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Uncertainty and Irreversibility in Groundwater Resource Management

Yacov Tsur and Amos Zemel

DEPARTMENT OF AGRICULTURAL AND APPLIED ECONOMICS

COLLEGE OF AGRICULTURE

UNIVERSITY OF MINNESOTA

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Yacov Tsur[1] and Amos Zemel[2]

Abstract:

Optimal exploitation of renewable groundwater resources when extraction affects the probability of occurrence of an irreversible event is studied. The term irreversible signifies that the event occurrence renders the resource obsolete. It is found that uncertainty concerning the event occurrence has a profound effect. Under certainty—when the stock level below which the event occurs is known in advance—the optimal state process converges to a unique equilibrium state. Under uncertainty, when the event occurrence level is unknown, we identify equilibrium intervals and show that optimal processes initiated elsewhere converge to a boundary of one of these intervals. Inside an equilibrium interval, the expected loss due to the event occurrence is so high that it does not pay to extract in excess of recharge, even though under certainty doing so would be beneficial. These properties are illuminated by means of an example for which analytic solutions are derived.

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1 Department of Agricultural and Applied Economics, University of Minnesota, 1994 Buford Avenue, St. Paul, MN 55108

2 Center for Energy and Environmental Physics, The Jacob Blaustein Institute for Desert Research, Ben Gurion University of the Negev, Sede Boqer Campus, 84990, Israel

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1. Introduction

This study is concerned with the allocation of groundwater resources under uncertainty with regard to the occurrence of an influential event. Such an event may correspond, for example, to the intrusion of salt water when the groundwater table declines below some unknown threshold level. The event is irreversible in that the resource cannot be used after its occurrence. It is found that uncertainty concerning the event occurrence level has a profound effect on optimal exploitation policies.

While the focus here is on groundwater resources, the analysis extends to other renewable resource situations and thus should have wide applications. Possible extensions are outlined in the concluding section.

The paper is built on a few strands of literature. The first is concerned with the exploitation of a natural resource deposit of unknown size, investigated by Kemp (1976, 1977), Cropper (1976, Model II), Loury (1978), and Gilbert (1978). These authors considered the case in which the uncertain event corresponds to depletion. Extensions to situations involving more desirable events, such as the development of a substitute product or discoveries of new deposits, were offered by Kamien and Schwartz (1978), Dasgupta and Stiglitz (1981), and Deshmukh and Pliska (1980, 1985), among others. This literature deals with non-renewable resources and assumes that extraction costs are constant over time. The present work incorporates recharge processes and allows the extraction costs to vary with the resource state.

In her "Catastrophic Pollution" model, Cropper (1976, Model I) was the first to incorporate (the equivalent of) a recharge process (namely, a pollution reduction process) within models involving event uncertainty. The uncertainty in this model is partly exogenous, so the event may occur regardless of the pollution stock increasing, decreasing or remaining constant. In our model, the event occurs when the groundwater stock reaches some unknown critical level. This can happen only when the stock decreases below the lowest level ever encountered in the recorded history. This property has important implications for the formulation of the exploitation model and the associated optimal extraction policies.

Another precursor of this work is the literature on renewable resource exploitation with state-dependent extraction costs (Burt, 1964, Dasgupta and Heal, 1979, Feinerman, 1988, Tsur and Graham-Tomasi, 1991, and many others). A parallel line of research studies the conditions under which it is profitable to harvest a stock of renewable resource to extinction (Clark, 1973, Lewis and Schmalensee, 1977, Cropper, 1988). The extinction event in these works, however, involves no uncertainty, as the stock level at which extinction is bound to occur is known in advance. By incorporating future event uncertainty, the present effort provides a unified framework of analysis into which the variants mentioned above can be accommodated.

In the absence of recharge, the resource stock cannot increase. This implies (as observed by Long, 1975, in a related context) that information obtained over time cannot affect decisions prior to the event occurrence—a property which greatly simplifies the formulation of the decision problem. With positive recharge, however, the state process may not be monotonic, hence learning can play a role and the decision problem is more involved. This difficulty is removed when we show that the optimal stock evolves monotonically in time.

With positive recharge a delicate issue is raised of whether to extract in excess of the recharge rate, thereby advancing the (probability of the) event occurrence, or to extract at or below the recharge rate, thus avoiding the event occurrence risk. We specify the conditions under which it is optimal to extract above, at or below the recharge rate, thereby characterizing the dynamic behavior as well as the equilibrium states. Under certainty, when the event occurrence state is known in advance, the optimal process converges to a unique equilibrium state from any initial state. In contrast, under uncertainty we identify equilibrium intervals and characterize approach paths to them. In so doing, we give precise meaning to the intuitive notion that exploitation policies under uncertainty are more conservative.

The analysis is carried out via a relation established between the equilibrium states and the roots of simple functions of the state variable that depend on the structural relations and parameters (but require no knowledge of the optimal policy).

2. Formulation of the decision problem

The state S_t represents the aquifer stock, $R(S_t)$ denotes the recharge rate, i.e., the net water inflow excluding extractions, and g_t is the extraction rate at time t. The time evolution of the stock process is given by

$$dS_t/dt \equiv \dot{S}_t = R(S_t) - g_t. \tag{2.1}$$

The aquifer capacity limit is $\bar{S}$.

The cost of extracting g at the state S is C(S)g, and the benefit of consuming g is Y(g). The net benefit of consuming g at the state S is Y(g)−C(S)g. Note that by letting the unit extraction cost vary with the state level, we do not require that the instantaneous net benefit function Y(g)−C(S)g is concave in (g,S). This complicates the analysis but allows extending the results to general net benefit structures.

The following assumptions are made: (i) R(S) is decreasing and concave, and

$R(\bar{S}) = 0$; (ii) $C(S)$ is non-increasing and convex; and (iii) $Y(g)$ is increasing and strictly concave, and $Y(0) = 0$. The assumptions with regard to Y and C are common. The assumed properties of $R(S)$ reflect the common observation that as the aquifer's stock declines, recharge increases at a diminishing rate.

An extraction plan consists of the extraction process g_t and the associated state process S_t, $t \geq 0$. The decision maker, however, may not be able to carry out the original plan, as S_t cannot decrease below some unknown threshold level X: as soon as S_t falls below the level X, the event occurs following which extraction ceases. Our information concerning the location of X is described in terms of a probability distribution function $F(S) = \Pr\{X<S\}$ and the associated density $f(S) \equiv dF(S)/dS$. A plan is **feasible** if, for all t, g_t is piecewise continuous and nonnegative, and $S_t \geq \underset{\sim}{S}$, where $\underset{\sim}{S} \geq 0$ is the lower support of the distribution of X.

Let T represent the event occurrence time. The distribution on X induces a distribution on T. Given that the event has not occurred at t = 0, the expected benefit generated by an extraction plan is

$$E_T\left\{\int_0^T [Y(g_t)-C(S_t)g_t]e^{-\rho t}dt \,|\, T>0\right\},$$

where E_T represents expectation with respect to the distribution of T and ρ is the time rate of discount. The aquifer allocation problem entails finding a feasible plan corresponding to

$$V(S_0) = \underset{\{g_t\}}{\text{Max}}\; E_T\left\{\int_0^T [Y(g_t)-C(S_t)g_t]e^{-\rho t}dt \,|\, T>0\right\} \tag{2.2}$$

subject to $\dot{S}_t = R(S_t)-g_t$, $g_t \geq 0$, $S_t \geq \underset{\sim}{S}$, and S_0 given. We assume that an optimal plan exists.

As the process evolves in time, our assessment of the probabilities of X and T may be modified. At each point of time, the distribution of X, given that the event has not yet occurred, depends on the history of the S-trajectory up to time t. In particular, it depends on $\tilde{S}_t = \underset{\tau\in[0,t]}{\text{Min}} \{S_\tau\}$, as it is known that X must lie below this value. This complicates the allocation problem, since the expected benefit in (2.2) involves $\tilde{S}_t$ which depends on all history to time t. The situation is greatly simplified if the state trajectory S_t evolves monotonically in time, since then $\tilde{S}_t = S_0$ or $\tilde{S}_t = S_t$ for non-decreasing or non-increasing trajectories, respectively. It turns out that the optimal S-trajectory (at least one) is indeed monotonic, as established in

Proposition 2.1 (Monotonicity): At least one of the optimal state trajectories corresponding to problem (2.2) evolves monotonically in time.

(For a proof, see Appendix A.) When problem (2.2) admits multiple optima, it is possible that some are non-monotonic. Nonetheless, at least one of the optimal plans must be monotonic. Consequently, we restrict attention to monotonic state trajectories.

