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DISCRETE STOCHASTIC SEQUENTIAL

PROGRAMMING: A PRIMER

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The use of mathematical programming has been widespread in analyses of decision-making and economic behavior under risk. Most notably, quadratic programming (EV) and MOTAD techniques have been employed as a means of capturing random components of the objective functions of production problems (Markowitz, Hazell). Cocks (1968) and Rae (1971a) have presented the discrete stochastic sequential programming (DSSP) technique for extending the specification of risk beyond that which influences the objective function directly to include stochastic constraint parameters (typically technical coefficients and resource endowments). The DSSP model allows also for the incorporation of a sequential decision process in which the decision maker's knowledge of the outcomes of random events changes through time as production decisions are made. The purpose of this paper is to present an overview of discrete stochastic sequential programming and to illustrate the technique through a numerical example. The application of the technique to empirical problems involving farm decision making will be briefly discussed and an empirical application will be summarized.

Stochastic Programming

Stochastic programming refers to a class of constrained optimization problems in which some subset of constraint parameters (coefficients and RHS's) are stochastic. This class of decision problems may be subdivided into four categories involving the type of random variables and the nature of the decision process. The random variables may be discrete or continuous and the decision process may be nonsequential or sequential. An example of a non-sequential problem with a continuous random resource endowment appears in Anderson, Dillon, and Hardaker (pp. 216-221). The class of problems involving continuous random variables and a sequential decision process cannot be handled with the techniques discussed here. The reason is straightforward. Consider the vectors selected in a later stage in the decision process. Each of these vectors must be selected conditionally upon earlier stage decisions and the outcomes of earlier random events (this is why the problem is sequential). If the variables representing the earlier random events are continuous, the number of circumstances under which later decisions must be made is infinite. Thus, continuous random variables in sequential decision problems must be modeled as discrete variables which take on a finite number of values.[1/]

The focus of this paper is on the modeling of sequential decision problems involving discrete random state variables (or, as is more often the case in practice, the approximation of continuous random variables using discrete distributions). This subcategory of stochastic programming, discrete stochastic sequential programming (DSSP), was selected as the class of decision problems which holds the greatest potential for the heuristic conceptual or empirical address of farm decision problems.

1/ The solution technique employed in the non-sequential stochastic programming problem cited above gives solutions which would seem to have characteristics similar to solutions generated with a discrete treatment of the random variables. However, the authors have not examined the notion in detail.

Discrete Stochastic Sequential Programming

Within this general class of stochastic programming problems, discrete stochastic sequential programming (DSSP) includes a discrete specification of random problem parameters and a multi-stage decision process. In many cases, probability density functions of continuous random variables are approximated using discrete "states of nature". The use of a multi-stage decision process involves a specification of discrete time intervals (stages). Decisions in a particular stage are made with probabilistic knowledge of the occurrence of particular states of nature in various stages of the decision process. Central to the specification of several interdependent decision stages is the condition that the opportunity set restricting decisions in a later stage is influenced not only by the occurrence of particular random events in that stage, but also by random outcomes and decisions made in earlier stages. The discussion of DSSP which follows is based largely on concepts developed in papers by Cocks and Rae (1971a) and an article based on the application of these concepts to a farm production problem, also by Rae (1971b).

The nature of a sequential decision process under risk as captured in DSSP can be illustrated with a decision tree depicting the stages in the decision process and the states of nature in each stage. An exemplary decision tree for a two stage decision problem with two discrete states of nature in each stage is shown in Figure 1. Borrowing from the notation used in Rae (1971a) (and modifying that notion slightly), $e_{kn_t t}$ represents the occurrence of the kth state of nature in stage t subject to which the n_tth set of stage t activities will be selected. Rae points out that the structure of the mathematical programming matrix depends on the underlying information structure of the problem.

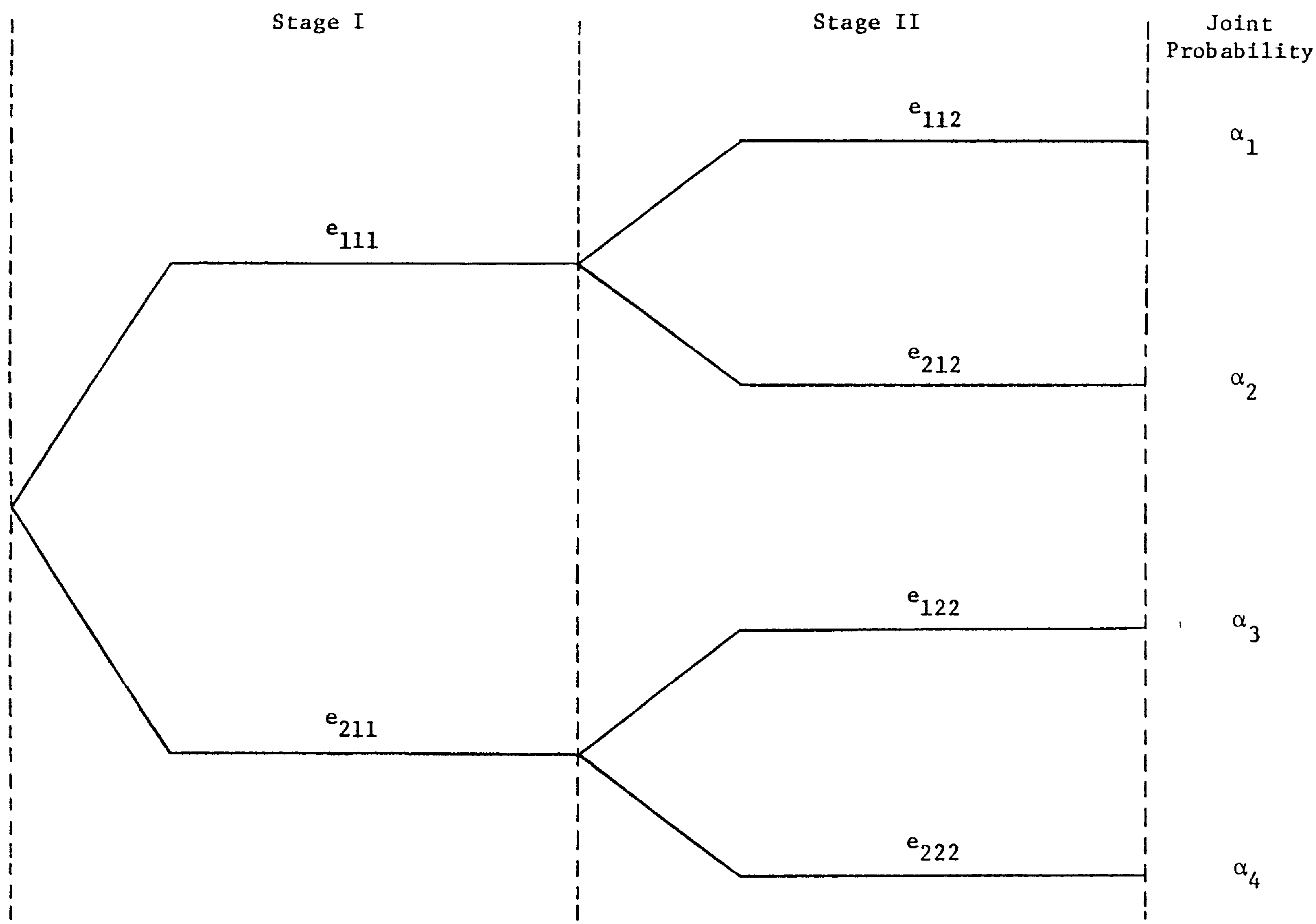

FIGURE 1: Decision Tree for the Two-Stage, Two-State Problem with Events $e_{kn_t t}$ for Complete Knowledge of the Past.

A discussion of three general information structures will help to illustrate the construction of a DSSP matrix; a particular decision model may incorporate elements of each. Decisions are assumed to be made at the beginning of each stage. If at the time stage t decisions are made the decision maker knows the outcomes of random events in stages $t-\ell$, $t-\ell-1$, $t-\ell-2,\ldots,1$; the information structure where $\ell=0$ is complete knowledge of the past and present. With $\ell=1$, the information structure is complete knowledge of the past and $\ell>1$ implies incomplete knowledge of the past. The events depicted in Figure 1 are for the case of complete knowledge of the past. The general LP formulation for this case will now be discussed.

