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CAPITAL DECISIONS WITH TEMPORAL RESOLUTION OF UNCERTAINTY: A MEAN-VARIANCE APPROACH

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The traditional approach to evaluating capital investments under uncertainty is capital budgeting. Traditional application involves maximizing the present value of expected net returns to a choice or choices from some discrete set of alternatives, generally subject to a budget constraint.

Much attention has been given to the appropriate risk adjusted discount rate to use in discounting future returns when standard assumptions underlying financial market models hold. (See Copeland and Weston (1983) or Bierman and Smidt (1980)). When the assumptions do hold, the appropriate discount rate is independent of the decision maker's preferences. Less attention has been given to an appropriate approach when the assumptions do not hold, and the decision maker's preferences do matter.

Appropriate risk adjusted discount rates (specific to each alternative) are those which reflect the systematic or market correlated risk inherent in the alternative under consideration. Appropriate risk adjusted discount rates can, at least in principle, be computed when the following conditions hold. Financial markets must be sufficiently complete to allow full diversification of investment specific risk. Financial market friction (transactions costs, differences between borrowing and lending rates, etc.) must be absent or insignificant. The decision maker must have full access to those markets.

Where any of these conditions are not met, the conceptual basis for determining a risk adjustment is lost, and the agent's risk preferences do matter. Further, when an alternative would severely limit the agent's access to existing diversification opportunities, all risk, systematic and unsystematic, is relevant to the evaluation[1].

This paper presents a variation of capital budgeting, called Certainty Equivalent Capital Budgeting (CECB), designed for evaluation of practically irreversible investment opportunities when appropriate risk adjusted discount rates are unavailable. Mutually exclusive opportunities are evaluated using the means and variances of net returns and the risk preferences of the agent as measured by the Arrow-Pratt risk aversion coefficient. The certainty equivalent framework and independence requirements to be discussed later impose restrictions upon the forms of risk that are appropriately incorporated into the model. However, the model does allow a variety of specifications of multiple sources of uncertainty.

The capital budgeting literature gives little attention to how expected future returns are to be determined. Expected returns, and potentially the ordering of alternatives, is dependent upon how the uncertainty is modeled. The effect of different approaches to modeling uncertainty is dramatized in CECB because variances of future returns enter the model explicitly. For example, consider first a case where returns at t, (R_t), are a function of the realization of some random variable ϵ_t that enters R_t linearly, and ϵ_t is revealed *after* the capital asset utilization decisions (related variable input decisions) at t are made. Here $E(R_t(\epsilon_t)) = R_t(E\epsilon_t)$ and computations at $t = 0$, when the capital investment is chosen are straight forward. A second possibility is where ϵ_t enters $R_t(\epsilon_t)$ in a nonlinear fashion, then $E(R_t(\epsilon_t))$

[1] Most individuals and many organizations would be hard pressed to diversify the unsystematic risks inherent in a business purchase.

$\neq R_t(E\epsilon_t)$, and $R_t(\epsilon_t)$ becomes the relevant random variable, adding significantly to the computational requirements.

A third case, typical of real world situations, is where some or all of the uncertainty faced at the time of the capital decision, $t = 0$, is resolved before periodic utilization decisions are made. Consider a firm that invests in employee training. When an employee enters the program, there is some expectation of ability upon completion. However, upon completion, the employee can be assigned work reflecting skills actually gained in the program, rather than work requiring ex ante expected ability. This subsequent ability to respond to actual outcomes increases expected returns to the program, and reduces the variability of the resulting returns. In general, as has been shown by Hartman (1976) in a different context, this flexibility will increase the optimal level of training, given usual marginal cost and productivity characteristics. The question then, is how to consider this subsequent flexibility in today's capital decision.

Recent literature regarding capital decisions under uncertainty considers sequential revelation of information and further, questions the inflexible timing of the capital choice upon which capital budgeting is based. The concern with the timing and level of investment given sequential opportunities to invest or add to previous investments. The work focuses upon the effects of forthcoming information on current investment decisions and generally finds that firms will delay investments when sequential investment opportunities exist. Cukierman (1980) uses this framework to show that this effect holds for risk neutral investors, as well as risk averse investors. Bernanke (1983) and Pindyck (1988) consider the effects on aggregate investment levels and fluctuations.

This approach provides a sound basis for evaluating capital investment opportunities when the capital investment decisions are, in fact, sequential. Many are not. Delay is sometimes not an option. Subsequent addition to the current investment is often practically precluded by the lumpy nature of the capital technology. In other cases, risks may be time independent, hence, no information affecting the capital decision is forthcoming[2].

In these cases, the capital budgeting approach, which is predicated upon the decision to make a choice currently, applies. However, even when the capital decision cannot or need not be delayed or subsequently adjusted, future information often will affect future utilization decisions and hence affect returns to current investments.

Subsequent sections of this paper will consider the modeling of uncertainty and develop the CECB model. For simplicity, the discussion of uncertainty will utilize the traditional capital budgeting model, where risk adjusted discount rates are used to account for the relevant risks inherent in the alternatives. The connection between capital budgeting and expected utility theory (EU), and hence certainty equivalence, will be deferred until the CECB model is developed. The final section of the paper briefly considers potential applications.