For non-decreasing trajectories, it is known with certainty that the event will never occur and the objective function in (2.2) reduces to

$$\int_0^\infty [Y(g_t)-C(S_t)g_t]e^{-\rho t}dt. \tag{2.3}$$

For non-increasing state processes, the distribution of T is given by

$$1-F_T(t) \equiv Pr\{T>t\,|\,T>0\} = Pr\{X<S_t\,|\,X<S_0\} = F(S_t)/F(S_0) \tag{2.4}$$

with the density

$$f_T(t) \equiv dF_T(t)/dt = f(S_t)[g_t-R(S_t)]/F(S_0).$$

The hazard rate associated with T is

$$\frac{f_T(t)}{1-F_T(t)} = \lambda(S_t)[g_t-R(S_t)],$$

where

$$\lambda(S_t) = \frac{f(S_t)}{F(S_t)} . \tag{2.5}$$

Express the expectation in (2.2) as $E_T\left\{\int_0^\infty [Y(g_t)-C(S_t)g_t]I(T>t)e^{-\rho t}dt \mid T>0\right\}$,

with $I(\cdot)$ denoting the indicator function that assumes the values one or zero when its argument is true or false, respectively. Since $E_T\{I(T>t)\mid T>0\} = 1-F_T(t) = F(S_t)/F(S_0)$, the objective function for non-increasing trajectories becomes

$$\int_0^\infty [Y(g_t)-C(S_t)g_t]e^{-\rho t} \frac{F(S_t)}{F(S_0)} dt. \tag{2.6}$$

The allocation problem for which (2.3) is the objective is denoted the *certainty problem*, while that for which (2.6) is the objective is denoted the *auxiliary problem*. It is verified in Appendix A that

Remark 2.1: Proposition 2.1 holds for the optimal state trajectories corresponding to the certainty and auxiliary problems.

In the following two sections we characterize the dynamic behavior and equilibrium states of the optimal trajectories corresponding to the certainty and auxiliary problems. Studying the dynamic and equilibrium properties of the optimal state trajectory of the aquifer problem (2.2), we show, in Section 5, that this trajectory coincides with that of the certainty problem when it increases, and that it coincides with the optimal trajectory of the auxiliary problem when it decreases. The complete characterization of the optimal process requires, therefore, to determine the conditions under which the process increases, decreases or remains constant; this task is undertaken in Section 5.

3. Certainty

In this case, X is known in advance and the allocation problem is to find the state process corresponding to

$$V^c(S_0;X) = \underset{\{g_t\}}{\text{Max}} \int_0^\infty [Y(g_t)-C(S_t)g_t]e^{-\rho t}dt \tag{3.1}$$

subject to: $\dot{S}_t = R(S_t)-g_t$, $g_t \geq 0$, and $S_t \geq X$, where $X < \bar{S}$ is a given constant and $S_0 \in (X,\bar{S})$ is the initial state. The current-value Hamiltonian and Lagrangian functions corresponding to (3.1) are

$$H(S_t,g_t,p_t,t) = Y(g_t) - C(S_t)g_t + p_t[R(S_t)-g_t]$$

and

$$\mathcal{L}(S_t,g_t,p_t,\alpha_t,\gamma_t,t) = H(S_t,g_t,p_t,t) + \gamma_t g_t + \alpha_t[S_t-X]$$

where p_t is the current value costate variable, and γ_t and α_t are the current value Lagrange multipliers associated with the constraints $g_t \geq 0$ and $S_t \geq X$. Necessary conditions for an optimal process include (Arrow and Kurz, 1970, pp. 48-49): $\partial\mathcal{L}/\partial g = 0$, giving

$$Y'(g_t^c) - C(S_t^c) = p_t - \gamma_t, \tag{3.2}$$

$\dot{p}_t-\rho p_t = -\partial\mathcal{L}_t/\partial S_t$, yielding

$$\dot{p}_t = p_t[\rho-R'(S_t^c)] + C'(S_t^c)g_t^c - \alpha_t \tag{3.3}$$

and the complimentary slackness conditions

$$\gamma_t \geq 0, \quad \alpha_t \geq 0, \quad \gamma_t g_t^c = 0, \quad \alpha_t[S_t^c-X] = 0. \tag{3.4}$$

(The superscript "c" indicates optimal quantities.)

From (3.2)-(3.3), we obtain

$$\dot{p}_t= [Y'(g_t^c)-C(S_t^c)+\gamma_t][\rho-R'(S_t^c)] + g_t^c C'(S_t^c) - \alpha_t. \tag{3.5}$$

Using the time evolution of the state variable

$$\dot{S} = R(S) - g \tag{3.6}$$

to eliminate g_t^c from (3.5), we find

$$\dot{p}_t = \{-C'(S_t^c)-[\rho-R'(S_t^c)]Y''(\tilde{g})\}\dot{S}_t^c + L(S_t^c) + \gamma_t[\rho-R'(S_t^c)] - \alpha_t, \tag{3.7}$$

where $\tilde{g}$ is a value between g_t^c and $R(S_t^c)$,

$$L(S) \equiv [\rho-R'(S)][Y'(R(S)) - C(S) - J(S)] \tag{3.8}$$

and

$$J(S) \equiv \frac{-C'(S)R(S)}{\rho-R'(S)} . \tag{3.9}$$

An "equilibrium (or steady) state" refers to the S member of the (S,p) pair for which $\dot{p} = \dot{S} = 0$. It follows from (3.7) that an equilibrium state $\hat{S}$ must satisfy

$$L(\hat{S}) + \gamma[\rho-R'(\hat{S})] - \alpha = 0. \tag{3.10}$$

This result could have been attained more directly by substituting $R(S_t^c)$ for g_t^c in (3.5) and equating $\dot{p}_t$ to zero. The above derivation, however, conveys additional information on the process evolution, since (3.7) holds also when the system is away from equilibrium, and is therefore presented.

If the equilibrium state falls in $(X,\bar{S})$, then (3.4) requires that $\alpha = \gamma = 0$, hence $L(\hat{S}) = 0$. For an equilibrium to occur at $\hat{S} = X$, where $\alpha \geq 0$ and $\gamma = 0$, it must be, according to (3.10), that $L(X) = \alpha \geq 0$. If an equilibrium occurs at a full aquifer $\bar{S}$, i.e., the aquifer does not admit profitable exploitation, then $\alpha = 0$, $\gamma \geq 0$ and (3.10) requires that $L(\bar{S}) = -\gamma[\rho-R'(\bar{S})] \leq 0$; noting (3.8)-(3.9), this case occurs when $Y'(0) \leq C(\bar{S})$.

The vanishing of L(S) at an interior equilibrium can be motivated by noting, from (3.3), that the costate (shadow price) variable p equals J(S) at this state. Since $\rho-R'(S) > 0$, the roots of L(S) are the same as the roots of $Y'(R(S)) - C(S) - J(S)$. Thus, $L(S) = 0$ is consistent with condition (3.2) at an interior equilibrium.

Since Y is strictly concave, R is decreasing and concave, and C is non-increasing and convex, the difference $Y'(R(S)) - C(S) - J(S)$ must increase with S and there must exist a unique state level $\hat{S}$ in $[X,\bar{S}]$, satisfying

$$\begin{cases} \hat{S} = X & \text{if } L(X) > 0 \\ \hat{S} = \bar{S} & \text{if } L(\bar{S}) < 0 \cdot \\ L(\hat{S}) = 0 & \text{otherwise} \end{cases} \tag{3.11}$$

It is evident from (3.10) and the above discussion that any equilibrium state must satisfy (3.11). Hence, $\hat{S}$ is the unique equilibrium state. Now, the optimal state trajectory is monotonic (Remark 2.1) and bounded (between X and $\bar{S}$), hence it must converge to an equilibrium state. We have thus established:

Proposition 3.1: When X is known with certainty, $\hat{S}$ is the unique steady state to which the optimal state process converges from any initial state.

The situation is depicted in Figure 1.

Figure 1

4. The Auxiliary Problem

In this section we assume that $\underset{\sim}{S} < \hat{S} < \bar{S}$. Let $K \in (\hat{S},\bar{S}]$ be a given constant. The auxiliary problem is formulated as:

$$V^a(S_0;\hat{S},K) = \underset{\{g_t\}}{\text{Max}} \int_0^\infty [Y(g_t)-C(S_t)g_t]\frac{F(S_t)}{F(S_0)}e^{-\rho t}dt \tag{4.1}$$

subject to: $\dot{S}_t = R(S_t)-g_t$, $S_t \geq \hat{S}$, $S_t \leq K$, and $g_t \geq 0$. $S_0 \in [\hat{S},K]$ is the initial state.