A general linear programming (LP) formulation of the two stage, two state DSSP problem is presented in Figure 2. At this point, stochastic components are accounted for in the constraint function coefficients ($A_{kn_t t}$), the constraint constants or righthand sides ($b_{kn_t t}$) and the objective function coefficients ($c_{kn_t t}$).[2/] A decision strategy is represented by the optimal solution values to vectors $X_{n_t t}$. Stage I decisions are represented by vector X_{11}. Because the outcome of stage I random events is unknown when vector X_{11}^* is selected, X_{11} must be "permanently feasible" - i.e., resource constraints (2.2) and (2.3) must be satisfied regardless of which state of nature occurs. In a similar way, stage II decisions must be permanently feasible, as well. However, two stage II vectors (X_{12} and X_{22}) are included since the decision maker having

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2/ Variability of objective function coefficients is, of course, handled in the more common EV and MOTAD models. The ability to deal with continuous distributions of activity gross margins within these models make a hybrid of the EV and DSSP approaches desirable.

OBJECTIVE (MAX): $\alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3 + \alpha_4 Y_4$ (2.1)

SUBJECT TO:

$$
\begin{array}{lllllllllll}
 & & & & A_{111}X_{11} & & & & \leq & b_{111} & (2.2) \\
 & & & & A_{211}X_{11} & & & & \leq & b_{211} & (2.3) \\
 & & & & & & A_{112}X_{12} & & \leq & b_{112} & (2.4) \\
 & & & & & & A_{212}X_{12} & & \leq & b_{212} & (2.5) \\
 & & & & & & & A_{122}X_{22} & \leq & b_{122} & (2.6) \\
 & & & & & & & A_{222}X_{22} & \leq & b_{222} & (2.7) \\
 & & & & - D_{111}X_{11} & + & E_{12}X_{12} & & \leq & 0 & (2.8) \\
 & & & & - D_{211}X_{11} & & & + E_{22}X_{22} & \leq & 0 & (2.9) \\
Y_1 & & & & - C'_{111}X_{11} & - & C'_{112}X_{12} & & \leq & 0 & (2.10) \\
 & Y_2 & & & - C'_{111}X_{11} & - & C'_{212}X_{12} & & \leq & 0 & (2.11) \\
 & & Y_3 & & - C'_{211}X_{11} & & & - C'_{122}X_{12} & \leq & 0 & (2.12) \\
 & & & Y_4 & - C'_{211}X_{11} & & & - C'_{222}X_{12} & \leq & 0 & (2.13)
\end{array}
$$

$$Y_1, Y_2, Y_3, Y_4, X_{11}, X_{12}, X_{22} \geq 0 \qquad (2.14)$$

FIGURE 2: Matrix Construction for the Two Stage, Two State Problem Assuming an Information Structure of Complete Knowledge of the Past.

complete knowledge of the past, will know at the beginning of stage II which stage I state of nature has occurred. Thus stage II decisions will be made subject to the opportunities afforded jointly by stage II random events, by stage I decisions and the outcome of random events in stage I. The interdependence of decisions in the two stages is captured through constraints 2.8 and 2.9 which allow for the continuance of stage I activities in stage II and for the transfer of resources between the first and the second stage activities. Matrices D_{kn_tt} and E_{n_tt} are appropriately constructed to preserve these relationships between stages. Given the outcome of random events in stage I, constraints 2.4 and 2.5, and 2.6 and 2.7 render decision vectors X_{12} and X_{22}, respectively, permanently feasible.

Activities Y_1 through Y_4 represent total net revenue associated with each possible sequence of random events in the two stages (joint events (e_{111},e_{112}), (e_{111},e_{212}), (e_{211},e_{122}) and (e_{211},e_{222}), respectively). C_{kn_tt} are vectors of objective function coefficients corresponding to the associated events. Thus, through constraints 2.10, 2.11, 2.12 and 2.13, net revenue levels associated with the occurence of each combination of events are summed into Y. Joint probabilities α_1, α_2, α_3 and α_4 are objective function coefficients for Y, so the objective (2.1) is expected net revenue, which is maximized.[3/] With the problem formulated in this way, the optimal stage I vector is then selected with

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3/ Note that the appropriate marginal and joint probabilities could have been used to weight the vectors C_{kn_tt} and these coefficients could have been placed directly in the objective function. However, use of vector Y provides useful solution information and will facilitate latter discussions of expected utility models.

consideration of the expected explicit and implicit values of stage II decision vectors.

Under an information structure of complete knowledge of the past and present, stage t decisions are unique for each stage t state of nature - they need not be permanently feasible since the outcome of the stage t random events is known. Here, the combination of optimal decision vectors which constitutes an optimal strategy is comprised of solutions derived from separate optimization problems - one for each state of nature in the first decision stage. A decision tree showing events corresponding to the two stage, two state problem under complete knowledge of the past and present is shown in Figure 3. As the notation in Figure 3 indicates, the problem now involves two sets of activities in stage I: one set which will be selected in the event of state one and another which will be selected in the event of state two. Four sets of stage II activities are implied. A vector of activities is included for each stage II state of nature since the outcome of random events in that stage will be known when the decisions are made. Further, each set of decisions will be made as a consequence of one of two stage I random events and decision vectors.

Incorporating Utility Functions Into the DSSP Model

Because the probability distributions of monetary outcomes are explicitly considered in DSSP, the modeling technique can be readily extended from the expected net revenue formulation presented above to a formulation for the maximization of expected utility. The implications of the introduction of utility concepts into the DSSP model parallel

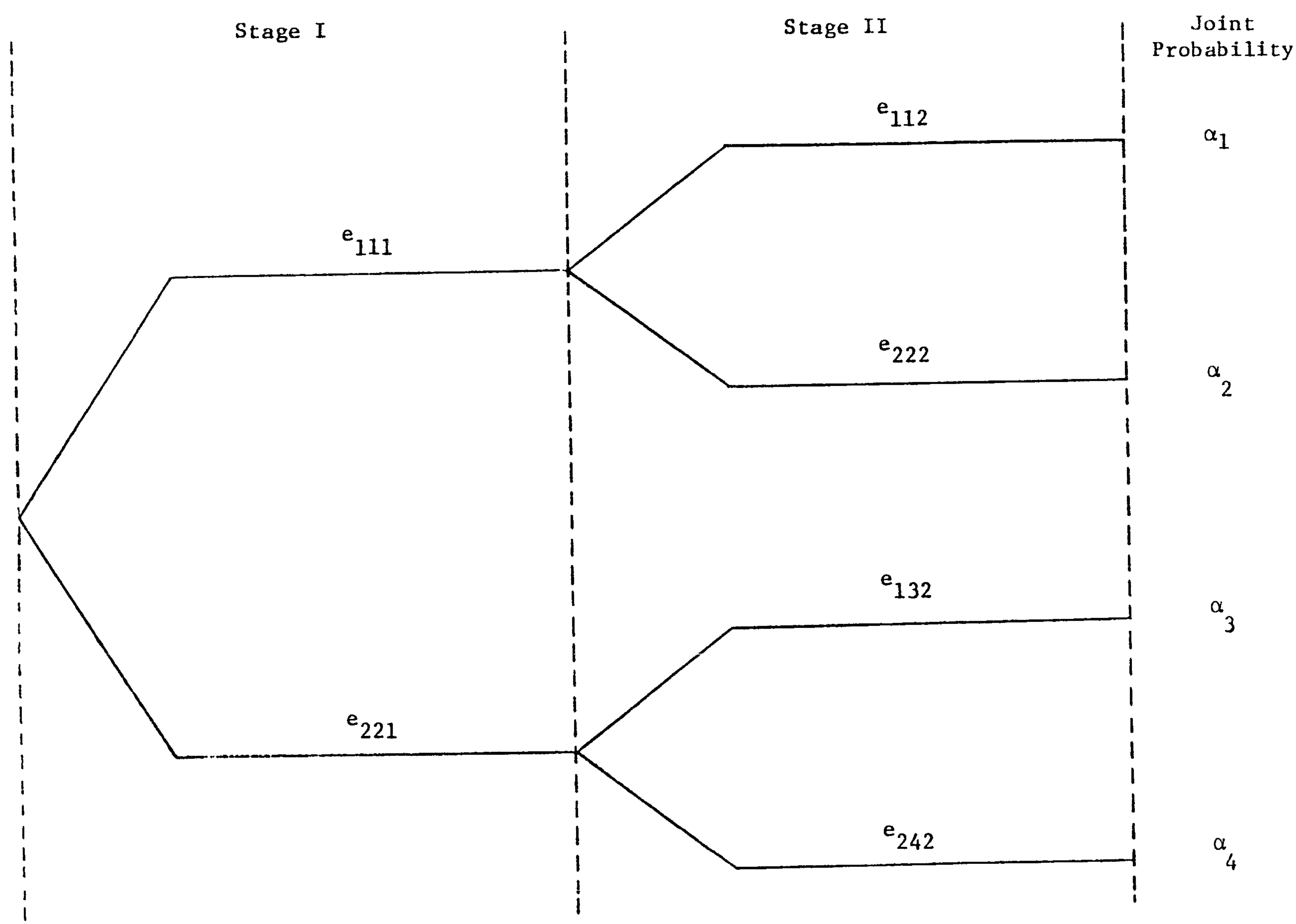

FIGURE 3: Decision Tree for the Two-Stage, Two-Stage Problem with Events $e_{kn_t t}$ for Complete Knowledge of the Past and Present.