II. UNCERTAINTY IN CAPITAL BUDGETING

The basic capital budgeting model is:

$$[1] \quad \underset{s \in S}{\text{MAX}} \sum_{t=1}^{T} \beta(s)^{t} \; ER_{t}(s) \quad - C(s)$$

where S is the discrete set of available actions, here limited to reflect any budget constraint, and s is an individual available choice. $\beta(s)$

[2] Consider the training program investment example. If an organization is relatively stable, and the need for people with certain skills is also relatively stable, delay is likely to carry substantial costs, and that delay is unlikely to result in much relevant information regarding the appropriate program to select. However, unplanned subsequent adjustments are not precluded in this particular example.

is the appropriate alternative specific risk-adjusted discount rate[3]. $ER_t(s)$ is expected returns at t, in t dollars, with the expectation taken at $t = 0$, given optimal variable inputs levels, and conditional upon having chosen s at $t = 0$. C(s) is the cost of s and is assumed to be known and stated in $t = 0$ dollars (present value of all costs). Note that returns are assumed to begin one period after s is chosen, an assumption which will be retained throughout the paper.

Three different methods of modeling uncertainty will be considered: (1) uncertainty enters the periodic return function linearly with respect to the periodic choice variable and is not resolved until after capital asset utilization decisions have been made, (2) uncertainty enters the periodic return function in a nonlinear fashion and again remains unresolved until all relevant decisions have been made, (3) uncertainty is fully or partially resolved before the periodic decisions are made, but after the capital choice has been made.

Throughout the discussion, the following example will be used. Consider an individual, otherwise unencumbered (no financial obligations or opportunities), with a contract to sell an unlimited number of widgets over some specified period at a certain price of $1.

The widgets cannot be bought, so any deliveries must be produced. Some discreet number of irreversible production system choices (s) (indexed s_j; $j=1..J$) are available at $t = 0$ for production beginning at $t = 1$. In addition, production requires the use of one variable input at each t, L_t, which is available in unlimited quantities in a competitive market at wage w_t. Choices of L at each t will completely determine how s is utilized. At

[3] The s argument is suppressed in subsequent discussion, as the example used involves selecting the appropriate capital intensity with which to produce a particular product. Presumably, the choices would carry the same rate.

any $t>0$, $y = f(L|s_j)$, where y is output, and f is a strictly increasing and strictly concave production function in L for all s_j. The s_j's are ordered such that if $i < k, \Rightarrow f(L|s_i) < f(L|s_k)\ \forall\ L$, and $C(s_i) < C(s_k)$ where $C(s)$ is the initial capital cost of the system choice. In words, the s's are indexed in order of cost, and more costly systems are more capital intensive, resulting in higher labor productivity.

The periodic return function, ignoring uncertainty, is:

[2] $\pi_t = f(L_t|s) - w_t L_t$

<u>CASE 1</u>:

In this case wages are uncertain, which is represented by the positive random variable v_t. The periodic returns are now:

[3] $\pi_t = f(L_t|s) - (w_t v_t)L_t$

$Ev = 1 \Rightarrow E\ w_t v_t = w_t$.

The firm's problem is now:

[4] $\underset{s\in S}{\text{MAX}} \sum \beta^t \underset{L_t}{\text{MAX}}\ E_{v_t}\left[f(L_t|s) - (w_t v_t)L_t\right] - C(s)$

First order conditions for optimality at any t, given any previous choice of s are:

[5] $f'(L_t^*|s) = w_t$ where L_t^* uniquely solves the problem. The solution here depends only upon the mean of v_t.

Procedurally, the T inner maximizations need to be solved for each s, the results plugged in to [4], and the sums compared. Since all information available at t was also available at $t = 0$, the problem can be written as:

[4'] $\underset{\substack{s\in S\\ \{L\}}}{\text{MAX}}\ E_{\{v\}}\sum \beta^t \left[f(L_t|s) - (w_t v_t)L_t\right] - C(s)$

where $\{.\}$ are sequences from $t = 0$ to T.

The simplicity of this case follows from the fact that $E\pi(v) = \pi(Ev)$, and that follows from the fact that v enters π linearly.

Case 2:

In this case, assume labor quality is uncertain. Nominal labor purchases is L_t, but it is of uncertain quality. Uncertainty is represented by z_t, actual labor is $L_t z_t$, and the agent's problem is specified as follows:

$$[6] \quad \underset{s \in S}{\text{MAX}} \sum_{t=1}^{T} \beta^t \underset{L_t}{\text{MAX}} E_{z_t} \left[f(L_t z_t | s) - w_t L_t \right] \quad - C(s)$$

Now, because f(.) is strictly concave in $L_t z_t$, z_t enters nonlinearly, and $E\pi_t(z_t) \neq \pi_t(Ez_t)$. Assuming $Ez_t = 1 \ \forall\ t$, and that z_t takes on two values, z_L and z_H with probabilities ρ and $(1-\rho)$, first order conditions for solution of an inner maximization problem are:

$$[7] \quad \rho f'(L_t^* z_L | s) + (1-\rho) f'(L_t^* z_H | s) = w_t .$$

The key point is that y at each realization of z enters in to the determination of ER_t. The capital choice problem remains procedurally the same, but computational requirements increase. Again, because information relevant to the L_t decision is the same at t as at t = 0, [6] could be rewritten as a simultaneous maximization problem at t=0, similar to [4'].