With $\Lambda(S_t) = \log[F(S_t)/F(S_0)]$ and $d\Lambda/dS_t = f(S_t)/F(S_t) \equiv \lambda(S_t)$, the current value Hamiltonian and Lagrangian functions corresponding to (4.1) are:

$$H(S,g,p,t) = [Y(g)-C(S)g]e^{\Lambda(S)} + p[R(S)-g]$$

and

$$\mathcal{L}(S,g,p,\alpha,\beta,\gamma,t) = H(S,g,p,t) + \alpha[S-\hat{S}] + \beta[K-S] + \gamma g,$$

where p, α and γ are as defined in Section 3 and β is the Lagrange multiplier corresponding to $S_t \leq K$. Necessary conditions for optimum include: $\partial\mathcal{L}/\partial g = 0$, giving

$$Y'(g_t^a) - C(S_t^a) = [p_t - \gamma_t]e^{-\Lambda(S_t^a)}, \tag{4.2}$$

$\dot{p} - \rho p = -\partial\mathcal{L}/\partial S$, yielding

$$\dot{p}_t = p_t[\rho - R'(S_t^a)] + \{g_t^a C'(S_t^a) - \lambda(S_t^a)[Y(g_t^a) - C(S_t^a)g_t^a]\}e^{\Lambda(S_t^a)} - \alpha_t + \beta t \tag{4.3}$$

and the complimentary slackness conditions

$$\alpha_t \geq 0,\quad \beta_t \geq 0,\quad \gamma_t \geq 0,\quad \alpha_t[S_t^a - \hat{S}] = 0,\quad \beta_t[K - S_t^a] = 0,\quad \gamma_t g_t^a = 0. \tag{4.4}$$

(The superscript "a" indicates optimal quantities.)

From (4.2) and (4.3) we obtain

$$\dot{p}_t = \left\{[Y'(g_t^a) - C(S_t^a)][\rho - R'(S_t^a)] + g_t^a C'(S_t^a) - \lambda(S_t^a)[Y(g_t^a) - C(S_t^a)g_t^a]\right\}e^{\Lambda(S_t^a)} + \gamma[\rho - R'(S_t^a)] - \alpha_t + \beta_t \tag{4.5}$$

Following Section 3, we use (3.6) to eliminate g_t^a from (4.5) and find

$$\dot{p}_t = e^{\Lambda(S_t^a)}\left(\lambda(S_t^a)[Y'(\tilde{g}) - C(S_t^a)] - C'(S_t^a) - Y''(\hat{g})[\rho - R'(S_t^a)]\right)\dot{S}_t^a + e^{\Lambda(S_t^a)}L_M(S_t^a) + \gamma[\rho - R'(S_t^a)] - \alpha_t + \beta_t. \tag{4.6}$$

In (4.6), $\tilde{g}$ and $\hat{g}$ are some values between g_t^a and $R(S_t^a)$,

$$L_M(S) \equiv [\rho - R'(S)][Y'(R(S)) - C(S) - J(S)] - M(S) \equiv L(S) - M(S), \tag{4.7}$$

$L(S)$ and $J(S)$ are defined in (3.8)-(3.9), and

$$M(S) \equiv \lambda(S)[Y(R(S)) - C(S)R(S)]. \tag{4.8}$$

Observing (4.6), one finds that at every steady state

$$L_M(S) + \gamma[\rho - R'(S)]e^{-\Lambda(S)} - (\alpha - \beta)e^{-\Lambda(S)} = 0 \tag{4.9}$$

must hold.

Consider the roots of $L_M(S)$ in the open interval $(\hat{S}, K)$. In this interval $\alpha = \beta = 0$ by virtue of conditions (4.4). Below $\bar{S}$, extractions cannot vanish along the $\dot{S} = 0$ curve (in the S,p phase plane), hence γ must also vanish along this curve. With $L_M(\hat{S}_M) = 0$, (4.6) implies that the $\dot{p} = 0$ and $\dot{S} = 0$ curves must intersect at $S = \hat{S}_M$. Thus, each of these roots corresponds to an equilibrium state. Conversely, we see in (4.4) and (4.9) that any equilibrium state in $(\hat{S},K)$ must be a root of L_M. We have thus established

Proposition 4.1: The equilibrium states in $(\hat{S},K)$ corresponding to the auxiliary problem (4.1) coincide with the roots of $L_M(S)$ in this interval.

This result is similar to Proposition 3.1 of the certainty case with $L_M(S)$ replacing $L(S)$. The difference between these two functions, namely $M(S)$, measures the expected loss due to the event occurrence, as discussed in detail in the next section. Thus, the shift of the equilibrium states is a direct manifestation of this expected loss.

In fact, when $\underline{S} < \hat{S} < \bar{S}$ and $\lambda(\hat{S}) > 0$, Proposition 4.1 can be extended to the closed interval $[\hat{S},\bar{S}]$, noting that

lemma 4.1: If $\underline{S} < \hat{S} < \bar{S}$ and $\lambda(\hat{S}) > 0$, then $L_M(\bar{S}) > 0$ and $L_M(\hat{S}) < 0$.

Proof: We note first that $M(\bar{S}) = 0$, hence (4.7) implies that $L(\bar{S}) = L_M(\bar{S})$. Thus, if $L_M(\bar{S}) \leq 0$, then $\hat{S} = \bar{S}$, contradicting our assumption. Next, observe that $M(S) > 0$ for all $\hat{S} \leq S < \bar{S}$ for which $\lambda(S) > 0$. To see this, recall that $L(S) = [\rho - R'(S)][Y'(R(S)) - C(S) - J(S)] \geq 0$ above $\hat{S}$ and $J(S) \geq 0$, hence $Y'(R(S)) - C(S) \geq L(S)/[\rho - R'(S)] \geq 0$. Using the concavity of Y,

$$Y(R(S)) - C(S)R(S) = \int_0^{R(S)} [Y'(g) - C(S)]dg > 0,$$

verifying that $M(S) > 0$ and $L_M(S) < L(S)$. If $\hat{S} > \underline{S}$, then $L_M(\hat{S}) < L(\hat{S}) = 0$. ∎

Setting $K = \bar{S}$, we can now establish

lemma 4.2: If $\underline{S} < \hat{S} < \bar{S}$ and $\lambda(\hat{S}) > 0$, then $\hat{S}$ and $\bar{S}$ are not equilibrium points.

Proof: Consider the slackness conditions (4.4). At $S = \bar{S}$, $L_M(S)$, β and γ are positive and α vanishes. Thus, the left hand side of (4.9) does not vanish, and $\bar{S}$ cannot be an equilibrium state. At $S = \hat{S}$, $L_M(S)$ is negative, β and γ vanish and α is positive. Thus, the left hand side of (4.9) does not vanish, ruling out the possibility that $\hat{S}$ is an equilibrium state.■

Two cases are of interest: (i) $K = \bar{S}$, $\hat{S} \in (\underline{S},\bar{S})$ and $\lambda(\hat{S}) > 0$, in which case $L_M(S)$ must have a root in $(\hat{S},\bar{S})$; (ii) $L_M(K) \le 0$ and $L_M(S)$ has no root in $[\hat{S},K)$. We discuss each of these cases separately.

For case (i), let $\hat{S}_L$ and $\hat{S}_U$ denote, respectively, the smallest and the largest roots in $(\hat{S},\bar{S})$. We can now prove

Proposition 4.2: Starting at some initial level S_0, the optimal state trajectory S_t^a corresponding to $V^a(S_0;\hat{S},\bar{S})$ decreases if $\hat{S}_U < S_0 \le \bar{S}$ and increases if $\hat{S} \le S_0 < \hat{S}_L$.

Proof: According to Proposition 4.1 and Lemma 4.2, S_0 is not an equilibrium state. If $S_0 > \hat{S}_U$ and S_t^a increases, the trajectory must reach a steady state above $\hat{S}_U$, violating proposition 4.1 or Lemma 4.2. If $S_0 < \hat{S}_L$ and S_t^a decreases, the trajectory must reach a steady state below $\hat{S}_L$, violating proposition 4.1 or Lemma 4.2 again.■

The situation is particularly simple when $L_M(S)$ has a single root in $[\hat{S},\bar{S}]$, denoted $\hat{S}_M$. In this case, any process initiated within $[\hat{S},\bar{S}]$ must converge to the unique equilibrium state $\hat{S}_M$. An example of such a situation, which is similar to the certainty problem with $\hat{S}_M$ replacing $\hat{S}$, is studied in Section 6. When several equilibrium states exist, the particular value to which the optimal stock process converges depends on the initial state. However, not all the roots of $L_M(S)$ should be considered as possible equilibrium states. Depending on the local behavior of $L_M(S)$ near its roots, the following lemma rules out some of them:

Lemma 4.3: If, for some $\tilde{S} \in (\hat{S},\bar{S})$, $L_M(\tilde{S}) = 0$ and L_M decreases in some neighborhood of $\tilde{S}$, then, starting at any $S \neq \tilde{S}$, the optimal state process corresponding to $V^a(S_0;\hat{S},\bar{S})$ will never converge to $\tilde{S}$.