OBJECTIVE (MAX): $\alpha_1 Y_1 + \alpha_2 Y_2$ (4.1)

SUBJECT TO:

$$A_{111}X_{11} \leq b_{111} \quad (4.2)$$

$$A_{112}X_{12} \leq b_{112} \quad (4.3)$$

$$A_{222}X_{22} \leq b_{222} \quad (4.4)$$

$$-D_{111}X_{11} + E_{12}X_{12} \leq 0 \quad (4.5)$$

$$-D_{211}X_{11} + E_{22}X_{22} \leq 0 \quad (4.6)$$

$$Y_1 - C_{111}X_{11} - C_{112}X_{12} \leq 0 \quad (4.7)$$

$$Y_2 - C_{111}X_{11} - C_{222}X_{22} \leq 0 \quad (4.8)$$

$$Y_1, Y_2, X_{11}, X_{12}, X_{22} \geq 0 \quad (4.9)$$

FIGURE 4: Matrix Construction for the Two-Stage, Two-State Problem Assuming an Information Structure of Complete Knowledge of the Past and Present (Given State 1 in Stage I).

OBJECTIVE (MAX): $\alpha_3 Y_3 + \alpha_4 Y_4$ (5.1)

SUBJECT TO:

$$A_{221}X_{21} \leq b_{221} \quad (5.2)$$

$$A_{132}X_{32} \leq b_{132} \quad (5.3)$$

$$A_{242}X_{42} \leq b_{242} \quad (5.4)$$

$$- D_{321}X_{21} + B_{32}X_{32} \leq 0 \quad (5.5)$$

$$- D_{421}X_{21} + B_{42}X_{42} \leq 0 \quad (5.6)$$

$$Y_3 - C_{221}X_{21} - C_{132}X_{32} \leq 0 \quad (5.7)$$

$$Y_4 - C_{221}X_{21} - C_{242}X_{42} \leq 0 \quad (5.8)$$

$$Y_3, Y_4, X_{21}, X_{32}, X_{42} \geq 0 \quad (5.9)$$

FIGURE 5: Matrix Construction for the Two-Stage, Two-State Problem Assuming an Information Structure of Complete Knowledge of the Past and Present (Given State 2 in Stage I).

those seen with other risk programming models except for special considerations of time in the decision making process. Following a brief discussion of time considerations, the incorporation of an implicit single-dimensioned utility function into the DSSP model will be demonstrated. In addition, the incorporation of an EV risk framework will be discussed.

Two general formats may be used to incorporate one-dimensional utility functions (i.e., utility as a function of monetary outcome) into a DSSP model (Rae, 1971a). The first approach involves the use of separate utility functions $U_t(Y_t)$ for each time period. The objective function, expressed as discounted utility, can then be specified as follows:

$$\text{Max:} \quad U = \sum_{t=1}^{T} U_t(Y_t)(1+r)^{-t} \qquad (1)$$

Where Y_t is the monetary payoff at time t and r is the agent's discount rate. This approach may be difficult to implement because it requires estimating utility functions for each period in the model. The second approach involves the use of a single utility function which operates on the present value of monetary payoffs. This formulation of the problem can be written mathematically as:

$$\text{Max:} \quad U = U\Big(\sum_{t=1}^{T} (Y_t)(1+r)^{-t}\Big). \qquad (2)$$

This method is more manageable because it only requires the estimation of one utility function. In both formats, the choice of the discount rate should reflect the decision-makers preference of present over future returns. There are two conditions where returns need not be discounted to present value terms. If the time period being analyzed is sufficiently short, or if the agent has no preference for present over future returns, the future payoffs should not be discounted.

Given an estimated utility function, U(Y), the objective function of the DSSP problem in Figure 2 can be replaced by the following expected utility function:

$$\text{Max:} \quad E[U(Y)] = \sum_{\ell=1}^{4} \alpha_\ell \, U(Y_\ell) \tag{3}$$

As in the original problem, Y_ℓ is net revenue under the ℓth joint event and α_ℓ is the corresponding joint probability. Thus the linear programming problem becomes a non-linear program (assuming $U(Y_\ell)$ is non-linear). However, if U is concave, the functions $\alpha_\ell \, U(Y_\ell)$, ℓ=1...4 may be approximated using separable programming and solved using an LP algorithm. [4/] The separable programming formulation of the problem appears in Figure 6.

The separable programming formulation provides a piecewise linear approximation of expected utility through the use of special activities $Q_{\ell i}$. An activity $Q_{\ell i}$ is specified for each of m discrete values of the monetary outcome associated with the *th joint event ($\hat{Y}_{\ell 1} < \hat{Y}_{\ell 2} < \ldots \hat{Y}_{\ell m}$). By imposing constraints 6.9 through 6.12 and "convexity" constraints 6.13 through 6.16, net revenue under joint event ℓ is constrained to equal $Y_{\ell i}$ (for some i=1...m; where $Q_{\ell i}=1$ and $Q_{\ell j}=0$ for $j \neq i$) or a convex combination of two adjacent values $\hat{Y}_{\ell i}Q_{\ell i} + \hat{Y}_{\ell i+1}Q_{\ell i+1}$ (where $Q_{\ell i} + Q_{\ell i+1} = 1$

4/ Unless $U(Y_\ell)$ is concave over the opportunity set, separable programming will generally yield only a local solution. Further, adjacency restrictions on the special separable programming variables would be necessary if $U(Y_\ell)$ is not concave.

OBJECTIVE (MAX):

$$\sum_{i=1}^{m} U_1(\hat{Y}_{1i})Q_{1i} + \sum_{i=1}^{m} U_2(\hat{Y}_{2i})Q_{2i} + \sum_{i=1}^{m} U_3(\hat{Y}_{3i})Q_{3i} + \sum_{i=1}^{m} U_4(\hat{Y}_{4i})Q_{4i} \qquad (6.1)$$

SUBJECT TO:

$$A_{111}X_{11} \leq b_{111} \qquad (6.2)$$

$$A_{211}X_{11} \leq b_{211} \qquad (6.3)$$

$$A_{112}X_{12} \leq b_{112} \qquad (6.4)$$

$$A_{212}X_{12} \leq b_{212} \qquad (6.5)$$

$$A_{122}X_{22} \leq b_{122} \qquad (6.6)$$

$$A_{222}X_{22} \leq b_{222} \qquad (6.7)$$

$$- D_{111}X_{11} + E_{12}X_{12} \leq 0 \qquad (6.8)$$

$$- D_{211}X_{11} + E_{22}X_{22} \leq 0 \qquad (6.9)$$

$$\sum_{i=1}^{m} \hat{Y}_{1i}Q_{1i} - C'_{111}X_{11} - C'_{112}X_{12} \leq 0 \qquad (6.10)$$

$$\sum_{i=1}^{m} \hat{Y}_{2i}Q_{2i} - C'_{111}X_{11} - C'_{212}X_{12} \leq 0 \qquad (6.11)$$

$$\sum_{i=1}^{m} \hat{Y}_{3i}Q_{3i} - C'_{211}X_{11} - C'_{122}X_{22} \leq 0 \qquad (6.12)$$

$$\sum_{i=1}^{m} \hat{Y}_{4i}Q_{4i} - C'_{211}X_{11} - C'_{222}X_{22} \leq 0 \qquad (6.13)$$

$$\sum_{i=1}^{m} Q_{1i} = 1 \qquad (6.14)$$

$$\sum_{i=1}^{m} Q_{2i} = 1 \qquad (6.15)$$

$$\sum_{i=1}^{m} Q_{3i} = 1 \qquad (6.16)$$

$$\sum_{i=1}^{m} Q_{4i} = 1 \qquad (6.17)$$

$$Q_{1i},\ Q_{2i},\ Q_{3i},\ Q_{4i},\ X_{11},\ X_{12},\ X_{22} \geq 0 \qquad (6.18)$$

FIGURE 6: Matrix Construction for the Two Stage, Two Stage Problem Assuming Complete Knowledge of the Past with Separable Programming Approximation of Expected Utility.

and $Q_{\ell j}=0$ for $j \neq i$, $j \neq i+1$). The value of the utility function is thus $U(\hat{Y}_{\ell i})Q_{\ell i}$ where $Q_{\ell i}=1$, or utility is approximated as $U(\hat{Y}_{\ell i})Q_{\ell i} + U(\hat{Y}_{\ell i+1})Q_{\ell i+1}$ where $Q_{\ell i} + Q_{\ell i+1} = 1$. For a more complete discussion of separable programming, see Hillier and Lieberman (pp. 581-586), Duloy and Norton, or Roe.