Case 3:

In the third case, at least some of the uncertainty at t = 0, when s is chosen, is resolved before L_t is chosen. Reconsider case 1, except assuming that v_t is revealed at the beginning of t. At t=0 the information set remains the same (except for the knowledge that v_t will be revealed). Assuming that v_t takes on values v_L and v_H with probabilities ω and $(1-\omega)$, the problem at t will be either:

$$[8] \quad \underset{L_t}{\text{MAX}}\ f(L_t | s) - (w_t \ v_L) L_t \quad \text{,or}$$

$$[9] \quad \underset{L_t}{\text{MAX}}\ f(L_t | s) - (w_t \ v_H) L_t$$

Denote solutions to these problems as L_t^L and L_t^H, respectively.

The determination of ER_t requires that problems be solved for each possible realization of v, a significant increase in computational requirements over either case 1 or 2. However, the computational requirements do not increase if, as in case 2, the uncertainty enters the problem in a nonlinear fashion.

Compare ER_t under cases 1 and 3, recalling that ER_t is viewed from today, t = 0, and that $Ev_t = 1$:

[10] In case 1: $f(L_t^*|s) - w_t L_t^* =$

$\omega (f(L_t^*|s) - (w_t v_L)L_t^*) + (1-\omega)(f(L_t^*|s) - (w_t v_H)L_t^*)$

[11] In case 3: $\omega(f(L_t^L|s) - (w_t v_L)L_t^L) +$

$(1-\omega)(f(L_t^H|s) - (w_t v_H)L_t^H)$

Quite clearly, returns at t will be higher with revelation of v, except in the case where the realization of v = Ev (not included in this specification), and ER_t will be higher whenever v is not degenerate. Thus, net returns to each s are higher given revelation of v. Nothing can be said about the effect of revelation of v upon the optimal capital choice, s_j without specifying the problem further. However, this clearly shows that proper specification of uncertainty is important to the capital choice problem.

Multiple sources of uncertainty can be incorporated, and will be, in the next section. The uncertainty need not be time independent. However, the T inner maximization problems would need to be solved for each time path of the random variables, and the expectation taken over the sum of the present value of returns. The problem could also be set up to accommodate a case where a noisy signal about the random variable(s) is observed. In this case, the periodic problems would be solved for each post signal distribution to determine expected returns viewed from the beginning of t. Expected returns at t = 0 would then be determined by taking expectations over the prior distribution of the signals.

The appendix provides a two period numerical example to compare the results with and without revelation of information.

III. CERTAINTY EQUIVALENT CAPITAL BUDGETING

Certainty Equivalent Capital Budgeting (CECB) is a form of capital budgeting designed for evaluation of discrete capital investment opportunities when risk adjusted discount rates cannot appropriately be used. Alternatives are compared based upon the means and variances of the present value of net returns. Since the model uses a certainty equivalence framework, the connection between capital budgeting and expected utility theory is considered before the model is presented. The example used previously is retained here. Assume our agent satisfies the Expected Utility Hypothesis and maximizes a strictly increasing function of the present value of lifetime income. Also assume that T, the time horizon of returns to choosing any s_j, is shorter than our agent's lifetime.

The problem is then:

$$[12] \quad \max_{\substack{s \in S \\ \{L\}}} EU(W_0 + \sum \beta^t (f(L_t|s) - w_t L_t) - C(s))$$

where W_0 is initial wealth, and other components are as previously defined.

In the absence of uncertainty, and given the assumption that U is strictly increasing over the entire range of possible outcomes, maximizing U will result in the same ordering of s and {L} as the capital budgeting setup discussed in the previous section. Introduction of uncertainty, and the existence of financial market imperfections, generally implies that all characteristics of the distribution of returns to each investment opportunity are important to the choice, requiring direct solution of the agent's problem ([12]).

If, however, elements of the opportunity set (the s_j's) can be appropriately or acceptably compared on the basis of the means and variances of their net returns, the utility maximization problem can be transformed into a problem of maximizing the certainty equivalents to the risky choices in the opportunity set. When the mean variance approach is used, the certainty equivalent Y_{CE} to a risky prospect Y is:

$$[13] \quad Y_{CE} = EY - \frac{\lambda}{2}\sigma_Y^2$$

where Y is the monetary argument in the agent's utility function, E is the expectation operator, λ is the agent's coefficient of absolute risk aversion, and σ_Y^2 is the variance of monetary returns. For a detailed discussion of certainty equivalence modeling, see Robison and Barry (1987).

Classical theoretical justification for the mean-variance, and hence, certainty equivalent, approach is provided by the assumption of a quadratic utility function and/or the normality of the random outcomes. The quadratic form of the utility function implies increasing absolute risk aversion. Of primary importance here is that the absolute risk aversion coefficient is dependent upon overall outcome levels including endowed wealth. CECB requires the assumption of constant absolute risk aversion, precluding the assumption of a quadratic utility function.

Meyer (1987), Meyer (1988), and Robison and Meyer(1987) provide and explore conditions under which problems can be formulated in a mean standard deviation (MS) framework to yield results consistent with direct application of EU theory, regardless of the form of the utility function or the distribution of the random outcomes. If the distributions of all random outcomes in the choice set differ only by location (expected value) and scale (standard deviation) parameters, then MS decision rules will yield results which are consistent with EU theory. Even when the mean-variance framework is not theoretically justified, Y_{CE}, as specified in equation [13], has been shown (Pratt, 1964) to locally approximate the certainty equivalent regardless of the form of the utility function or the underlying distribution of the random outcomes.

The CECB Model

Recall that we wish to compare alternative capital investment opportunities and choose the one which maximizes the present value certainty equivalent income. This choice is made in an environment where some of the uncertainty at $t = 0$ about future returns will be resolved before we actually use the equipment at $t = 1,..,T$.

