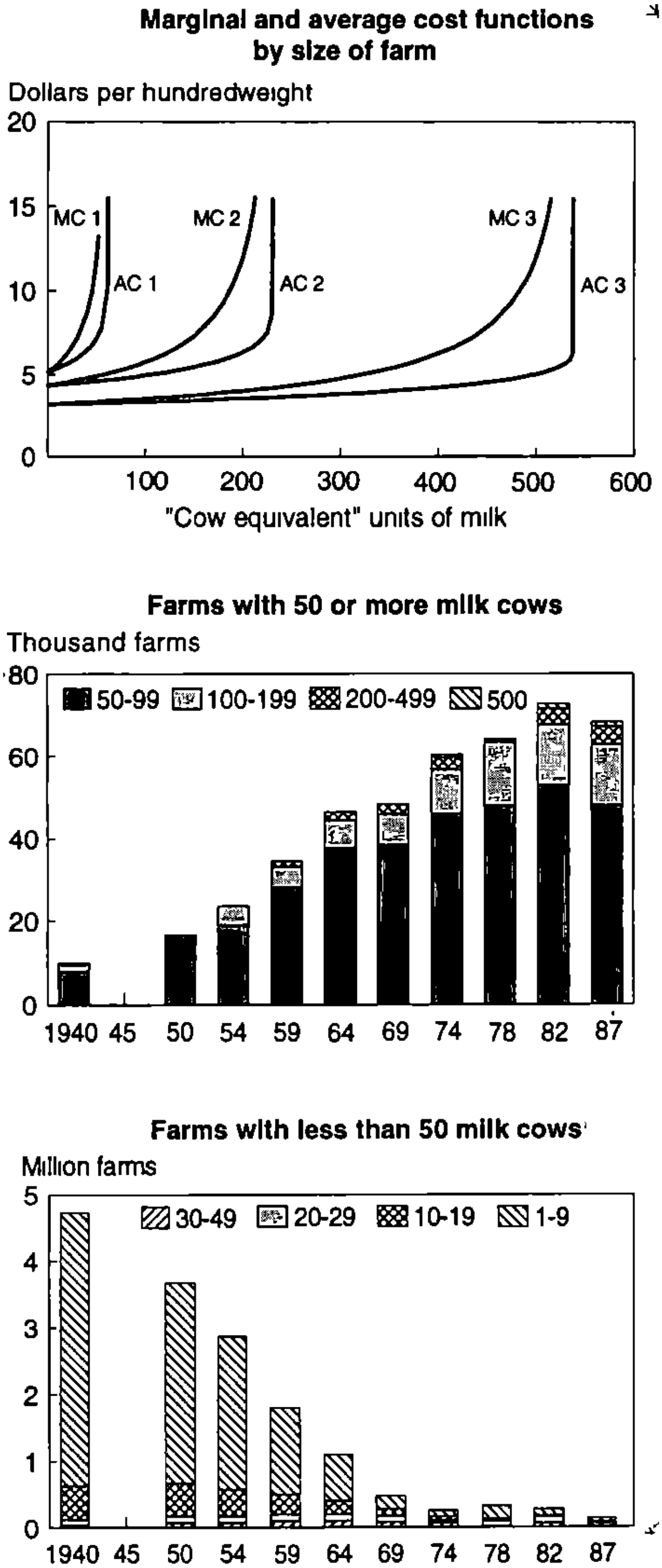
[1]This note originated as a commentary on "The Conceptual Model of Agricultural Development" in Cochrane's history of American agriculture

* Note that the smallest supply will be c when c>0 This can be achieved with X=0 and no cost

Figure 1
Cost and structural change in the dairy industry



their derivation When a firm has a quadratic production function, its supply function under profit-maximizing assumptions is a quadratic form in relative prices This quadratic supply function is given by

$$Y = S(w/p) = a + 5\ w'Hw/p^2 \quad (1)$$

There is a corresponding input demand function that is a linear form in relative prices, given by

$$X = D(w/p) = b + H(w/p) \quad (2)$$

The quadratic production function is given by

$$Y = F(X) = c + d'X + 5\ X'H^{-1}X \quad (3)$$

In these equations, a, c, Y, and p are scalars, and b, d, X, and w are (column-) vectors, and H is a (nonsingular) negative definite, symmetric matrix The quantity of output and the quantity of inputs are represented by Y and X The price of output and the price of inputs are represented by p and w The parameters a and b are related to the parameters c and d by the following