The EV approach requires the measurement of expected returns and the variance returns. The occurrence of a particular joint even in the DSSP model is characterized by the multinomial distribution (Cocks). That is, one of m joint events will occur (for each trial) with probabilities α_j, $j=1 \ldots m$. The expected value of the jth joint event is α_j, where $\alpha_j \geq 0$ and $\sum_{j=1}^{m} \alpha_j = 1$, and the variance is $\sigma_j^2 = \alpha_j(1-\alpha_j)$. The covariance of joint events i and j is $\sigma_{ij} = -\alpha_i \alpha_j$ $(i \neq j)$. Referring to the DSSP problem in Figure 2, expected returns equals $\sum_{j=1}^{4} \alpha_j Y_j$ (as noted earlier) and the variance of returns is $\sum_{i=1}^{4} \sum_{j=1}^{4} V_{ij} Y_i Y_j$, where:

$$V_{ii} = \alpha_i\ (1-\alpha_i),\ i=1 \ldots 4 \tag{4}$$

$$\text{and } V_{ij} = -\alpha_i \alpha_j,\ i=1 \ldots 4,\ j=1 \ldots 4,\ i \neq j \tag{5}$$

Then, the EV objective function corresponding to the problem in Figure 2 is:

$$\text{Max:} \quad \sum_{i=1}^{4} \alpha_i Y_i - \Phi \sum_{i=1}^{4} \sum_{j=1}^{4} V_{ij} Y_i Y_j \tag{6}$$

where: Φ is a coefficient of risk aversion. An alternative formulation could be used to find the minimum variance solution (i.e. Min: $\sum_{i=1}^{4} \sum_{j=1}^{4} V_{ij} Y_i Y_j$) with an additional constraint on expected returns (e.g., $\sum_{i=1}^{4} \alpha_i Y_i \geq \overline{Y}$). Since $[V_{ij}]$ is positive semi-definite, objective function (6) is concave and a global solution to the DSSP/EV model is ensured. Two procedural concerns may be apparent: the DSSP/EV quadratic programming (QP) problem may be too large for the available QP solver(s) and the available QP solution algorithm(s) may require a positive definite quadratic form. Therefore, an alternative solution procedure may be desirable. McCarl and Tice present such a procedure.

Consider the following nonlinear and nonseparable portion of objective function (6):

$$\sum_{i=1}^{4} \sum_{j=1}^{4} V_{ij} Y_i Y_j \qquad (7)$$

Matrix $V = [V_{ij}]$ is positive semi-definite and symmetric. Let W be a matrix made up of eigenvectors of V (as columns) and let vector Z be a transformation of vector Y such that Y=WZ. Note that the variance Y'VY now becomes:

$$Z'W'VWZ \qquad (8)$$

A desirable result of the transformation stems from the fact that Ω=W'VW is a matrix with diagonal elements equal to the eigenvalues of V and off-diagonal elements equal to zero. Therefore, the variance (7) is now:

$$Z'\Omega Z \qquad (9)$$

Which can be restated as:

$$\sum_{i=1}^{4} \lambda_i Z^* \qquad (10)$$

Where λ_i is the ith diagonal element of Ω (ith eigenvalue of V). Thus, (10) can be substituted into (6) and, with the following constraint added to the problem ...

$$Y - WZ = 0 \qquad (11)$$

... an exact transformation of the original QP is formed. The transformed problem has a separable quadratic objective function which can be approximated using separable programming and solved with an LP code.[5/]

Forecast Information

Thus far, various time patterns of information about the outcome of random events have been discussed. For some empirical problems, it may be appropriate to include information gathering as a decision variable or to access the value of information made available to the decision maker. Forecast information may be incorporated into a DSSP problem if a finite number of discrete outcomes of the forecast ($f_{kn_t t}$) can be specified. Consider, for example, the two-stage two-state problem. Suppose that at the beginning of each stage (when decision vectors are selected), the decision maker has available the outcome of a forecast of random events in that stage.[6/] Let there be two such outcomes for each forecast. The decision tree for the new stochastic sequential problem is shown in Figure 7. Note that given forecast outcome k, either event 1 or 2 may occur -- that is, the forecast is not perfect. The problem now has two decision vectors in stage I -- one for each forecast outcome. When the stage II decisions are made, the outcome of the first stage random process will be known (as before), and the outcomes of both forecasts will be known. Thus the new problem has eight stage II

5/ See McCarl and Tice for a more complete discussion of the procedure. Intriligator includes a useful discussion of the diagonalization of quadratic forms (pp. 495-497).

6/ Generally, the forecast can augment information about any random process in past, current, or future stages when the outcome of that process is unknown at the time the forecast is received.