Certainty Equivalent Capital Budgeting is essentially the incorporation of the certainty equivalent framework, as generally specified by Y_{CE} above, into the capital budgeting framework used in Section II. The situation considered is of the Case 3 type, with two sources of uncertainty. One, wage uncertainty, represented by v_t, $t = 1,...,T$, is uncertain at $t = 0$, but becomes known prior to the choice of L_t at t. The other, underlying production uncertainty, represented by ϵ_t, $t = 1,...,T$, remains uncertain until the end of t. Assume further that the random variables exhibit the following independence conditions: $E(\epsilon_t \epsilon_s) = 0$, and $E(v_t \epsilon_s) = 0 \ \forall \ t \neq s$, and $E(v_t \epsilon_t) = 0 \ \forall \ t$.

First consider the problem that the producer will face at any $t \geq 1$, given some previous choice of s, and the objective of maximizing contributions to the overall certainty equivalent:

$$[14]\ \underset{L_t}{\text{MAX}}\ E_{\epsilon_t}\left(f(L_t|s)\epsilon_t-(w_t\bar{v}_t)L_t\right)\ -\ \frac{\lambda}{2}\ \text{VAR}(f(L_t|s)\epsilon_t-(w_t\bar{v}_t)L_t).$$

where $\bar{v}_t$ is the observed outcome, λ is the agent's coefficient of absolute risk aversion, and VAR is the variance of returns viewed from the beginning of t, when only ϵ_t remains uncertain. The solution to this problem is L_t^*. Define the following components of [14], evaluated at L_t^*.

$$[15]\ CER_t(s,\bar{v}_t) = E_{\epsilon_t}\left(f(L_t^*|s)\epsilon_t-(w_t\bar{v}_t)L_t^*\right)$$

$$[16]\ CV_t(s,\bar{v}_t) = \text{VAR}(f(L_t^*|s)\epsilon_t-(w_t\bar{v}_t)L_t^*).$$

CER_t is the contribution to overall expected returns conditional on an s and an observation $\bar{v}_t$, in t dollars viewed from t. CV_t is, similarly, the undiscounted contribution to variance of total returns.

The problem , however, is choosing s at t = 0, when neither v nor ϵ is known:

$$[17]\ \underset{s\in S}{\text{MAX}}\ E_v\left[\sum_{t=1}^{T}\beta^t\ CER_t(s,v)\right] - \frac{\lambda}{2}\left[\text{VAR}\left[\sum\beta^t\ CER_t(v)\ \right]\right.$$
$$\left. +\ E_v\left[\ \sum\beta^{2t}CV_t(s,v)\right]\right]\ -\ C(s).$$

where the first term represents expected returns viewed from t = 0, when v is unknown. The first variance term is the variance in returns associated with v being unknown at t = 0. The second is the expected variance of returns associated with the underlying production uncertainty, ϵ, which is determined by solving the periodic problems specified in expression [14].

At t = 0, the agent knows that v_t will be known when L_t is chosen $\forall$ t $\geq$ 1. Hence $\{L_t^*(v|s)\}_{t=1}^{T}$, and the resulting sums of the present values of expected returns and contributions to the overall variance $\left[\sum\beta^t CER_t(v,s)\right.$ and $\left.\sum\beta^{2t}CV_t(v,s)\right]$ can be calculated for each time path of v. The certainty equivalent associated with the s under consideration is the expectation, over time paths of v, of the present value of the sum of expected returns, less C(s), less the risk premium associated with the variance of total returns. Since v and ϵ_t are independent, the total variance is the sum of the variances introduced by the two sources of uncertainty. That is, it is the sum of: (1) the expectation over v of the sum of the variances of the periodic returns resulting from production uncertainty, CV_t's, and (2) the variance of $\sum \beta^t CER_t$ resulting from v being unknown at t = 0.

Note that $CV_t(s,v_t)$ is the variance of returns viewed from the beginning of t given a realization of v_t. Since CER_t enters the sum of expected returns viewed from t = 0 as $\beta^t CER_t$, its ϵ_t related variance enters total variance as $\beta^{2t} E_v CV_t$. Taking the expectation over v is required, because the v is unknown at t = 0.

That the CECB objective function [17] appropriately reflects the ex ante expected returns and variance of those returns of a given capital choice is not immediately obvious. The appendix demonstrates that it does.

The process must be completed for each s, and the results compared to complete the process specified in the objective function.

As a practical matter, the v distribution must be discrete and s must be a discrete choice set. No such conditions are imposed upon L or ϵ. Quite clearly, the ability to assume that v takes on a relatively small number of time paths, and/or the v_t's are independently and identically distributed would assist in keeping computational requirements to a minimum. If v_t's are i.i.d., the objective function can be written:

$$[18] \quad \underset{s}{MAX} \left[\sum_{t=1}^{T} \beta^t E_{v_t} \; CER_t(s,v_t)\right] - \frac{\lambda}{2}\beta^{2t}\left[VAR\left[\sum CER_t(v)\right] + \left[\sum E_{v_t} CV_t(s,v_t)\right]\right] - C(s).$$

Further, if the periodic return function is time invariant, and the ϵ_t's can be assumed to be i.i.d., CER_t's and CV_t's will be time invariant, greatly reducing computations required.