$$a = c - 5\ d'Hd \text{ and} \quad (4)$$

$$b = -Hd \quad (5)$$

The asymptote of the supply function is a, which measures production capacity It is the maximum attainable output, namely that which occurs when input use equals b, the intercept of the input demand function The optimal level of profit attainable by this firm is given by

$$\pi = \pi^*(p,\ w/p) = p[\ a - b(w/p) - 5\ w'Hw/p^2\ ] \quad (6)$$

The firm's supply function can be parametrized by assuming a price elasticity (with respect to product price) at a particular price and quantity point—treating all input prices as fixed in the analysis Setting the point elasticity to e, and the price/quantity point as (p_o, Q_o), the parameters of the supply function are given by

$$a = Q_o(1 + e/2) \text{ and} \quad (7)$$

$$w'Hw = -Q_o\ p_o^2\ e \quad (8)$$

The supply function intersects the price (cost) axis at the value MC_o, equal to the marginal cost of the first unit of output, which is given by

$$MC_o = p_o \sqrt{[e/(2+e)]} \quad (9)$$

The marginal cost at any output, Q, for a given input price vector w, is the inverse of the supply function, given by

$$MC(Q,\ w) = \sqrt{[\ 5(w'Hw)/(Q-a)]},\ \text{for } Q < a \quad (10)$$

* is derived by substituting (7) and (8) into (1) and solving for p with Y=0 If c>0, (9) is wrong.

The integral of the marginal cost is the total variable cost, given by

$$TVC(Q, w) = [\sqrt{a} - \sqrt{(a-Q)}]\sqrt{(-2w'Hw)},$$

$$\text{for } Q < a \quad (11)$$

The average variable cost is the ratio of total variable cost to level of output, given by

$$AVC(Q, w) = Q^{-1}[\sqrt{a} - \sqrt{(a-Q)}]\sqrt{(-2w'Hw)},$$

$$\text{for } Q < a \quad (12)$$

The maximum average variable cost occurs as Q approaches a, and is equal to twice the marginal cost of the first unit of output, namely

$$\lim_{Q->a}AVC(Q, w) = \sqrt{[(-2w'Hw)/a]}$$

$$= 2\ p_o\sqrt{[e/(2+e)]} = 2\ MC_o \quad (13)$$

What if the firm doesn't have a quadratic production function? If the production function is not quadratic, but has continuous derivatives of second order, the function can be locally approximately by a quadratic function The approximation error is the Taylor series remainder term, and approaches zero as we approach the point of approximation Moreover, the derivatives of the true function are analogous to the parameters of the quadratic function Thus, whatever the true production function, its derivatives should enter the firm's supply, input demand, and profit function in a manner analogous to equations 1, 2, and 6

Does the quadratic function exhibit constant returns to scale? Not globally But product exhaustion, in the sense of Euler's theorem, is possible With the right combination of parameters, there is a set of inputs that exhausts total revenue when they are paid their marginal products Namely, when c is positive, product exhaustion occurs for any $X \in \{X \in R^n \ 2 c + X'H^{-1}X = 0\}$ This set defines an n-dimension ellipsoid If c is negative, the set is empty Equation 6 determines the set of relative prices at which the product-exhausting X's would be chosen Imposing constant returns to scale globally results in singularity of the H^{-1} matrix, consequently H would not exist under those circumstances

How do these firm-level functions relate to the market aggregates? Market supply and profit is the sum of the supply or profit originating on each of the firms in the industry Similarly, input use is the sum of the input demand functions across firms It is important to distinguish between the sum of the functions (which itself is a function) and the sum of the values taken by the functions (which is a number) To indicate the summation, the subscript i denotes the individual and the subscript k denotes the distinct technologies in the industry Further, suppose the n_k individuals use the k-th technology Finally, assume that all firms face the same relative price vector, (w/p), and see the same product price p Then the aggregate supply function is given by

$$\Sigma_i Y_i = \Sigma_i S_i(w/p)$$

$$= \Sigma_k n_k a_k + 5\ w' (\Sigma_k n_k H_k)w/p^2 \quad (1a)$$

The aggregate input demand function is given by

$$\Sigma_i X_i = \Sigma_i D_i(w/p)$$

$$= \Sigma_k n_k b_k + \Sigma_k n_k H_k (w/p) \quad (2a)$$

The aggregate profit function is given by

$$\Sigma_i \pi_i = \Sigma_i \pi_i^*(p, w/p)$$

$$= p[\Sigma_k n_k a_k - (\Sigma_k n_k b_k)(w/p) -$$

$$5\ w'(\Sigma_k n_k H_k)w/p^2] \quad (6a)$$

The industry "marginal cost" function is the price-dependent form of the aggregate supply function [2] This is obtained by inverting equation (1a) and solving for p, holding relative input prices constant To do this, the function in (1a) is linearized in the neighborhood of a point, and the linear function inverted Let w_o/p_o be the point around which the function, $\Sigma_i S_i(w/p)$, is linearized Let $\nabla F(x)$ represent the gradient of a function F, evaluated at x The gradient of F is the (column-) vector whose components are the respective partial derivatives of the function F Denote the aggregate output by Q, then the industry "marginal cost" is given by