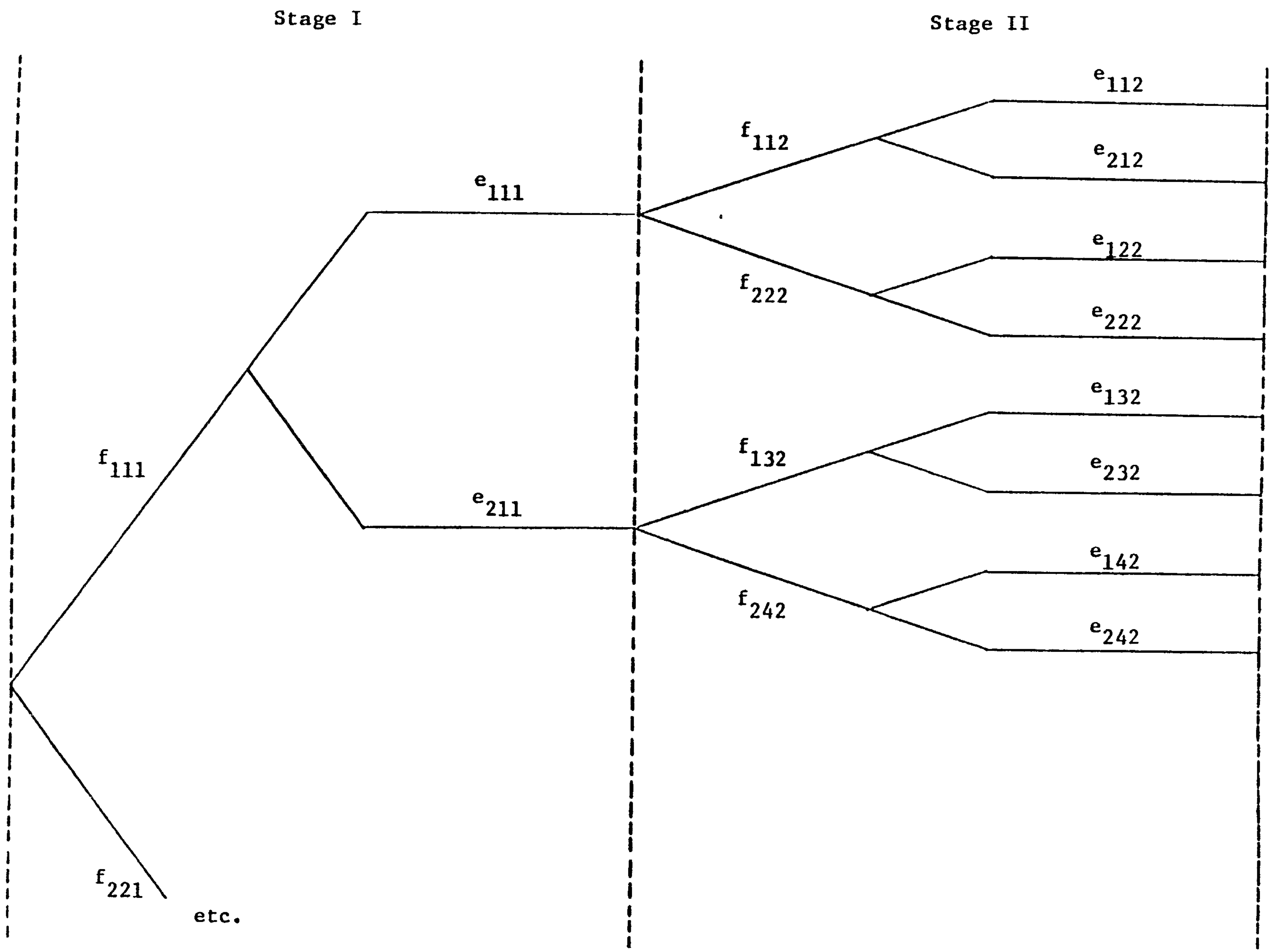

FIGURE 7: Decision Tree for the Two Stage, Two State Problem with Forecasts.

decision vectors -- one for each combination of first stage states of nature, forecast one outcomes and forecast two outcomes.

The posterior probabilities of random events can now be used -- the probabilities of states of nature conditional on each forecast outcome. By solving the problem with and without the forecast results, the value of the forecast can be estimated. Rae points out that if receipt of the forecast results imposes an added demand on scarce resources (for example, a cash payment when cash flow restrictions exist), it will be necessary to include the incidence of this resource requirement in the model.

Some General Comments on the Use of DSSP

Dimensionality problems remain a concern with the DSSP model. A stochastic programming matrix will generally grow in size more than proportionally with increases in the number of sources of risk (random variables), the number of discrete values taken by random variables and the number of stages in the decision process. One might argue that farm decision making is carried out in the face of hundreds of sources of risk which are most accurately represented as continuous random variables and that the decision making process is continuously sequential. From this premise, the building of a DSSP model which incorporates "the risk inherent in farm production and marketing decisions" is more than a merely ambitious task. The central focus of model building using DSSP must be on selecting an economical representation of the problem with the greatest level of detail specified in components critical to the analysis. Further, Anderson, Dillon and

Hardaker point out that while decisions made in stage t are influenced by prospects in later stages, they are influenced less by prospects in more distant stages. Thus, it may be most appropriate to sacrifice detail in later decision stages as the earlier strategies are derived. A "rolling" process is implied because a complete strategy of optimal decisions in each stage will eventually require that the sacrificed detail be restored. These more exact later strategies must then be derived for several optimization problems - one for each earlier stage outcome (upon which the later strategies will be conditional). Such a process may increase the involvement of the analyst in the solution process, but may bring the model within the capacities of available solution software. Also, additional decision stages of only indirect interest which otherwise may have been ignored may be added to a model. Model size may be reduced by eliminating activities which under certain states of nature can be determined as non-optimal prior to model solution. It may be useful to employ models of sub-problems to generate sets of efficient activities which can be used in the more general DSSP model. For example, a feed formulation model may be used to create feeding strategies which are efficient (by some criteria) and the strategies may be used as alternate activities in a whole-farm DSSP problem. Or, efficient marketing activities (generated by techniques such as generalized Monte Carlo programming (King and Oamek), for example) may be used to economize on the formulation of a model with both production and marketing decisions.

While construction of the matrix data file for a large DSSP model may in itself seem too burdensome, the replications in coefficient placement and parameter use inherent with these models make the use of matrix generating computer programs especially helpful. When matrix generators have been written for a deterministic version of a particular system, modification of the software to allow for stochastic parameters and a sequential decision process may be a straightforward process. The use of computer programs to generate the DSSP matrix may be especially useful in that techniques for manipulating probability distributions and calculating coefficients for separable programming activities can be readily automated in such programs. Similarly, report generating computer programs may be useful for analyzing the formidable set of solution values associated with a DSSP model.

Summary

Stochastic programming has not been used frequently in empirical applications to agriculture. Model size and complexity is probably the most often cited reason that the technique is not employed. The computer software for constructing DSSP models tends to be relatively problem specific when compared to more commonly used risk programming techniques. Thus the development of such software is costly. In summarizing this discussion of DSSP, it may be appropriate to cite some applications of stochastic programming to empirical problems in agriculture, and to comment about the potential for future applications of DSSP.

Stochastic programming has been used to analyze growth of farm firms (Johnson, Tefertiller and Moore, 1967). Rae (1971) reported results of a DSSP based analysis of crop mix decisions for vegetable farms under stochastic weather conditions. The potential for integrating simulation and DSSP approaches was identified by Trebeck and Hardaker (1972) in a paper reporting the application of these tools to pasture management and cattle feeding decisions. More recent applications of DSSP have focused on optimal fertilization strategies (Tice, 1979), crop residue production (Apland, 1979; Apland, McCarl and Baker, 1981) and on-farm grain drier investments (Klemme, 1980). A bibliography of stochastic programming theory and applications from 1955 through 1975 has been prepared by Stancu-Minasian and Wets.

A few observations may be made about the potential for further applications of DSSP. As mentioned earlier, software development is a critical issue. The increased availability of more general matrix generating programs would, of course, enhance the use of DSSP. Investment in the skills needed to design and implement computerized matrix generators and other software will be paramount to both the development of general and problem-specific models. Existing software for solving large linear and non-linear programming problems must be accessible. Simulation techniques can be effectively integrated and broaden the applicability of DSSP, especially in generating stochastic problem parameters. Such an integration will rely heavily on further efforts to quantify the stochastic elements of agricultural decision problems.

A numerical example of DSSP is presented in the Appendix of this paper. This simple decision problem was designed to illustrate critical aspects of sequential decision making in a risky environment as they are addressed in a DSSP model.

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APPENDIX

A Hypothetical DSSP Problem

A numerical example of discrete stochastic sequential programming will be presented in this section. As with the general model used earlier, the example has two stages with two states of nature in each stage. The model depicts a firm which produces two products -- one in stage I and the other in stage II. Four alternative production activities can be used for each product. Each production activity uses two inputs which are available in fixed supply. The stochastic component of the problem is limited to the levels of inputs available -- RHS's of the input use constraints. The decision problem is sequential in nature in that endowed quantities of input two in excess of that used in the first stage may be held and used in stage II production. Thus, the expected value of input one in the second stage represents an opportunity cost to its use in stage I.

Figure 7 illustrates the problem in a decision tree framework with the random events and the associated resource endowments and joint probabilities given.[7/] Note that in each stage, input one is relatively abundant under state one and relatively scarce under state two. An LP tableau for the problem assuming an information structure of complete knowledge of the past is provided in Table 1. The tableau is organized

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7/ Following the notation presented earlier ($e_{kn_t t}$, $b_{kn_t t}$), an information structure of complete knowledge of the past and present is implied in Figure 7 (since n_1 = 1,2 and n_2 = 1...4). Other information structures will be considered for the problem, also.