The proof is presented in Appendix B, which also extends Lemma 4.3 to situations where both $L_M(S)$ and its derivative vanish:

Remark: (i) If a root $\tilde{S}$ is a local maximum of $L_M(S)$, the same argument shows that it cannot be optimal to approach $\tilde{S}$ from above and stop there. (ii) If a root $\tilde{S}$ is a local minimum of $L_M(S)$, the same argument shows that it cannot be optimal to approach $\tilde{S}$ from below and stop there.

We turn now to case (ii), in which $L_M(K) \leq 0$ and $L_M(S)$ has no root in $[\hat{S},K)$. For this case, which occurs when $K \leq \hat{S}_L$, the following result holds:

Lemma 4.4: If $L_M(S) < 0$ for every S in $[\hat{S},K)$, then K is the unique equilibrium state associated with $V^a(S_0;\hat{S},K)$.

Proof: $L_M(S)$ has no roots, hence Proposition 4.1 implies that the process has no equilibrium states in $(\hat{S},K)$. $L_M(\hat{S}) < 0$, hence Lemma 4.2 implies that $\hat{S}$ is not an equilibrium state. Since the process is monotonic and bounded in $[\hat{S},K]$, it must increase to K.■

The results obtained for the auxiliary problem can be used to analyze the aquifer problem. This is done in the next section.

5. Uncertain irreversible events

Under certainty, the optimal state process possesses the appealing property that it converges to a unique steady state $\hat{S}$ regardless of the initial state level. The situation is quite different under uncertainty. Here, we identify equilibrium intervals such that optimal state processes initiated within these intervals remain constant. If the initial state lies outside these intervals, the particular choice of the equilibrium state depends on the initial state.

We consider first the case $S < \hat{S} < \bar{S}$. Let S_t^* indicate optimal trajectories corresponding to the aquifer problem (2.2), and recall that the optimal processes corresponding to the certainty and auxiliary problems are denoted S_t^c and S_t^a, respectively. At low stock levels, when S_t^c increases, the trajectories S_t^c and S_t^* coincide. This result is formulated as

Lemma 5.1: Starting at some initial S satisfying $L(S) < 0$, i.e., $S < \hat{S}$, $S_t^* = S_t^c$.

Proof: Consider some initial stock $S < \hat{S}$ and suppose that the optimal trajectory S_t^* does not increase. Let V(S) be the value of the optimal plan under uncertainty and $U(S) = \int_0^\infty [Y(g_t^*)-C(S_t^*)g_t^*]e^{-\rho t}dt$ represent the benefit derived from S_t^* in the favorable situation where the plan is never interrupted by the event. As shown in Appendix A, occurrence of the event cannot be desirable and $V(S) \le U(S)$. For $S < \hat{S}$, the optimal S-trajectory under certainty, S_t^c, increases (Proposition 3.1), hence $U(S) < V^c(S;X)$. But S_t^c is feasible under uncertainty as well, yielding the same value $V^c(S;X) > V(S)$. Hence, the non-increasing S-trajectory cannot be optimal under uncertainty. Thus, for states $S < \hat{S}$, the optimal trajectories for certainty and uncertainty coincide, and are increasing.■

We study now the properties of S_t^* in the complimentary interval $[\hat{S},\bar{S}]$:

Lemma 5.2: Starting at some initial level S satisfying $L(S) \ge 0$, i.e., $S \ge \hat{S}$, the optimal trajectory S_t^* cannot increase.

Proof: Consider the benefit $U(S) = \int_0^\infty [Y(g_t^*)-C(S_t^*)g_t^*]e^{-\rho t}dt$ associated with any increasing trajectory. When $S \geq \hat{S}$, i.e., when $L(S) \geq 0$, $U(S) < W(S) \equiv [Y(R(S))-C(S)R(S)]/\rho$, where W(S) is the benefit obtained from the steady state policy $g_t \equiv R(S)$. To see this, consider the certainty problem for which the event level X is taken to be equal to the initial level S. For this problem, every increasing trajectory is feasible, yet Proposition (3.1) implies that, with $L(X) = L(S) > 0$, the plan $S_t \equiv X$ is optimal, and the inequality follows. Now, uncertainty does not affect the benefit associated with non-decreasing trajectories, hence the steady state policy outperforms all increasing plans under uncertainty as well, and S_t^* cannot increase.■

The results above stress the common features of the optimal trajectories corresponding to the certainty and the aquifer problems. For certain states above $\hat{S}$ we can strengthen the similarity by ruling out the possibility that they are steady states. This is done by partitioning the interval $[\hat{S},\bar{S}]$ according to the roots of $L_M(S)$, in similarity with case (i) of the previous section.

Lemma 5.3: Any S satisfying $L(S) > 0$ and $L_M(S) > 0$ does not qualify as a steady state of S_t^*.

Proof: The idea is to construct a decreasing extraction plan which yields an expected benefit higher than the value $W(S) = [Y(R(S))-C(S)R(S)]/\rho$, obtained under the steady state plan. For some arbitrary small constants $h>0$ and $\delta>0$, define the extraction plan, starting at the state S

$$g_t = \begin{cases} R(S) + \delta, & 0 \leq t < h \\ R(S_h), & t \geq h \end{cases} . \tag{5.1}$$

With this plan, for all $t \le h$, $\Delta S_t = S_t - S = \int_0^t [R(S_{t'}) - R(S) - \delta]dt' = -\delta t + o(\delta t)$. The expected benefit associated with g_t is

$$V^{\delta h}(S) = \int_0^h [Y(R(S)+\delta) - C(S_t)(R(S)+\delta)]e^{\Lambda(S_t)}e^{-\rho t}dt + e^{\Lambda(S_h)}e^{-\rho h}W(S_h) + o(\delta h) \quad (5.2)$$

The first term on the right-hand side of (5.2) is expanded as

$$\int_0^h [Y(R(S)+\delta) - C(S_t)(R(S)+\delta)]e^{\Lambda(S_t)}e^{-\rho t}dt = W(S)[1-e^{-\rho h}] + [Y'(R(S)) - C(S)]\delta h + o(\delta h),$$

where use has been made of $\Delta S_t = -\delta t + o(\delta t)$, knowing that C' and λ are uniformly bounded in $[S, S_t]$. The second term on the right-hand side of (5.2) is expanded as $W(S_h)e^{\Lambda(S_t)}e^{-\rho h} = W(S)e^{-\rho h} - W(S)\lambda(S)\delta h - W'(S)\delta h + o(\delta h)$.

Combining terms, using $\rho W'(S) = [Y'(R(S)) - C(S)]R'(S) - C'(S)R(S)$, we obtain

$$V^{\delta h}(S) - W(S) = L_M(S)\delta h/\rho + o(\delta h). \quad (5.3)$$

Observing (5.3), we see that when $L_M(S) > 0$, there exist $h > 0$ and $\delta > 0$ such that $V^{\delta h}(S) - W(S) > 0$. Thus, the steady state plan that yields the value $W(S)$ is not optimal, ruling out the possibility that S is a steady state.■

Together, Lemma 5.2 and Lemma 5.3 imply that when $L_M(S) > 0$, S_t^* and S_t^c evolve in the same direction:

Lemma 5.4: Starting at some initial level S satisfying $L(S) \ge 0$ and $L_M(S) > 0$, the optimal trajectory S_t^* must decrease.

Denoting, as in Section 4, the smallest root of $L_M(S)$ in $[\hat{S}, \bar{S}]$ by $\hat{S}_L$, we consider the interval in which the effect of uncertainty is most pronounced:

Lemma 5.5: Any state S in $[\hat{S}, \hat{S}_L]$ must be a steady state.

Remark: Recall that the interval $[\hat{S},\hat{S}_L]$ does not reduce to a single point unless $\lambda(\hat{S}) = 0$.

Proof: Let $S_0 \in [\hat{S},\hat{S}_L]$ be the initial state. According to Lemma 5.2, S_t^* cannot increase. To show that it cannot decrease, consider the auxiliary problem associated with $V^a(S_0;\hat{S},S_0)$, for which the process cannot increase above the initial value. For this problem, $L_M(S) < 0$ for all $S \in [\hat{S},S_0)$. According to Lemma 4.4, only the end point S_0 can be an equilibrium state for this auxiliary problem, hence the optimal process associated with it must remain at the initial value S_0. It follows that the steady state benefit $W(S_0)$ exceeds the benefit obtained from any feasible decreasing trajectory. For non-increasing trajectories, the benefits associated with the auxiliary and the aquifer problems are the same, hence the steady state policy outperforms any decreasing plan for the aquifer problem as well. Thus, S_t^* cannot decrease.■

To gain insight, consider the case in which $L_M(S) = 0$ has a unique solution in $[\hat{S},\bar{S}]$, denoted $\hat{S}_M$. For this case, the dynamics and equilibrium structure of S_t^* can be characterized in a simple manner:

Proposition 5.1: If $L_M(S)$ has a unique root $\hat{S}_M$ in $[\hat{S},\bar{S}]$, then:
(i) S_t^* increases while passing through S levels below $\hat{S}$, where $L(S) < 0$; (ii) S_t^* decreases while passing through S levels above $\hat{S}_M$, satisfying $L(S) > 0$ and $L_M(S) > 0$; (iii) state levels S in $[\hat{S},\hat{S}_M]$, for which $L(S) \geq 0$ and $L_M(S) < 0$, are equilibrium states of S_t^*.