The β should be the risk free rate, as risk is considered directly. In keeping with that, C(s) should reflect the present value of cash outlays discounted at the risk free rate.

The appendix compares a numerical example using CECB and a case where neither v_t nor ϵ_t are revealed before L_t is chosen. In other words s and $\{L_t\}_{t=1}^{T}$ can be simultaneously chosen at $t = 0$, because all relevant information that will be available at any t is available at $t = 0$. Though the example is contrived, the effects are dramatic, both in terms of expected returns and certainty equivalent income.

IV. APPLICATIONS

Many, if not most, problems to which capital budgeting is applied or applicable involve comparisons of mutually exclusive capital investment alternatives. Available alternatives generally differ in expectation and dispersion of returns. Dispersion of returns is commonly important to the decision maker. Quite often, some of the risk inherent in the capital decision will be resolved prior to related periodic decisions regarding the use of the capital choice. In other words, CECB appears to apply to a wide variety of real world situations.

Consider the classic textbook example. A firm is comparing two or more alternative machines for producing a product. The machines differ in initial cost, with more expensive machines providing lower per unit (of output) variable costs (e.g., labor). Possible sources of uncertainty at the time the machine is chosen include unknown future demand (quantities and/or prices) and/or unknown future wages or labor availability. In most cases, wages and, at least to some extent, demand, will be known when future output decisions are made. Choices affect the fixed vs. variable structure of future product costs, implying machine specific ex-ante distributions of returns. Given revelation of wage or demand information prior to future labor choices, the alternatives provide varying levels of subsequent flexibility, further affecting the machine specific return distributions. Wherever variability is important to the decision maker and revelation of information provides subsequent flexibility, CECB is applicable.

Production often has a stochastic element. In the language of the example above, the machines and labor may operate at varying rates or produce varying numbers of defective products. The distribution of the stochastic production elements might be machine specific. For example, labor intensive processes may allow early detection of defective operations, thereby reducing expected defective output and its variability over time. CECB provides an opportunity to directly consider alternative specific random variables. In this case, the production function might be specified as $f(x|s)\epsilon(s)$, where x is variable inputs, s is the capital choice, and $\epsilon(s)$ is a random variable reflecting capital choice specific stochastic production elements.

Another possibility is that, either through production of intermediate goods or investments to enhance purchased inputs, the distribution of input quality might be affected. (See case 2 in section 2.) For example, an organization may consider staff training alternatives which affect both expected quantity and quality of a unit of labor purchased. This could be incorporated into the model by specifying ρ as a random variable, L as nominal inputs purchased, and $L\rho$ as actual labor input. To reflect the effects of training alternatives, the distribution of ρ could be specified as follows $\rho \sim (\mu(s), \sigma^2(s))$, where s is the training alternative.

A troublesome aspect of CECB is that comparison of alternatives is limited to differences in the first two moments of the distributions of their respective returns. Of course, if one or more of the previously discussed assumptions are met, this is not a problem. If the assumptions do not hold, the validity of the approximation to expected utility outcomes must be carefully considered.

In summary, proper specification of uncertainty is important to the formulation of the capital budgeting problem, whether the traditional setup or CECB is used. Certainty Equivalent Capital Budgeting provides a method of directly incorporating risk and the risk preferences of the investor into the capital budgeting problem. Considerable flexibility is provided for including multiple sources of risk under a variety of assumptions regarding information structure. Requirements regarding the independence of various random variables and the consideration of only the means and variances of their distributions does, however, limit applicability.

BIBLIOGRAPHY

Bernanke, Ben S., 1983. "Irreversibility, Uncertainty, and Cyclical Investment", Quarterly Journal of Economics, 97, pp. 85-106.

Bierman, Harold, Jr. and Seymour Smidt, 1980. The Capital Budgeting Decision: Economic Analysis of Investment Projects, Macmillan Publishers, New York.

Cukierman, Alex, 1980. "The Effects of Uncertainty on Investment Under Risk Neutrality with Endogenous Information", Journal of Political Economy, 88(3), pp.462-475.

Copeland, Thomas E. and J. Fred Weston, 1983. Financial Theory and Corporate Policy, Addison-Wesley Publishing Co., Reading, MA

Hartman, Richard, 1976. "Factor Demand with Output Price Uncertainty", American Economic Review, 66, pp.675-681.

Meyer, Jack, 1987. "Two-Moment Decision Models and Expected Utility Maximization", American Economic Review, 77(3), pp.421-430.

Meyer, Jack, 1988. "Two Moment Decision Models and Expected Utility Maximization: Some Implications for Applied Research", presented at the Annual Meeting of the Southern Regional Project S-180 "An Economic Analysis of Risk Management Strategies for Agricultural Production Firms", Savannah, Georgia, March 20, 1988.

Pindyck, Robert S., 1988. "Irreversible Investment, Capacity Choice, and the Value of the Firm", American Economic Review, 78(5), pp.969-985.

Pratt, John W., 1964. "Risk Aversion in the Small and the Large", Econometrica, 32(1), pp. 122-136.

Robison, Lindon J. and Peter J. Barry, 1987., The Competitive Firm's Response to Risk, Macmillan Publishers, New York.

Robison, Lindon J. and Jack Meyer, 1987. "Agricultural Policies, Land Values, and Agricultural Production Under Risk: A Net Present Value Certainty Equivalent Approach", Michigan State University Department of Agricultural Economics Staff Paper 87-105, East Lansing, MI.