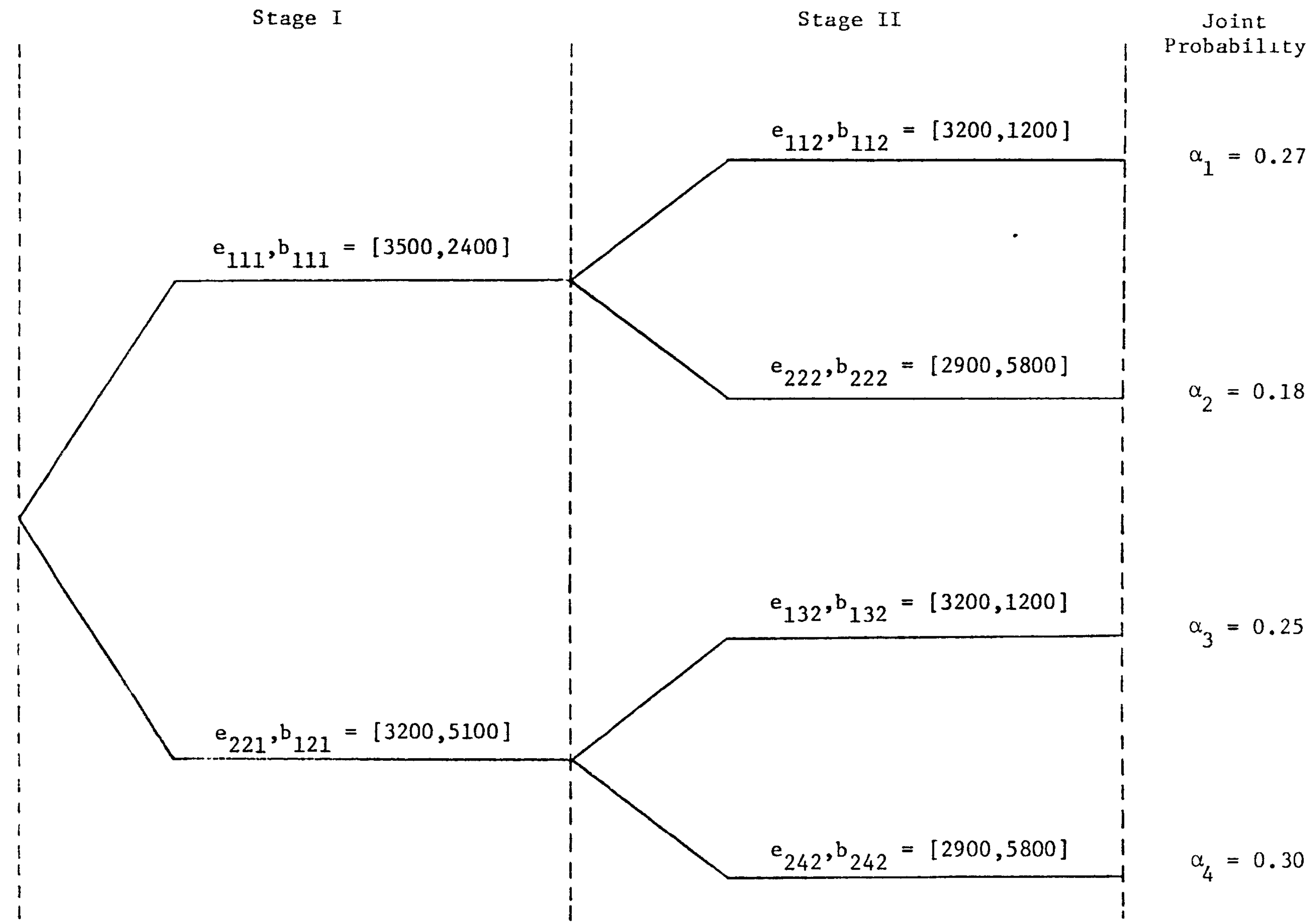

FIGURE 8: Decision Tree for the Hypothetical DSSP Problem.

TABLE 1: TABLEAU FOR A HYPOTHETICAL DSSP PROBLEM (COMPLETE KNOWLEDGE OF PAST).

	Y1	Y2	Y3	Y4	X111	X112	X113	X114	X115	X116	X121	X122	X123	X124	X125	X221	X222	X223	X224	X225		RHS	SLACK	DUAL
1) OBJ (MAX)	.27	.18	.25	.30																				
2) INPUT 1, 111					.80	.50	.29	.18													<=	3500	300	0
3) INPUT 2, 111					.10	.22	.45	.73	1												<=	2400	0	5.46
4) INPUT 1, 211					.80	.50	.29	.18													<=	3200	0	5.47
5) INPUT 2, 211					.10	.22	.45	.73		1											<=	5100	0	4.83
6) INPUT 1, 112											.85	.58	.35	.22							<=	3200	300	0
7) INPUT 2, 112											.24	.32	.52	.75	-1						<=	1200	0	5.46
8) INPUT 1, 212											.85	.58	.35	.22							<=	2900	0	4.75
9) INPUT 2, 212											.24	.32	.52	.75	-1						<=	5800	4600	0
10) INPUT 1, 122																.85	.58	.35	.22		<=	3200	300	0
11) INPUT 2, 122																.24	.32	.52	.75	-1	<=	1200	0	4.83
12) INPUT 1, 222																.85	.58	.35	.22		<=	2900	0	8.54
13) INPUT 2, 222																.24	.32	.52	.75	-1	<=	5800	4600	0
14) TRANSFER 11->12									-1						1						<=	0	0	5.46
15) TRANSFER 11->22										-1										1	<=	0	0	4.83
16) Y1 (e111,e112)	1				-5	-5	-5	-5			-10	-10	-10	-10							<=	0	0	0.27
17) Y2 (e111,e212)		1			-5	-5	-5	-5			-10	-10	-10	-10							<=	0	0	0.18
18) Y3 (e211,e122)			1		-5	-5	-5	-5								-10	-10	-10	-10		<=	0	0	0.25
19) Y4 (e211,e222)				1	-5	-5	-5	-5								-10	-10	-10	-10		<=	0	0	0.30
OPTIMAL VALUE	89181	89181	119978	119978	0	6400	0	0	992	3692	0	3907	1811	0	992	0	0	7419	1379	3692				
MARGINAL	0	0	0	0	-.41	0	-1.2	-3.5	0	0	-.84	0	0	-.64	0	-2.9	-1.0	0	0	0				

OPTIMAL VALUE OF THE OBJECTIVE FUNCTION= $106120

in the same way as the general DSSP problem in Figure 2. Activities Y_1 through Y_4 are net revenues associated with joint events (e_{111}, e_{112}), (e_{111}, e_{212}), (e_{211}, e_{122}), and (e_{211}, e_{222}), respectively. X_{111} through X_{114} are alternative stage I production activities, and X_{115} and X_{116} provide for the transfer of input two from stage I to stage II under each of the stage I states of nature. X_{121} through X_{124} and X_{221} through X_{224} are alternative stage two production activities given stage I states of nature one and two, respectively. Activities X_{125} and X_{225} complete the transfer of input two from stage I to stage II under the corresponding stage I states of nature.

Constraints (2) and (3) restrict the use of inputs one and two in stage I to the endowed levels associated with state of nature one. Constraints (4) and (5) limit resource use in stage I to the state two endowment. Constraints (6), (7), (8), (9), and (10), (11), (12), (13) are, similarly, the stage II resource constraints rendering stage II production activities permanently feasible following stage I states one and two, respectively. Constraint (14) transfers input two from stage I to stage II given state of nature one in stage I. Constraint (15) performs the same function given state two in the first stage. In constraints (16) through (19), total net revenue for each of the four possible joint events is summed into vector Y. Coefficients on the production activities in these rows are unit net revenues. Notice that resource requirements and activity net revenues are unchanged when the states of nature change (that is, these parameters are nonstochastic). Only the resource endowments (RHS's $b_{kn_t t}$) are random.

The tableau shown in Table 2 is for the same DSSP problem under an information structure of complete knowledge of the past and present, and assuming the occurrence of state one in stage I. As pointed out before, the optimal strategy under this information structure includes the solutions to separate optimization problems for each stage I state of nature. Thus, for this example, a second LP problem is formulated for the sequence of decisions implied under state two in stage I. The LP tableau for this case is shown in Table 3.

Optimal solutions for the complete knowledge of the past and for the complete knowledge of the past and present information structures are given with the tableaus in Tables 1, 2, and 3. Three other decision problems were formulated to demonstrate the significance of information structures as well as the sequential nature of the problem. First, the problem was solved for the case of perfect foresight. Here four solutions were generated -- one for each joint event -- with the outcomes of random events in both stages known at the beginning of the decision process. Second, the problem was solved with an information structure of complete knowledge of the past with the sequential component of the problem ignored (i.e., the first stage vector was selected without considering the opportunity to retain input two for use in stage II). Finally, a non-sequential problem was solved under complete knowledge of the past and present. The solutions to the five problems are summarized in Table 4.

The optimal strategy derived under complete knowledge of the past involves producing 6,400 units of product one in the first stage. Given that state of nature one occurs in stage I, this production strategy leaves

TABIE 2: TABLEAU FOR A HYPOTHETICAL DSSP PROBLEM (COMPLETE KNOWLEDGE OF THE PAST AND PRESENT, STATE ONE IN STAGE 1).