This situation is illustrated in Figure 2.

Figure 2

Proposition 5.1 implies that the optimal stock process converges to the boundaries of $[\hat{S},\hat{S}_M]$ from any initial state outside this interval, and remains constant when initiated at any state inside this interval.

In fact, the continuity of S_t^* implies that the interior $(\hat{S},\hat{S}_M)$ is forbidden for all optimal trajectories initiated outside this interval. It is in this open interval (which is not empty unless $\lambda(\hat{S})$ vanishes) that the effect of uncertainty is most vividly seen. Here, the expected loss due to event occurrence is so high that entering the interval cannot be optimal, even if under certainty doing so would yield a higher benefit.

This observation is borne out by (4.7)-(4.8), which show that the steady state interval is due to the difference between L(S) and $L_M(S)$, namely M(S). Indeed, $M(S) = \lambda(S)[Y(R(S))-C(S)R(S)]$, the former term of which measures the risk that the event will follow immediately a decision to extract above recharge, while the latter term is the permanent benefit stream which could have been enjoyed had a steady state policy been adopted and which is lost due to the event occurrence. Within $(\hat{S},\hat{S}_M)$, the expected loss more than outweighs the positive value of L(S), and extraction above recharge is too risky to be optimal. The exact relation between $L_M(S)$ and the steady state value W(S) is elucidated by (5.3).

We close the discussion by considering the extreme cases $\hat{S} = \underset{\sim}{S}$ and $\hat{S} = \bar{S}$. If $\hat{S} = \underset{\sim}{S}$ and $L(\hat{S}) > 0$ it can happen that $L_M(S)$ possesses no root in $[\underset{\sim}{S},\bar{S}]$, in which case, according to Lemma 5.4, the aquifer will be depleted under all circumstances. If $\hat{S} = \bar{S}$, then Lemma 5.1 applies to the entire interval $[\underset{\sim}{S},\bar{S}]$, and the aquifer does not admit profitable exploitation under all circumstances. In fact, this case occurs only if $Y'(0)-C(\bar{S}) \le 0$ which implies, in turn, that the instantaneous benefit $Y(g)-C(S)g$ is negative for all states S and all non-vanishing extraction rates.

6. Example

We present an example for which analytic expressions for the optimal state processes are derived. Our aim is to illuminate the effects of uncertainty on optimal exploitation processes. In particular, we wish to examine the complete time dependence of the processes and not limit attention to the steady states. To that end, we consider the simple case of constant cost, $C(S) = c$, constant hazard function $\lambda(S) = \lambda$, and linear recharge rate, $R(S) = r(\bar{S}-S)$. With $C(S)$ constant, $J(S) = 0$ for all S. We assume that $Y'(0) > c$, so that $L_M(\bar{S}) = L(\bar{S}) = (\rho+r)[Y'(0)-c] > 0$ and the aquifer admits profitable exploitation. We further assume that $\hat{S} > \underset{\cdot}{S}$. Finally, we consider an initial state $S_0 > \hat{S}$, so that the optimal stock process decreases (cf. Proposition 5.1).

The state process evolves according to

$$\dot{S}_t = r(\bar{S}-S_t)-g_t. \tag{6.1}$$

The condition of maximal Hamiltonian (cf. (4.2)) gives

$$p_t e^{-\Lambda(S_t)} = Y'(g_t) - c. \tag{6.2}$$

Taking the time derivative of (6.2), we find

$$\dot{p}_t e^{-\Lambda(S_t)} = Y''(g_t)\dot{g}_t + \lambda\dot{S}_t[Y'(g_t) - c]. \tag{6.3}$$

From $\dot{p}_t - \rho p_t = -\partial\mathcal{L}/\partial S$, we obtain,

$$[\dot{p}_t - p_t(\rho+r)]e^{-\Lambda(S_t)} = -\lambda[Y(g_t) - cg_t]. \tag{6.4}$$

Using (6.2) and (6.3) to eliminate p_t and $\dot{p}_t$, (6.4) becomes

$$Y''(g_t)\dot{g}_t - (\rho+r)[Y'(g_t) - c] + \lambda\{Y(g_t)-cg_t+[Y'(g_t)-c]\dot{S}_t\} = 0. \tag{6.5}$$

Under certainty, $\Lambda(S_t^c)$ and λ vanish, and (6.4) reduces to $\dot{p}_t = p_t(\rho+r)$. At the steady state $\hat{S}$, $\dot{p}_\infty = 0$, implying that p_t must vanish at all times. From (6.2) we see that the extraction rate g_t^c is also independent of time, its constant value given by the solution of $Y'(g)-c = 0$. The steady state $\hat{S}$ is determined in the same manner: $Y'(r(\bar{S}-\hat{S}))-c = 0$.

For arbitrary times, we use (6.1) to eliminate g_t^c from (6.2) and obtain $Y'(r(\bar{S}-S_t^c)-\dot{S}_t^c) = Y'(r(\bar{S}-\hat{S}))$, or, in view of the strict concavity of Y, $\dot{S}_t^c + rS_t^c = r\hat{S}$. Thus,

$$S_t^c = \hat{S} + (S_0-\hat{S})e^{-rt} \tag{6.6}$$

In order to derive the time evolution under uncertainty, we restrict attention to the parabolic approximation $Y(g) = bg - ag^2$ (it is assumed that g is never large enough to enter the decreasing branch of this function.) Using (6.1) to eliminate g_t^*, we find

$$Y''(g_t^*)\dot{g}_t^* - (\rho+r)[Y'(g_t^*)-c] = 2a\ddot{S}_t^*-2a\rho\dot{S}_t^*+2a(\rho+r)r(\bar{S}-S_t^*)-(b-c)(\rho+r), \tag{6.7}$$

and

$$Y(g_t^*) - cg_t^* + [Y'(g_t^*) - c]\dot{S}_t^* = a\dot{S}_t^{*2} - ar^2(\bar{S}-S_t^*)^2 + (b-c)r(\bar{S}-S_t^*). \tag{6.8}$$

Setting $q_t = \bar{S}-S_t^*$ and $\hat{q} = (b-c)/(2ar)$, we can use (6.7)-(6.8) and reduce (6.5) to

$$\ddot{q}_t - \lambda\dot{q}_t^2/2 - \rho\dot{q}_t - (\rho+r)rq_t + \lambda r^2q_t^2/2 - \lambda r^2\hat{q}q_t = -(\rho+r)r\hat{q}. \tag{6.9}$$

The effect of uncertainty is manifest through the terms involving λ, which introduce nonlinearities and shift the steady state. The steady state $\hat{q}_M$ corresponds to the root of $L_M(S) = 0$, or

$$\lambda r\hat{q}_M^2/2 - (\rho+r + \lambda r\hat{q})\hat{q}_M + (\rho+r)\hat{q} = 0, \tag{6.10}$$

yielding

$$\hat{q}_M = \hat{q} + [(\rho+r)/(\lambda r)] - \sqrt{\hat{q}^2 + [(\rho+r)/(\lambda r)]^2}. \tag{6.11}$$

For small λ, (6.11) can be approximated by $\hat{q}_M \cong \hat{q} - \lambda\hat{q}^2r/[2(\rho+r)]$.

Setting $Q_t = \lambda(q_t-\hat{q}_M) = -\lambda(S_t^*-\hat{S}_M)$ and using (6.9) and (6.10), we obtain

$$\ddot{Q}_t - \dot{Q}_t^2/2 + r^2Q_t^2/2 - \rho\dot{Q}_t - [(\rho+r)r+\varepsilon]Q_t = 0, \tag{6.12}$$

where $\varepsilon = \lambda r^2(\hat{q}-\hat{q}_M) > 0$.