$$MC(Q, p_o, w_o/p_o) = p$$

$$= p_o\ [1 + (\Sigma_i S_i(w_o/p_o) - Q)/$$

$$\{(w_o/p_o)'(\Sigma_i \nabla S_i(w_o/p_o))\}] \quad (14)$$

Those troubled by the minus sign in front of the Q are reminded that the expression in curly braces ({}) is negative ∇S contains the partial derivatives of supply with respect to input prices, which are usually negative Concavity of S(x) ensures that $x'\nabla S(x)$ is negative S(x) is concave because each of the H_k matrices is negative definite

Is there an aggregate production function consistent with these relationships? Subject to a proviso,

[2]Chambers' (1988) concept of cost aggregation across firms (p 182) employs notation (equation 5 26) that is mathematically suspect, and differs from the industry marginal cost curve presented here as equation 14

yes The restriction is that the d vector of parameters must not change across firms Klein (1946, pp 94-95) sets forth the conditions necessary for the existence of the aggregate production function The aggregate function must relate aggregate output to aggregate input, and its marginal products must equal the relative input prices whenever the marginal products of individual firms equal relative prices The aggregate function which satisfies these criteria is given by

$$\Sigma_i Y_i = A(\Sigma_i X_i)$$

$$= \Sigma_k n_k c_k + d'(\Sigma_i X_i) +$$

$$5 (\Sigma_i X_i)' (\Sigma_k n_k H_k)^{-1}(\Sigma_i X_i) \qquad (3a)$$

Note, however, that this aggregate function depends intrinsically on the distribution of firms and their individual parameters Parameters of this aggregate function depend on the distribution of firms in the industry, and are not constant over time Consequently, its estimation presents a host of problems, which most empirical work sidesteps Only when all firms possess the same parameters is the estimation simplified, since the number of firms would then factor out of the expressions

Measures of technical change, in the tradition of Solow attribute to time or technology changes in the aggregate function (3a) that result from a different (or changing) distribution of firms in the industry Solow's Nobel Prize notwithstanding, Staehle's (1942) assessment is more correct " concerning technological change, the difficulties are truly insurmountable There comes into being, of course, a new cost, if not a new production function, whenever such change occurs, and no amount of assuming, or fitting of trends to residuals will really do " (p 271) Solow anticipates this criticism, stating " it takes something more than the usual 'willing suspension of disbelief' to talk seriously of the aggregate production function " Although he did not cite Klein, Solow seems aware of the tenuous link between his aggregate relationship and the rational decision units in the economy

Other economic constructs are also linked to an aggregate relationship The exactness of index numbers is defined in relation to an aggregate function similar to that expressed in equation 3a If such a function does not exist, or has parameters that are not stable, exact index numbers have little meaning Any other index number is nearly as meaningful

Returning to costs and elasticities, the more elastic a firm s response, the more unused capacity it has That is, the difference between a and Q_o increases with the size of the firm's supply elasticity with respect to output price In equation 7, a is the absolute production capacity, and Q_o the production level, and the difference is unused capacity Firms with higher price elasticities have higher average cost levels, based on equations 9 and 13 Figure 1 illustrates curves where the initial point elasticities are 0 45, 0 30, and 0 15, for firms subscripted 1, 2, and 3

Why those particular elasticities? Elasticities of onfarm response are not common in the literature As Cochrane and Butz (1951) said, "The aggregate output function of a representative commercial, family farm, whether a single or multiple-enterprise unit, is perfectly inelastic or approximately so, but this inelastic aggregate output function shifts to the right as technological developments are adopted on farms (p 469)" My own econometric estimates of equations like equation 1a imply elasticities of milk supply with respect to milk price—holding input prices and farm numbers constant—that are quite small, typically near 0 15 Elasticities in the literature, which do not hold farm numbers constant, are much larger The range of elasticities is meant to illustrate some of the possibilities Larger elasticities would render a "cobweb model' of the milk market dynamically unstable