		Y1	Y2	X111	X112	X113	X114	X115	X121	X122	X123	X124	X125	X221	X222	X223	X224	X225	RHS	SLACK	DUAL
1)	OBJ (MAX)	.27	.18																		
2)	INPUT 1, 111			.80	.50	.29	.18												<= 3500	0	2.36
3)	INPUT 2, 111			.10	.22	.45	.73	1											<= 2400	0	4.86
4)	INPUT 1, 112								.85	.58	.35	.22							<= 3200	0	2.85
5)	INPUT 2, 112								.24	.32	.52	.75	-1						<= 1200	0	3.28
6)	INPUT 1, 222													.85	.58	.35	.22		<= 2900	0	2.80
7)	INPUT 2, 222													.24	.32	.52	.75	-1	<= 5800	0	1.58
8)	TRANSFER 11->12							-1					1						<= 0	0	3.28
9)	TRANSFER 11->22							-1										1	<= 0	0	1.58
10)	Y1 (e111,e112)	1		-5	-5	-5	-5		-10	-10	-10	-10							<= 0	0	0.27
11)	Y2 (e111,e212)		1	-5	-5	-5	-5							-10	-10	-10	-10		<= 0	0	0.18
	OPTIMAI VALUE	93745	138498	0	7000	0	0	860	0	4974	901	0	860	0	0	4793	5557	860			
	MARGINAL	0	0	-.13	0	-.62	-1.7	0	-.51	0	0	-.38	0	-.96	-.33	0	0	0			

OPTIMAL VALUE OF THE OBJECTIVE FUNCTION= $50241

TABLE 3: TABLEAU FOR A HYPOTHETICAL DSSP PROBLEM (COMPLETE KNOWLEDGE OF THE PAST AND PRESENT, STATE TWO IN STAGE I).

	Y3	Y4	X211	X212	X213	X214	X215	X321	X322	X323	X324	X325	X421	X422	X423	X424	X425		RHS	SLACK	DUAL
1) OBJ (MAX)	.25	.30																			
2) INPUT 1, 221			.80	.50	.29	.18												<=	3200	0	3.38
3) INPUT 2, 221			.10	.22	.45	.73	1											<=	5100	0	4.83
4) INPUT 1, 132								.85	.58	.35	.22							<=	3200	0	3.88
5) INPUT 2, 132								.24	.32	.52	.75	-1						<=	1200	0	2.19
6) INPUT 1, 242													.85	.58	.35	.22		<=	2900	0	4.66
7) INPUT 2, 242													.24	.32	.52	.75	-1	<=	5800	0	2.63
8) TRANSFER 21->32							-1					1						<=	0	0	2.19
9) TRANSFER 21->42							-1										1	<=	0	0	2.63
10) Y3 (e221,e132)	1		-5	-5	-5	-5		-10	-10	-10	-10							<=	0	0	0.25
11) Y4 (e211,e242)		1	-5	-5	-5	-5							-10	-10	-10	-10		<=	0	0	0.30
OPTIMAL VALUE	124637	160357	0	6400	0	0	3692	0	0	8938	325	3692	0	0	586	12250	3692				
MARGINAL	0	0	-.43	0	-.40	-1.4	0	-1.3	-.45	0	0	0	-1.6	-.54	0	0	0				

OPTIMAL VALUE OF THE OBJECTIVE FUNCTION= $79266

TABLE 4: SUMMARY OF SOLUTIONS TO THE HYPOTHETICAL DSSP PROBLEM UNDER VARIOUS INFORMATION STRUCTURES.

	PRODUCTION OF PRODUCT ONE (PRODUCT TWO) BY JOINT EVENT				UNITS OF INPUT TWO TRANSFERED FROM STAGE I TO STAGE II BY JOINT EVENT				NET REVENUE BY JOINT EVENT				EXPECTED NET REVENUE
INFORMATION STRUCTURE	1	2	3	4	1	2	3	4	1	2	3	4	
COMPLETE KNOWLEDGE OF THE PAST	6,400 (5,718)	6,400 (5,718)	6,400 (8,798)	6,400 (8,798)	992	992	3,692	3,692	89,181	89,181	119,978	119,978	$106,120
COMPLETE KNOWLEDGE OF THE PAST AND PRESENT	7,000 (5,875)	7,000 (10,350)	6,400 (9,264)	6,400 (12,836)	860	860	3,692	3,692	93,745	138,498	124,637	160,357	$129,507
PERFECT FORESIGHT	4,375 (7,211)	7,000 (10,350)	6,400 (9,263)	6,400 (12,836)	1,963	860	3,692	3,692	93,994	138,498	124,637	160,357	$129,574
COMPLETE KNOWLEDGE OF THE PAST, NON-SEQ	7,692 (4,187)	7,692 (4,187)	7,692 (7,790)	7,692 (7,790)	0	0	2,700	2,700	80,329	80,329	116,362	116,362	$100,147
COMPLETE KNOWLEDGE OF THE PAST AND PRESENT, NON-SEQ	8,120 (4,367)	8,120 (9,595)	11,147 (4,368)	11,147 (9,595)	0	0	0	0	84,276	136,550	151,687	117,693	$117,693

992 units of input two for use in stage II and 5,718 units of product two will be produced. However, if state of nature two occurs, 3,692 units of input two remain and the stage II strategy will be to produce 8,798 units. Under complete knowledge of the past and present, 7,000 units of product one will be produced given state one in stage I and 860 units of input two will be transferred for use in stage II. If, subsequently, state one occurs in stage II, 5,875 units of product two will be produced. If state two occurs, 10,350 units will be produced. Given state two in the first stage, output of product one will be 6,400 and 3,692 unit of input two will remain for later use. The resulting stage II strategy will involve production 9,264 (12,836) units given the occurrence of state one (two) in that stage.

Consider the differences in the optimal strategies under complete knowledge of the past and complete knowledge of the past and present. Under the second information structure, stage I output is greater under state one (note that an output of 7,000 is permanently feasible). With current stage resource endowments known, the selection of production activities is less restricted. Further, given state one, the relative scarcity of the transferable resource (input two) in the first stage is known when stage I production activities are selected. As a consequence, fewer units of input two are held for use in the second stage. Despite this, stage II production (which need not be permanently feasible with complete knowledge of the past and present) was greater under every joint event. Net revenue was, likewise, greater under all four joint events and expected net revenue was $129,507 compared to $106,120 under complete knowledge of the past.

Perfect foresight represents a higher information structure than complete knowledge of the past and present. Although the additional knowledge did not drastically change the expected net revenue maximizing solution, the differences in the two solutions are worth noting. When perfect foresight was assumed, decisions changed only under the first joint event (state one in both stages). Recall that under this joint event, input two is relatively scarce in both stages. With prior knowledge of this resource endowment, the decision maker produces less in the first stage, transfers more of the input to stage II and produces more in the second stage when compared to the case of complete knowledge of the past and present (where the relative scarcity of input two in stage II is unknown).

The last two solutions show the results for a myopic decision-maker. Here the expected value of the transferable resource in the second stage is ignored. The holding of input two for later use is, thus, only a consequence of the isolated stage I decision. Under complete knowledge of the past, 2,700 units of input two are left for use in the second stage when state two occurs in stage I. With complete knowledge of the past and present, the endowments of input two are totally enhausted under both states of nature. When these two strategies are compared to strategies derived for sequential decisions under the same information structures, stage I production is greater, stage II production is lower and net revenue is lower under every joint event. Correspondingly, expected net revenue is lower and the misallocation of resources associated with the non-sequential treatment of the decision process is demonstrated.