Let ζ be the negative root of the characteristic equation $x^2-\rho x-[(\rho+r)r+\varepsilon] = 0$ of the linear part of (6.12). It is easily verified that $\zeta < -r$. The non-diverging solution of (6.12) is expanded as

$$Q_t = \sum_{k=1}^{\infty} N^k a_k e^{k\zeta t}, \tag{6.13}$$

where $a_1 = 1$ and for $k>1$ a_k are constructed recursively:

$$a_k = \frac{\sum_{n+m=k} [nm-(r/\zeta)^2]a_n a_m}{2(k-1)(k+1-\rho/\zeta)} \quad (n,m>0). \tag{6.14}$$

The constant N is determined by the normalization condition

$$\sum_{k=1}^{\infty} N^k a_k = Q_0 = -\lambda(S_0-\hat{S}_M). \tag{6.15}$$

The uncertainty formulation (6.12) is valid only if $S_0 > \hat{S}_M$, hence Q_0 and N must be negative and the series in (6.13) and (6.15) consist of terms with alternating signs. Therefore, (6.15) can be solved for N only if $-Q_0$ is not too large, and (6.13) is a valid convergent representation of the solution to (6.12) only when S_0 is close enough to the asymptotic value $\hat{S}_M$. For higher values of $-Q_0$, the nature of the solution is similar, but a simple analytic expression, analogous to (6.13), is not available. Standard techniques to reduce to order of the equation can be used in this case.

Within its domain of validity, the solution (6.13) displays several modifications to the simple exponential solution (6.5) corresponding to certainty: First, uncertainty changes the asymptotic steady state from $\hat{S}$ to $\hat{S}_M$. Secondly, it increases the basic decay constant from r to $-\zeta$. Finally, the extraction rate g_t^* is no longer independent of time, and the steady state resource price p does not vanish:

$$p_\infty = M(\hat{S}_M)e^{\Lambda(\hat{S}_M)}/(\rho+r). \tag{6.16}$$

Examples of solutions obtained in this way are displayed in Figure 3 for $S_0 = \bar{S}$, $\bar{S}-\hat{S} = 1$, $\rho/r = 1$ and $\lambda = 2$. It is seen that uncertainty implies a more conservative extraction policy. Most notable is the increase of the steady state level $\hat{S}_M$ relative to $\hat{S}$. In the particular example at hand, the steady state interval corresponding to uncertainty covers over 40% of the interval $[\hat{S},\bar{S}]$, for which certainty conditions imply decreasing optimal trajectories.

7. Closing comments

The effects of irreversible uncertain events on the exploitation of groundwater resources are studied. We characterize the dynamic behavior and the equilibrium states of the optimal policy in terms of the evolution functions L and L_M. These functions, it is found, are useful in studying optimal exploitation processes when the presence of recharge, uncertainty and state dependent cost terms complicates the analysis based on phase plane configurations.

Irreversible events, after which the resource can no longer be used, pose a severe problem to the resource manager in that mistakes are too costly to fix; hence this case is of interest. The analysis, however, can be extended to situations in which the event is partly reversible, i.e., the resource can be used during the post-event period at the expense of some curing activities.

One can also consider resources other than groundwater and situations in which the probability of the event occurrence is wholly or partly exogenous. This is the case, for example, when the event corresponds to the extinction of a threatened animal population. The probability of an extinction event depends, inter alia, on exogenous factors such as inter-species dependencies and natural disasters (forest fires, diseases). A complete study of such situations, as well as of the partly reversible case mentioned above, is left for future research.

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Appendix A: Monotonicity of the state processes

Proposition 2.1 (Monotonicity): At least one of the optimal state trajectories corresponding to problem (2.2) evolves monotonically in time.

We begin with the simpler problems:

Remark 2.1: Proposition 2.1 holds for the optimal state trajectories corresponding to the certainty and auxiliary problems.

Proof: Consider first the case in which the optimal trajectory is unique. Suppose that S_t is not monotonic. For concreteness, consider three distinct time values, $t_1 < m < t_2$, such that $S_{t_1} < S_m$ and $S_{t_2} < S_m$. Since S_t is time-continuous, there must exist some $t_3 \in (t_1,m)$, at which S_t increases, and some $t_4 \in (m,t_2)$, at which S_t decreases, such that $S_{t_3} = S_{t_4}$. However, Y, C, R, F and ρ do not depend on t explicitly, hence the same decision problem is encountered at t_3 and at t_4. Thus, one cannot arrive at conflicting decisions concerning the sign of $g_t - R(S_t)$ at these times, since the optimality of both decisions violates the uniqueness of the optimal plan. This argument applies also when S_m corresponds to a minimum rather than to a maximum.

For problems with multiple optimal solutions, it is not possible to show that every optimal S trajectory is monotonic. We shall show, however, that at least one optimal path is monotonic. Observe, first, that the optimality of the decisions at t_3 and t_4 implies that one can choose either g_{t_3} or g_{t_4} at t_3 and t_4 and obtain the same value. Furthermore, this freedom of choice prevails at any state level between S_{t_3} and S_m. Thus, the existence of a local extremum of S implies the existence of a continuum of feasible plans, all yielding the optimal value. To construct a monotonic plan, one needs to specify, for any state S permitting several optimal extraction rates, a particular selection rule ensuring that whenever S is encountered, the same extraction rate is adopted. For example, one can demand that among the optimal extraction rates, the minimal optimal extraction rate is selected. The ensuing plan is optimal and monotonic, because non-monotonic plans involve conflicting choices of extraction rates at the same state levels.∎

The above discussion shows that non-monotonic optimal plans are associated with problems that are somewhat degenerate, in that they permit a continuum of optimal solutions, hence are unlikely to be encountered in realistic applications.

Proof of Proposition 2.1: The aquifer allocation problem (2.2) differs from the certainty and auxiliary problems in that decisions may depend on history. This means that passing through the same state at different times may lead to conflicting decisions. Nevertheless, we show that the monotonicity property of some optimal trajectory is preserved.

Consider first problems admitting a unique solution. As in the certainty and auxiliary problems, if the optimal state process passes through a maximum, the continuity of S_t implies that it obtains the same value at some $t_3 < m$, when extraction is below recharge, and at some $t_4 > m$, when extraction exceeds recharge, so that $S_{t_3} = S_{t_4} < S_m$. Indeed, t_3 and t_4 can be chosen so that S_t obtains its minimum value at the end points of $[t_3, t_4]$, ensuring that $\tilde{S}_t = \underset{\tau \in [0,t]}{\text{Min}} \{S_\tau\}$ is constant in that interval. The decision problems at t_3 and t_4 are the same, contradicting the different decisions taken at these two times. Thus, the possibility of a local maximum conflicts with the assumption of a unique solution, and once S_t starts increasing, it cannot decrease at later times.

The analysis of a possible local minimum is more involved. Suppose that extraction exceeds recharge at $t_3 < m$ but falls short of recharge at $t_4 > m$, and

S_m is the minimum value obtained during $[t_3,t_4]$. Although $S_{t_3} = S_{t_4}$, the decision problems at t_3 and t_4 may not be the same, since it is possible that $\tilde{S}_{t_4} = S_m < \tilde{S}_{t_3}$. To see that such "learning effects" cannot take place, consider any time t and the corresponding state S_t along the optimal path, and let

$$U(S_t) = \int_0^\infty [Y(g_{t+t'})-C(S_{t+t'})g_{t+t'}]e^{-\rho t'}dt'.$$

be the benefit from the **uninterrupted** plan starting at S_t. For non-monotonic state process, (2.4) changes to

$$1-F_T(t) \equiv Pr\{T>t|T>0\} = Pr\{X<\tilde{S}_t|X<\tilde{S}_0\} = F(\tilde{S}_t)/F(\tilde{S}_0),$$

and the expected benefit at time t generated by the optimal plan starting with the state S_t is given by

$$V(S_t) = \int_0^\infty [Y(g_{t+t'})-C(S_{t+t'})g_{t+t'}]\frac{F(\tilde{S}_{t+t'})}{F(\tilde{S}_t)} e^{-\rho t'}dt'.$$

Strictly speaking, both U and V do not depend on S_t alone, but rather have an explicit time dependence. (For V, the time dependence enters through the $\tilde{S}_t$ term in the probability factor.) For notational convenience, the time dependence is suppressed; once monotonicity is established, the explicit notation is no longer required.

Observe that for every $t' > 0$, $\tilde{S}_t \geq \tilde{S}_{t+t'}$, hence $U(S_t) \geq V(S_t)$ for all S_t along an optimal path. This result, which reduces to a trivial equality for $t \geq m$, corresponds, for $t<m$, to the intuitive notion that the interruption of the optimal path by an event cannot increase the benefit. It can be used to eliminate the possibility of a local minimum by comparing the values of four feasible paths (Figure 4):

a) S_t^{33}, starting at t_3 and following the path S_{t+t_3}; (optimal).

b) S_t^{34}, starting at t_3 and following the path S_{t+t_4}; (suboptimal).

c) S_t^{43}, starting at t_4 and following the path S_{t+t_3}; (suboptimal).

d) S_t^{44}, starting at t_4 and following the path S_{t+t_4}; (optimal).

Figure 4

Note that the time index t of S_t^{ij} measures the time elapsed from the corresponding start time t_i. In fact, $S_t^{33} = S_t^{43}$ for all t, and the two paths differ only with respect to the prior information involved: $\tilde{S}_t^{43} = S_m$ and S_t^{43} is carried out knowing that the event will never occur, whereas $\tilde{S}_0^{33} > S_m$ and S_t^{33} is planned under the risk that it will be interrupted by an event before the minimum level S_m is arrived at.