APPENDIX

This appendix illustrates CECB using a two period model with a simple specification of two sources of uncertainty. First, a step by step approach to solving the problem is illustrated. The results are then used to show that the CECB objective function reflects the ex-ante mean and variance of returns associated with a particular capital choice, and therefore provides an appropriate method of comparing alternatives from a discrete set of capital investment alternatives. As further illustration and as a basis for comparison, the problem is then reconsidered under the assumption that no relevant information will be revealed subsequent to $t = 0$ (a combination of Cases 1 and 2 from Section II.). Finally we consider a numerical example and compare the solutions under both sets of assumptions.

<u>The Setup</u>

The specification and assumptions of the example used in the paper are retained, and the following are added:

o At $t = 0$, the producer considers and chooses between two systems for producing the product; $s = 0$ or 1.

o Actual production takes place at $t = 1$. Output requires a variable input L and is conditional upon the previous choice of s. The production function is specified as $f(L|s)\epsilon$, where ϵ is a random variable representing output uncertainty as described below. f(.) is smooth, increasing, and strictly concave.

o L is available in unlimited quantities at a competitive wage w where w is random.

o w, unknown at $t = 0$, will be revealed at the beginning of $t = 1$. At $t = 0$, the agent believes that wages will be low (w_L) with probability ρ, or high

(w_H) with probability $(1-\rho)$.

o Similarly, ϵ will take on one of two values, ϵ_L or ϵ_H with probabilities ω and $(1-\omega)$, respectively. ϵ remains uncertain at the time L is chosen. To simplify subsequent algebra, assume that $E\ \epsilon = 1$ (so $E\ f(L|s)\epsilon = f(L|s)$.

o ϵ and v are independent.

The Solution

In this two period model, with two elements in the capital choice set, the producer's problem is simply:

$$[A1]\quad \underset{s\in\{0,1\}}{MAX}\left[\ \Phi(s)\right]\ ;\text{where}$$

$$[A2]\quad \Phi(s) = E_w\ \beta\ CER(s,w)\ -\ \frac{\lambda}{2}\ \beta^2\left[VAR(CER)\ +\ E_w CV(s,w)\right]\ -\ C(s)$$

and CER(s,w) and CV(s,w) are values of respective portions of the following objective at $t = 1$, given a realization of w and a previously made system choice, evaluated at the optimal solution.

$$[A3]\quad \underset{L}{MAX}\ E_\epsilon\left[\underbrace{f(L|s)\epsilon\ -\ wL}_{*}\right]\ -\ \frac{\lambda}{2}\ VAR(*).$$

Consider the steps in calculating $\Phi(0)$:

(1) At $t = 1$, wages will be known and L will be chosen accordingly. Solve the problem that will be faced at $t = 1$, objective [A3], for w_L and w_H, conditional upon having chosen $s = 0$. Explicitly stated, the problem and first order condition for an optimal solution, assuming w_L, are as follows (recalling our assumption that $E\epsilon = 1$):

$$[A4]\quad \underset{L}{MAX}\ f(L|s{=}0)\ -\ w_L L\ -\ \frac{\lambda}{2}\left[\ \omega\ (f(L|0)\epsilon_L - f(L|0))^2\ +\right.$$

$$\left.(1-\omega)\ (f(L|0)\epsilon_H\ -\ f(L|0))^2\right]$$

The first order condition for a maximum, assuming an interior solution, is:

[A5] $f'(L^*|s=0) - w_L -$

$$\lambda\Big[\omega\,(f(L^*|0)\epsilon_L - f(L^*|0)) \times (f'(L^*|0)\epsilon_L - f'(L^*|0)) +$$

$$(1-\omega)(f(L^*|0)\epsilon_H - f(L^*|0)) \times (f'(L^*|0)\epsilon_H - f'(L^*|0))\Big] = 0.$$

where L^*, hereafter $L^*(w_L,0)$, solves the condition and maximizes objective [A4]. (Existence of a maximum is guaranteed by the assumed strict concavity of f(.)).

(2) $CER(w_L,0)$ and $CV(w_L,0)$, and similarly $CER(w_H,0)$ and $CV(w_H,0)$, are then obtained by substituting $L^*(w_L,0)$ into the objective function as follows:

[A6] $CER(w_L,0) = f(L^*(w_L,0)|s=0) - w_L L^*(w_L,0)$

[A7] $CV(w_L,0) = \omega\,(f(L^*(w_L,0)|0)\epsilon_L - f(L^*(w_L,0)|0))^2 +$

$$(1-\omega)\,(f(L^*(w_L,0)|0)\epsilon_H - f(L^*(w_L,0)|0))^2$$

(3) $\Phi(0)$ can now be calculated:

[A8] $\Phi(0) = \beta\Big[\rho\, CER(w_L,0) + (1-\rho)\, CER(w_H,0)\Big] -$

$$\frac{\lambda}{2}\beta^2\Bigg[\Big[\rho\,(CER(w_L,0) - E_w\, CER)^2 + (1-\rho)\,(CER(w_H,0) - E_w\, CER)^2\Big] +$$

$$\Big[\rho\, CV(w_L,0) + (1-\rho)\, CV(w_H,0)\Big]\Bigg] - C(0).$$

where the bracketed variance terms correspond to those in equation [A2].

(4) Steps (1) to (3) need to be completed for s = 1, and the results compared as specified by the objective function [A1].