The cost structure in figure 1 reflects increasing productivity on larger farms Census of Agriculture data document the difference in productivity or efficiency between large farms and small farms This pattern extends as far back as 1929 (table 1) Measured on a per-cow basis, milk production or sales on farms with 100 or more milk cows is about 20 percent greater than that on farms with 30-49 cows, and sales on the smallest farms (1-29 cows) about 20 percent less In addition, many small farms produce only for onfarm consumption and have no sales at all

The yield difference and the changing distribution of farms explain a major part of the rising productivity of the U S dairy sector However, growth accountants don't currently partition the yield gains between structural change and new technology About a third of the yield growth between 1939 and 1987 came from structural change (in the size distribution of farms and cows), while about two-thirds came from technological change (in the production possibilities on farms of a given size) If the 1987 distribution of milk cows prevailed then, the 1929 yield per cow would have been 36 9 percent higher, and the 1939 yield would have been 44 3 percent higher The national average milk yields were 4,500 pounds in 1929,

Table 1—Sales per milk cow and relative efficiency, by size of herd.

Census year	Number of milk cows on the farm: 500 +	200-499	100-199	50-99	30-49	20-29	10-19	1-9
				Milk production per cow (gallons)				
1929	←	832	700	690	681	650	574	468
1939	←	842	776	738	724	681	577	448
				Whole milk sales per cow (pounds)				
1939	←	6,711	6,036	5,572	5,034	4,332	2,868	847*
1949			←	6,320	5,632	5,159	3,768	1,315*
1959		←	8,259	7,443	7,480	4,730	*→	
1964	←	9,952	8,982	8,646	8,510	7,701	6,199	1,852*
				Dairy product sales per cow (dollars)				
1969	←	630	590	552	511	427	349	406
1974	1,048	949	878	818	747	648	571	556
1978	1,357	1,255	1,202	1,127	1,043	892	783	828
1982	1,852	1,715	1,604	1,523	1,402	1,189	1,061	866
1987	1,849	1,713	1,688	1,577	1,488	1,225	1,109	1,080
				Relative efficiency, compared with 30-49 cow farms				
				Milk production per cow (gallons)				
1929		← 1 222	1 027	1 013	1 000	0 955	0 843	0 687
1939		← 1 164	1 072	1 019	1 000	942	798	619
				Whole milk sales per cow (pounds)				
1939		← 1 333	1 199	1 107	1 000	861	570	168*
1949				← 1 122	1 000	916	669	233*
1959			← 1 104	995	1 000	632*→		
1964		← 1 169	1 055	1 016	1 000	905	728	218*
				Dairy product sales per cow (dollars)				
1969		← 1 231	1 154	1 081	1 000	836	682	795
1974	1 403	1 270	1 175	1 095	1 000	867	765	744
1978	1 300	1 203	1 152	1 080	1 000	854	750	794
1982	1 321	1 224	1 145	1 087	1 000	849	757	618
1987	1 277	1 183	1 166	1 089	1 000	846	766	746

← Last tabulated entry describes all larger farms
→ Last tabulated entry describes all smaller farms
(*) Not adjusted for farms without sales
Source *Census of Agriculture* various years

4,512 pounds in 1939, and 13,819 pounds in 1987 Adjusted for the 1987 distribution of cows, the yields would have been 6,161 pounds in 1929 and 6,512 pounds in 1939 Similarly, if the 1929 distribution of cows prevailed in 1987, the national average yield would have been less—about 28-29 percent lower (between 9,832 and 9,937 pounds per cow)

The envelope representing the longrun average cost for the firms illustrated in figure 1 is the cost structure in AC3, the lowest of the three average cost curves Ultimately firms using technologies 1 and 2 go out of business, and all surviving firms employ technology 3 regardless of the milk price This is the consequence of the increasing inelasticity of price response with firm size If the price response were increasingly elastic as size increased, the cost curves would more closely resemble the textbook envelope curves In those cases, smaller firms would have the least cost under low milk prices and larger firms would have the cost advantage under higher milk prices, in effect swapping curves 1 and 3 in figure 1 If all three firms had the same price elasticity of supply, all the curves would intersect the cost axis at the same point

Summary

Cost curves for firms of different sizes, when drawn to scale, show clear differences in their gross and net income positions, which explain the growth and decline in numbers of farms by size The relationship between supply in the aggregate and supply on the farm has been made explicit The relationship between the firm-level supply function and the firm's marginal and average variable cost is derived The "duality" relationships among the production, marginal cost, and profit functions on the farm is illustrated The explicit aggregation process shows how aggregate functions depend on the number and distribution of farms, as well as parameters of the firm-level response In the U S dairy industry, change in the number and size of farms accounts for about one-third of the growth in per-cow milk yield between 1929 and 1987

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