Let $V(S^{ij})$ denote the value associated with each path, evaluated at its start time t_i. Judging by the decisions taken, $V(S^{33}) > V(S^{34})$ and $V(S^{44}) > V(S^{43})$. We also know that $S_t^{34} = S_t^{44}$ and these paths are increasing, hence $V(S^{34}) = V(S^{44})$. (For increasing paths the term representing the probability of non-occurrence reduces to unity and does not affect the value.) It follows that $V(S^{33}) > V(S^{43})$. However, $V(S^{33}) = V(S_{t_3})$ while $V(S^{43}) = U(S_{t_3})$, hence the latter inequality contradicts our finding that $U(S_t) \geq V(S_t)$.

For problems admitting multiple optima, the strong inequalities of the previous paragraph may be replaced by equalities, and non-monotonic plans cannot be ruled out. Yet, the construction of a monotonic optimal path from a non-monotonic plan follows the discussion of the certainty and the auxiliary problems: One chooses a selection rule according to which, for each state level, a particular extraction rate is chosen among all optimal rates. The resulting optimal plan is monotonic, because conflicting decisions at the same state level are not allowed.■

Appendix B: stability of the equilibrium states

Lemma 4.3: If, for some $\tilde{S} \in (\hat{S},\bar{S})$, $L_M(\tilde{S}) = 0$ and L_M decreases in some neighborhood of $\tilde{S}$, then, starting at any $S \neq \tilde{S}$, the optimal state process corresponding to $V^a(S_0;\hat{S},\bar{S})$ will never converge to $\tilde{S}$.

Proof: We show that starting from any state other than $\tilde{S}$, it cannot be optimal to arrive at $\tilde{S}$ and stop there. Consider some state $S = \tilde{S} + \Delta$, through which the process has passed before arriving at $\tilde{S}$. We show that when Δ is small enough, it is more advantageous to stop at S than to proceed to $\tilde{S}$ and stop there. Let $\delta_t = g_t - R(S_t)$ and $\Delta_t = \int_0^t \delta_{t'} dt'$. Setting the origin of time at the passage time through S, we find $S_t = S - \Delta_t$ and $\Delta = \Delta_\infty$. It is convenient to introduce $\bar{\Delta} = \int_0^\infty \delta_t e^{-\rho t} dt = O(\Delta)$. Note that $\int_0^\infty \Delta_t e^{-\rho t} dt = \bar{\Delta}/\rho$. The value associated with the trajectory leading from S to $\tilde{S}$ is

$$V(S,\tilde{S}) = \int_0^\infty \{Y(R(S_t)+\delta_t) - C(S_t)[R(S_t)+\delta_t]\} e^{\Lambda(S_t)} e^{-\rho t} dt =$$

$$[Y(R(S)) - C(S)R(S)] \int_0^\infty e^{-\rho t} dt + [Y(R(S)) - C(S)R(S)] \int_0^\infty [e^{\Lambda(S_t)} - 1] e^{-\rho t} dt +$$

$$Y'(R(S)) \int_0^\infty [R(S_t) - R(S)] e^{-\rho t} dt - \int_0^\infty [C(S_t)R(S_t) - C(S)R(S)] e^{-\rho t} dt +$$

$$[Y'(R(S)) - C(S)] \int_0^\infty \delta_t e^{-\rho t} dt + \frac{1}{2} \int_0^\infty Y''(\tilde{g})[R(S_t) - R(S) + \delta_t]^2 e^{-\rho t} dt + O(\Delta^2),$$

where $\tilde{g}$ is some intermediate value between R(S) and $R(S_t)+\delta_t$. The Y'' term is negative, hence

$$V(S,\tilde{S}) - W(S) < -W(S)\lambda(S)\bar{\Delta} - \{Y'(R(S))R'(S)-C'(S)R(S)-C(S)R'(S)\}\bar{\Delta}/\rho + [Y'(R(S))-C(S)]\bar{\Delta} + O(\Delta^2)$$

so that

$$V(S,\tilde{S}) - W(S) < \bar{\Delta}L_M(S)/\rho + O(\Delta^2). \qquad \text{(B.1)}$$

Since $L_M(\tilde{S}) = 0$ and is decreasing, it follows that $\bar{\Delta}L_M(S) < 0$, hence $V(S,\tilde{S}) - W(S) < 0$ when Δ is small enough. Thus, stopping at S yields a benefit larger than that obtained by going to $\tilde{S}$ and stopping there.■

Remark: (i) If a root $\tilde{S}$ is a local maximum of $L_M(S)$, the same argument shows that it cannot be optimal to approach $\tilde{S}$ from above and stop there. (ii) If a root $\tilde{S}$ is a local minimum of $L_M(S)$, the same argument shows that it cannot be optimal to approach $\tilde{S}$ from below and stop there.

Figure Captions

Fig. 1: The relation between the function L(S) and the time evolution of the optimal state process of the certainty problem. The single root $\hat{S}$ of L(S) is the unique steady state. The arrows indicate the direction in which the process evolves.

Fig. 2: The relation between the functions L(S) and $L_M(S)$ when the latter function has a single root. The arrows indicate the direction in which the optimal state process S_t^* evolves. The interval $(\hat{S},\hat{S}_M)$, in which L(S) is positive and $L_M(S)$ is negative, is an equilibrium interval.

Fig. 3: Optimal state processes for the certainty and the aquifer (uncertainty) problems vs. the dimensionless time rt. The values $S_0 = \bar{S} = 1.5$, $\bar{S}-\hat{S} = 1$, $\rho/r = 1$ and $\lambda = 2$ are used.

Fig. 4: Four hypothetical feasible paths that could be constructed if the optimal process S_t had a minimum at t = m. S^{33} and S^{44} are parts of the original process, initiated at t_3 and t_4, respectively. S^{43} and S^{34} are suboptimal copies of these paths, shifted by the time increments $\pm(t_4-t_3)$, respectively.