<u>Verification</u>

As noted in the paper, it is not obvious that $\Phi(s)$ ([A2]) actually reflects the ex-ante mean and variance of returns associated with a particular s, and hence provides an appropriate vehicle for comparing alternatives[4]. To show that it does, the ex-ante means and variances of returns are constructed from the basic information about ultimate states of nature and the return function. These are then shown to be equivalent to those reflected in $\Phi(s)$. (Throughout, we ignore the constant discount factor, β).

First, consider the four possible outcomes, given a choice of $s = 0$, assuming that our producer solves the problem as specified above:

(1) w_L, ϵ_L with probability $\rho\omega$; in which case $L^*(w_L,0)$ would have been chosen, resulting in returns $= f(L^*(w_L,0)|0)\epsilon_L - w_L L^*(w_L,0)$.

(2) w_L, ϵ_H with probability $\rho(1-\omega)$; in which case $L^*(w_L,0)$ would have been chosen, resulting in returns $= f(L^*(w_L,0)|0)\epsilon_H - w_L L^*(w_L,0)$.

(3) w_H, ϵ_L with probability $(1-\rho)\omega$; in which case $L^*(w_H,0)$ would have been chosen, resulting in returns $= f(L^*(w_H,0)|0)\epsilon_L - w_H L^*(w_H,0)$.

(4) w_H, ϵ_H with probability $(1-\rho)(1-\omega)$; in which case $L^*(w_H,0)$ would have been chosen, resulting in returns $= f(L^*(w_H,0)|0)\epsilon_H - w_H L^*(w_H,0)$.

From this information we can construct the mean (expression [A9]) and variance (expression [A10]) of returns. {1},...,{4}, refer to the respective bracketed terms in expression [A9].

$$[\text{A9}]\quad \rho\omega\Big[f(L^*(w_L,0)|0)\epsilon_L - w_L L^*(w_L,0)\Big] + \rho(1-\omega)\Big[f(L^*(w_L,0)|0)\epsilon_H - w_L L^*(w_L,0)\Big] + (1-\rho)\omega\Big[f(L^*(w_H,0)|0)\epsilon_L - w_H L^*(w_H,0)\Big] + (1-\rho)(1-\omega)\Big[f(L^*(w_H,0)|0)\epsilon_H - w_H L^*(w_H,0)\Big]$$

$$[\text{A10}]\quad \rho\omega\{1\}^2 + \rho(1-\omega)\{2\}^2 + (1-\rho)\omega\{3\}^2 +$$

[4] given, of course, that the underlying assumptions of the technique are appropriate for the specific problem under consideration.

$$(1-\rho)(1-\omega)\{4\}^2 - \Big[[A9]\Big]^2.$$

Now, consider expected returns. By the assumption that $E\epsilon = 1$, expression [A9] can be rewritten:

$$[A11]\quad \rho\left[f(L^*(w_L,0)|0) - w_L L^*(w_L,0)\right] + (1-\rho)\left[f(L^*(w_H,0)|0) - w_H L^*(w_H,0)\right]$$

which, using equation [A6], is equivalent to the appropriate portion of equation [A2]:

$$[A12]\quad \rho\ CER(w_L,0) + (1-\rho)CER(w_H,0) = E_w\ CER(w,0) = [A11].$$

Finally, consider the variance. First note that $CV(w_L,0)$ as reflected in equation [A7], can be written:

$$[A13]\quad \left[\omega\left[f(L^*(w_L,0)|0)\epsilon_L - w_L L^*(w_L,0)\right]^2 + (1-\omega)\left[f(L^*(w_L,0)|0)\epsilon_H - w_L L^*(w_L,0)\right]^2 - CER(w_L,0)^2\right]$$

Hence, $E_w CV(w,0)$, can be written:

$$[A14]\quad \rho\left[w\left[f(L^*(w_L,0)|0)\epsilon_L - w_L L^*(w_L,0)\right]^2 + (1-\omega)\left[f(L^*(w_L,0)|0)\epsilon_H - w_L L^*(w_L,0)\right]^2 - CER(w_L,0)^2\right] + (1-\rho)\left[\omega\left[f(L^*(w_H,0)|0)\epsilon_L - w_H L^*(w_H,0)\right]^2 + (1-\omega)\left[f(L^*(w_H,0)|0)\epsilon_H - w_H L^*(w_H,0)\right]^2 - CER(w_H,0)^2\right]$$

The VAR(CER), as written in equation [A8] is:

$$[A15]\quad \rho\ CER(w_L,0)^2 + (1-\rho)\ CER(w_H,0)^2 - \left[E_w CER(w,0)\right]^2$$

Summing expressions [A14] and [A15] to arrive at the total variance reflected in $\Phi(0)$ (equation [A2]):

$$[A16]\quad \rho\omega\left[f(L^*(w_L,0)|0)\epsilon_L - w_L L^*(w_L,0)\right]^2 +$$

$$\rho(1-\omega)\left[f(L^*(w_L,0)|0)\epsilon_H - w_L L^*(w_L,0)\right]^2 +$$

$$(1-\rho)\omega\left[f(L^*(w_H,0)|0)\epsilon_L - w_H L^*(w_H,0)\right]^2 +$$

$$(1-\rho)(1-\omega)\left[f(L^*(w_H,0)|0)\epsilon_H - w_H L^*(w_H,0)\right]^2 - \left[E_w CER(w,0)\right]^2$$

which, using the results obtained in equation [A12], is identical to expression [A10]. Q.E.D.