The example problem was reformulated to account for risk using an EV framework. The complete knowledge of the past information structure was assumed. Two objective function formats were used. First, the following objective function was specified:

$$\text{Max:} \quad .27Y_1 + .18Y_2 + .25Y_3 + .30Y_4 - \Phi \sum_{i=1}^{4} \sum_{j=1}^{4} V_{ij} Y_i Y_j \qquad (12)$$

Where, as before ...

$$V_{ii} = \alpha_j \,(1-\alpha_i),\ i=1\ldots4 \qquad (13)$$

$$V_{ij} = -\alpha_i \alpha_j,\ i=1\ldots4,\ j=1\ldots4,\ i\neq j \qquad (14)$$

For the probabilities used in this problem, matrix V takes on the following values:

$$V = \begin{bmatrix} .1971 & -.0486 & -.0675 & -.0810 \\ -.0486 & .1476 & -.0450 & -.0540 \\ -.0675 & -.0450 & .1875 & -.0750 \\ -.0810 & -.0540 & -.0750 & .2100 \end{bmatrix} \qquad (15)$$

The risk coefficient Φ was set at zero and a very large value to reveal the extreme points on the EV frontier (the risk neutral and risk free solutions, respectively). [8/] Then a series of solutions was generated with a minimum constraint on expected net revenue ...

$$.27Y_1 + .18Y_2 + .25Y_3 + .30Y_4 \geq \overline{Y} \qquad (16)$$

... and using the following objective function:

$$\text{Min:} \quad \sum_{i=1}^{4} \sum_{j=1}^{4} V_{ij} Y_i Y_j \qquad (17)$$

8/ It can easily be reasoned that a risk free solution to the problem exists. Variability of net revenue can be reduced to zero by reducing output under joint events 3 and 4 to a level which would make $Y_1=Y_2=Y_3=Y_4$. Such an adjustment is clearly feasible.

Thus, intermediate points of the EV frontier were derived. The resulting solutions are shown in Table 5.

Within the framework of this problem, reducing net revenue variability essentially involves adjustments in activities which will allow the levels of net revenue associated with each joint outcome to converge. For the example under complete knowledge of the past, only two unique net revenue outcomes occur — one for each stage I state of nature. Specifically, Y_1 equals Y_2 and Y_3 equals Y_4. Relatively few adjustments in decision vectors can be made to reduce the difference between the net revenue outcomes. The contribution of stage I production to the objective function will be the same regardless of the state of nature because the activities are permanently feasible and the activity net revenues are deterministic. However, by reducing the use of the second input in stage one, a greater amount of that input can be transferred for use in the second stage — thus more opportunities exist for stage II production. The two unique net revenue outcomes become closer in value if stage II production levels given each of the stage I states become closer in value (and thus, the variance of net revenue can decrease).

The first EV solution shown in Table 5 gives the risk neutral result which was found before. The variance of net revenue is 234,742,664. When expected net revenue was constrained to be no less than $105,000 and variance was minimized, stage I output decreased and more of input two was transferred to stage II under both stage I states. Production in stage II increased under both states, however, the difference between the two production levels declined and thus the variance

TABLE 5: SUMMARY OF SOLUTIONS TO THE QUADRATIC PROGRAMMING PROBLEM.

EXPECTED NET REVENUE	VARIANCE OF NET REVENUE	PRODUCTION OF PRODUCT ONE (PRODUCT TWO) BY JOINT EVENT				UNITS OF INPUT TWO TRANSFERED FROM STAGE I TO STAGE II BY JOINT EVENT				NET REVENUE BY JOINT EVENT			
		1	2	3	4	1	2	3	4	1	2	3	4
$106,120	234,742,664	6,400 (5,718)	6,400 (5,718)	6,400 (8,798)	6,400 (8,798)	992	992	3,692	3,692	89,181	89,181	119,978	119,978
$105,000	200,703,135	4,752 (6,558)	4,752 (6,558)	4,752 (9,405)	4,752 (9,405)	1,684	1,684	4,384	4,384	89,338	89,338	117,814	117,814
$100,000	91,770,428	4,000 (6,941)	4,000 (6,941)	4,000 (9,405)	4,000 (9,405)	2,000	2,000	4,700	4,700	89,409	89,409	108,665	108,665
$95,000	25,573,318	4,000 (6,941)	4,000 (6,941)	4,000 (8,867)	4,000 (8,867)	2,000	2,000	3,838	3,838	89,409	89,409	99,574	99,574
$90,000	285,503	4,000 (6,941)	4,000 (6,941)	4,000 (7,048)	4,000 (7,048)	2,000	2,000	2,994	2,994	89,409	89,409	90,483	90,483
$89,409	0	4,000 (6,941)	4,000 (6,941)	4,000 (6,941)	4,000 (6,941)	2,000	2,000	2,894	2,894	89,409	89,409	89,409	89,409

of net revenue declined. When the minimum expected net revenue was decreased to $100,000, stage I production declined again. Stage II production following state one in stage one increased again. Given state two in the first stage, stage II production is unchanged from the previous solution. As further decreases in expected net revenue were allowed, net revenue variability was minimized by reducing stage II production following state two in stage I. Other production vectors remained unchanged.

The DSSP-EV problem was transformed into an equivalent problem with a separable objective function using the technique discussed earlier. The first step was to find the eigenvalues and eigenvectors of matrix V. The eigenvalues $\lambda_1 \ldots \lambda_4$ are 0.00000, 0.19610, 0.25868, and 0.28742, respectively. The matrix used to transform variables Y_i into Z_i (with columns equal to the eigenvectors of V) is:

$$W = \begin{bmatrix} -0.50000 & -0.28173 & -0.62470 & 0.52250 \\ -0.50000 & 0.86205 & 0.05594 & 0.05725 \\ -0.50000 & -0.35766 & 0.75497 & 0.22826 \\ -0.50000 & -0.22265 & -0.19021 & -0.81501 \end{bmatrix} \qquad (18)$$

Special separable programming activities Q_{ij} were used, with Q_{ij} the activity corresponding to the jth value of Z_i ($\hat{Z}_{ij}$) used in the approximation. The resulting approximation of variance is:

$$\sum_{i=1}^{4} \sum_{j=1}^{4} \lambda_i \hat{Z}_{ij}^2 Q_{ij} \qquad (19)$$

$$\sum_{i=1}^{4} \sum_{j=1}^{4} Q_{ij} = 1 \qquad (20)$$

Where m values are used in the approximation of each separable term. Using prior information about the solution values of Y from the QP exercise, the following ranges were used for variables Z_i.

$$\hat{Z}_{11} = -225{,}000, \ldots, \hat{Z}_{1m} = -150{,}000$$
$$\hat{Z}_{21} = -20{,}000, \ldots, \hat{Z}_{2m} - 0$$
$$\hat{Z}_{31} = 0, \ldots, \hat{Z}_{3m} = 20{,}000$$
$$\hat{Z}_{41} = -20{,}000, \ldots, \hat{Z}_{4m} = 0$$

A matrix generating FORTRAN program was written to calculate m values $\hat{Z}_{i1} < \hat{Z}_{i2} < \ldots \hat{Z}_{im}$ at equal intervals for a user defined value of m. Two models were used -- model one with five steps in each interval and model two with 11 steps in each interval (i.e. m=5 and m=11, respectively). In addition to using more steps with model two, solution information from model one was used to further restrict the range of $\hat{Z}_{ij}$ values over which the objective function was approximated. Solutions along the EV frontier generated with the QP model are shown with the approximated separable programming (model one and model two) solutions in Table 6.

Note that the separable programming models over-estimate the variance of net revenue at each level of expected net revenue (results for the risk free solution were the same for all three models since zero was used as a grid point for each Z_i). However, with model two, the approximation is improved by using more values for Z_i over a smaller range. Generally, an acceptable level of accuracy can be achieved by using a large number of values in the piecewise linear approximation. The associated increase in the number of parameters to be calculated will be less burdensome if matrix generating computer programs are

TABLE 6: POINTS ON THE EV FRONTIERS USING QUADRATIC AND SEPARABLE PROGRAMMING.

EXPECTED NET REVENUE	VARIANCE OF NET REVENUE		
	QUADRATIC PROGRAMMING MODEL	SEPARABLE PROGRAMMING MODEL ONE	SEPARABLE PROGRAMMING MODEL TWO
$106,120	234,742,664	257,693,715 9.78%	237,888,787 1.34%
$105,000	200,703,135	204,455,948 1.87%	200,731,744 0.01%
$100,000	91,770,428	94,745,388 3.24%	91,807,072 0.04%
$95,000	25,573,318	28,118,649 9.95%	25,597,959 0.10%
$90,000	285,503	2,270,663 695.32%	317,679 11.27%
$89,409	0	0 0.0%	0 0.0%

* THE PERCENT ERROR IS SHOWN FOR EACH SEPARABLE PROGRAMMING SOLUTION.

employed. If relatively few values can be used because the matrix is "hand built" or the LP code imposes effective restrictions on matrix size, improved accuracy can be achieved by solving with a few grid points and solving again with the same number of steps in a closer grid around the first solution. For the numerical example used here, a decidedly modest increase in the number of steps and decrease in the range over which the functions are approximated gave a substantial improvement in the approximation.