Fig. 1

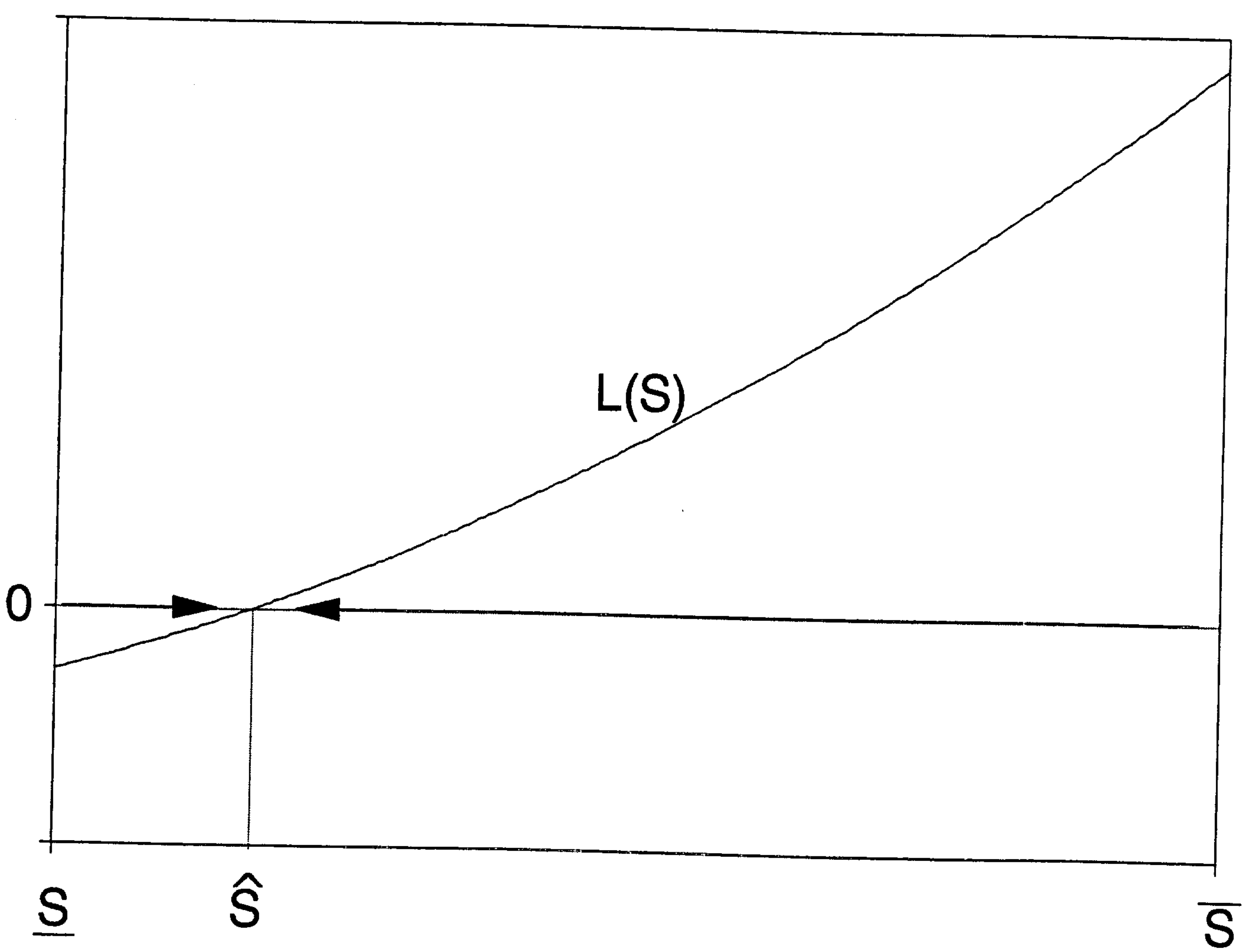

Fig. 2

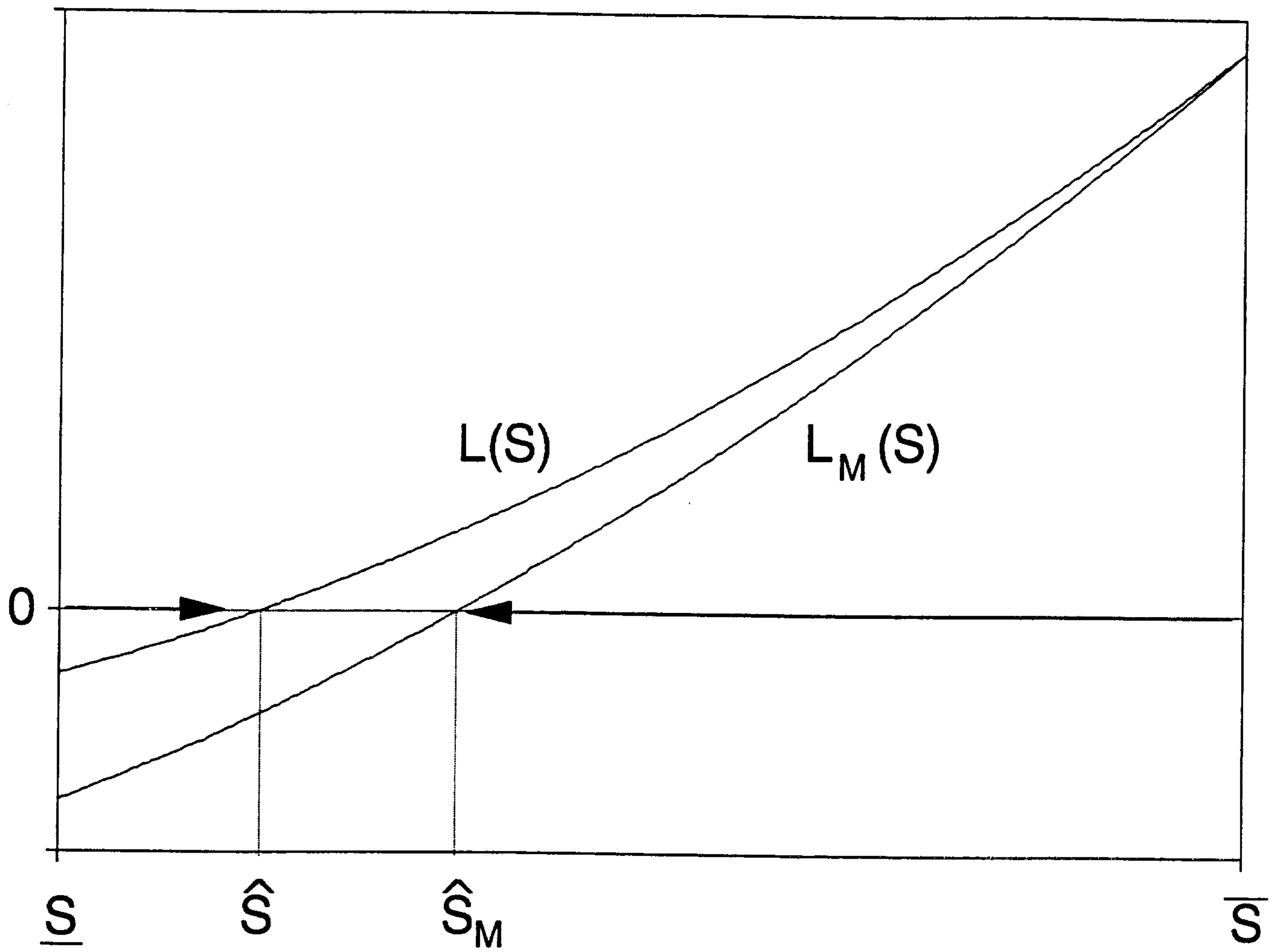

Fig. 3

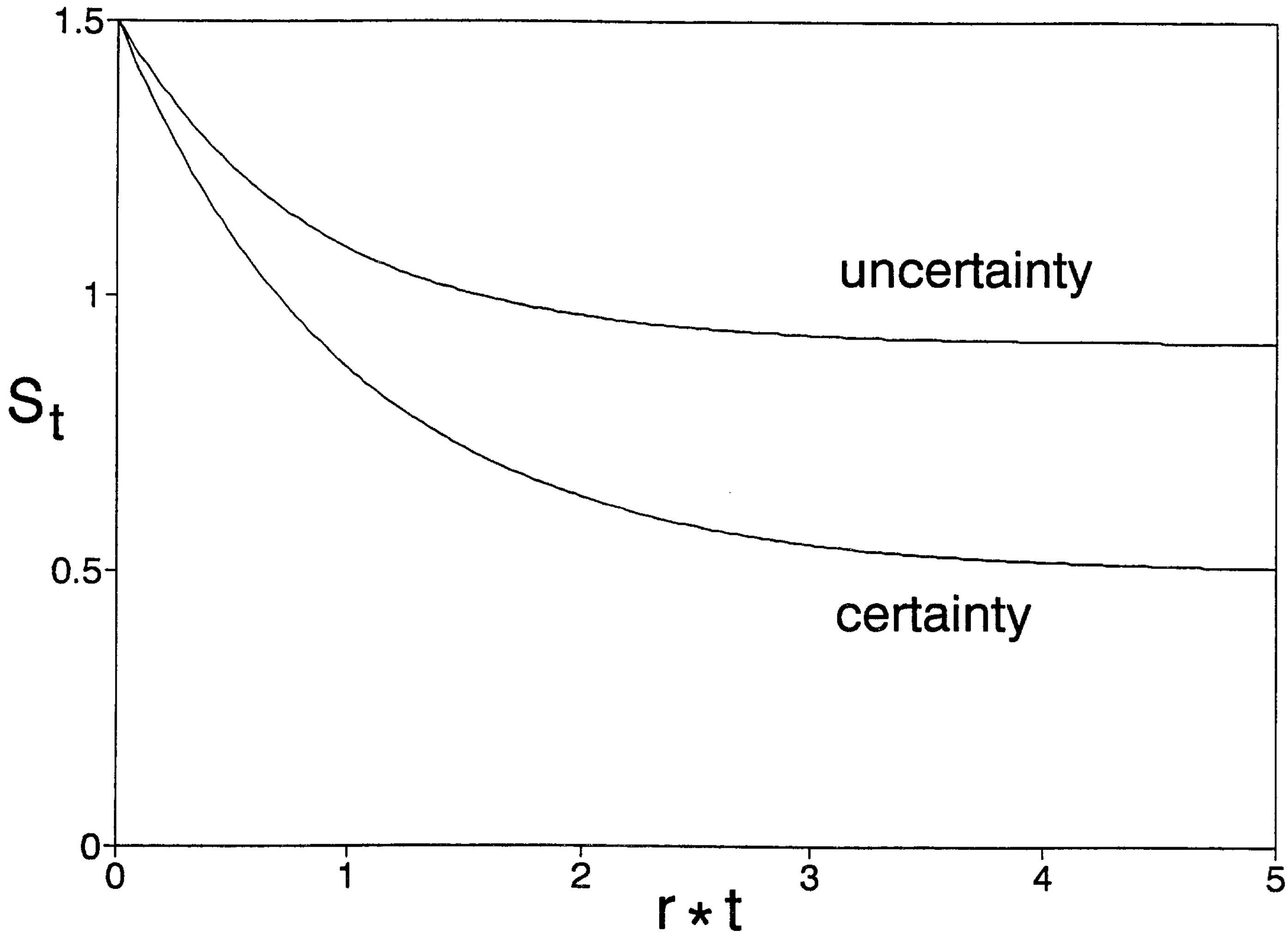

Fig. 4