Simultaneous Solution

Now consider a case where everything is as it was above, except that w is not revealed in time for the L choice at the beginning of t = 1. Since no information will be revealed between now and the beginning of t = 1, when L is hired, L may as well be chosen today, along with s. Our objective function can now be written:

$$[A16]\quad \underset{s,L}{MAX}\ E_{w,\epsilon}\ \beta\left[f(L,s)\epsilon + wL\right] - \frac{\lambda}{2}\beta^2\ VAR(f(L,s)\epsilon + wL)) - C(s).$$

$$s.t.\ s = 0 \text{ or } 1.$$

As a practical matter, since s is discrete, the problem can again be maximized over L, conditional on each s, the results compared, and appropriate settings of s and L chosen.

Now given our assumption on ϵ, $E\epsilon = 1$, undiscounted expected returns, conditional on s = 0, can be expressed as:

$$[A17]\quad f(L,0) - (\rho w_L + (1-\rho)w_H)L.$$

The variance can be written:

$$[A18]\quad \rho\omega\left[f(L,0)\epsilon_L - w_L L\right]^2 + \rho(1-\omega)\left[f(L,0)\epsilon_H - w_L L\right]^2 +$$

$$(1-\rho)\omega\left[f(L,0)\epsilon_L - w_H L\right]^2 + (1-\rho)(1-\omega)\left[f(L,0)\epsilon_H - w_H L\right]^2 -$$

$$\left[f(L,0) - (\rho w_L + (1-\rho)w_H)L\right]^2$$

Define $\varphi(0)$, analogous to $\Phi(0)$, as the certainty equivalent of choosing $s = 0$:

$$[A19]\quad \varphi(0) = \max_L \beta\ (\ [A17]) - \frac{\lambda}{2}\beta^2\ [A18].$$

The first order conditions for maximization are:

$$[A20]\quad 0 = f'(L,0) - (\rho w_L + (1-\rho)w_H) - \lambda\beta^2$$

$$\Big[\rho\omega\left[f(L,0)\epsilon_L - w_L L\right]\left[f'(L,0)\epsilon_L - w_L\right] +$$

$$\rho(1-\omega)\left[f(L,0)\epsilon_H - w_L L\right]\left[f'(L,0)\epsilon_H - w_L\right] +$$

$$(1-\rho)\omega\left[f(L,0)\epsilon_L - w_H L\right]\left[f'(L,0)\epsilon_L - w_H\right] +$$

$$(1-\rho)(1-\omega)\left[f(L,0)\epsilon_H - w_H L\right]\left[f'(L,0)\epsilon_H - w_H\right] -$$

$$\left[f(L,0) - (\rho w_L + (1-\rho)w_H)L\right]\left[f'(L,0) - (\rho w_L + (1-\rho)w_H)\right]$$

That the first order conditions differ from those in the previous instance [equation A5] is obvious. Since w is known in the previous instance, $L^*(w_H,0) < L^{**}(0) < L^*(w_L,0)$, where $L^{**}(0)$ solves the first order conditions ([A20]). As wage uncertainty was here modeled, i.e. $dVAR(w)/dL > 0$, $\varphi(s) < \Phi(s) \Rightarrow$ the capital choice will differ, at least under certain conditions. If s were continuous, the s chosen here would be unambiguously less than s in the previous formulation. Since s is discrete, the decision will differ only over some range of C(s). These comments are not general. The relationships between L^*'s and L^{**} depend upon the way uncertainty enters the model. However, $\varphi(s) \leq \Phi(s)$, under any nondegenerate specification of uncertainty.

Numerical Example

To illustrate the discussion, a numerical example of both formulations was constructed. All assumptions and conditions are as they were above. The specifics added are:

o $f(1,s) = L^{\gamma(s)}$; $\gamma(0) = .8$; $\gamma(1) = .82$.

o $\epsilon_L = .9$; $\epsilon_H = 1.1$; $\omega = .5$.

o $w_L = .8$; $w_H = 1.2$; $\rho = .5$.

o $\lambda = .002$.

o $\beta = .9$.

Rather than specify C(s), s = 0, continuing present practices is assumed to be costless. Ranges over which the switch to s = 1 would be optimal are considered.

Results under the assumption that w is revealed are:

	S = 0	S = 1
$\Phi(s)$	1.722	2.462
Expected returns		
E β CER	1.724	2.468
E CER	1.916	2.742
Variance of returns		
VAR	2.976	7.512
β^2VAR	2.410	6.085
CER(w|s)		
w_L	3.199	4.737
w_H	.632	.747
$L^*(w\|s)$		
w_L	31.955	53.792
w_H	4.213	5.668

Results of the simultaneous solution are:

	S = 0	S = 1
$\varphi(s)$	1.178	1.540
Present value of exp returns	1.180	1.543
Variance (comparable to β^2VAR)	1.522	3.320
L^{**}	10.460	15.553

Using CECB, if C(1) < .74, the producer will switch. In the simultaneously solved problem, if C(1) < .36, the producer will switch. When the assumption of revelation of v reflects reality (CECB is appropriate), but s and L are chosen simultaneously, C(1) between .36 and .74 would lead to suboptimally continuing current practices (choosing s = 0 when s = 1 is optimal). Outside of that range, the approaches would lead to identical choices for s.

Note that our producer responds quite dramatically to the revelation of w. The variation in w is quite high, so the revelation of w_L provides the producer with an opportunity to increase expected returns dramatically, particularly if s = 1.