Grazing a Common Pasture in a Random Environment[1,2]

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Abstract

A stochastic dynamic model of optimal stocking of a common range is presented. The model attempts to capture the predominant property rights features of mobile subsistence pastoralism in arid and semi-arid rangelands. Grazing rights are modelled on the one hand via the indirect effect of offtake on available pasture and on the other hand via resource sharing (niche overlap) between competing herds. Uncertainty is incorporated via environmental noise. The analysis is consistent with range dynamics being in disequilibrium.

1 Introduction

Ito stochastic control theory has been applied in agricultural and resource economics by a number of authors. Pindyck (1984) gives a survey of applications to renewable resources and Hertzler (1991) survey's applications to Agriculture. Applications involving common property have typically been restricted to non-renewable resources and have invariably involved postulating certainty to simplify the analysis[1].

Studies of optimal stocking have generally made use of either discrete time stochastic optimal control theory (e.g. Perrings (1994)) or stochastic dynamic programming (Wang (1991 and 1992)) and Passmore and Brown (1992), an exception is Beard (1995a) who used Ito control to study optimal stocking[2]. These studies have ignored the problem of property rights, in particular the grazing of common pastures which has been the focus of the static literature[3]. dynamic models that incorporate common property such as Clemhout and Wan (1985) have been more orientated towards, fisheries, forestry and mining problems.

In this paper we attempt to remedy this situation by developing a stochastic differential game of the grazing of a common pasture from the Ito perspective. The solution of stochastic differential games was first proposed by Friedman (1972). But it wasn't until the development of the technique of Davis and Varaiya that the solution of such problems became analytically tractable.

Despite the importance of both game theory and uncertainty in resource economics stochastic differential games have rarely been applied to resource problems. Clemhout and Wan (1985) are a notable exception[4].

The technical difficulties associated with solving systems of partial differential equations have led to the analysis of stochastic differential games being restricted to formal proofs of the existence of solutions and to numerical simulations.

A second approach would be to construct a solution and then verify that it is in fact a solution. This approach has been rarely used in the literature.

The literature on the existence of solutions involves two basic approaches. The Girsanov or weak-sense control approach and the occupation measure

[1]See Clemhout and Wan (1985)

[2]Other techniques have also been used, including discrete time deterministic optimal control Virtala (1992) and recently fuzzy control theory van Kooten and Kinyua (1995)

[3]For an overview of the static literature see Beard (1995b)

[4]It should however be noted that there is a developing literature on continuous-time principle agent problems that coinld be interpreted as games, although not in the sense I mean here.

approach. Both of these approaches have been used to study games in which noise follows a Wiener process and games in which noise follows a Poisson (jump) process.

The advantage of both these approaches is that they do not require the solution of a system of partial differential equations. Thus increasing the tractability of stochastic differential games. the disadvantage is that a concrete procedure for the construction of an equilibrium remains unclear.

The Girsanov approach involves substituting the original Wiener measure for a measure which guarantees a solution of the underlying system of stochastic differential equations and then solving the so-called Isaacs equation (a Hamiltonian) to find a solution to the game.

The occupation measure approach involves defining an "augmented state space" to include strategies as part of the state space. The result is that a dynamic programming problem can be transformed into a linear programming problem. Unfortunately, this method is not sufficiently developed to a stage where applications are possible. The standard treatment of this approach is Borkar and Ghosh (1992).

Given the prevalence of game theoretic problems in resource economics and the importance of uncertainty in many resource economic problems, the development of tractable techniques for solving stochastic differential games would be a significant development in improving the application and the popularity of stochastic differential game formulations of natural resource allocation problems.

In this paper, we present an application of a Girsanov approach to the solution of a stochastic differential game of grazing a common pasture The approach used involves applying the Martingale approach of Cox and Huang and Karatzas. Lehoczky and Shreve that is commonly used to solve stochastic control problems in finance, to the study of a stochastic differential game.

The approach therefore differs from the use of the Girsanov transform in the literature on stochastic differential games.

An objective of the paper is to analyze the tradeoff relationship between livestock numbers and the offtake rate in a common property setting. This is done by constructing a model based on a positive theory of economic behaviour (a theory of the rational pastoralist) and a normative theory of institutions (the theory of mechanism design). This approach differs from a New-Institutional approach in that institutions such as grazing rights are viewed from a normative rather than a positive perspective[5].

This objective must remain an intended objective pending the develop-

[5]I would like to thank Jyothi Vijaya Gali for clarifying the nature of the distinction between the new institutional economics and the theory of mechanism design.

ment of a theory of mechanism design for dynamic games of resource exploitation.

2 The Model

Consider the situation of common property grazing in a subsistence pastoral sector. A common situation in large parts of Africa and Asia. We restrict the analysis to a two player game.

The objective functional of each player is characterized by discounted expected utility, whereby utility in each period is represented by a Cobb-Douglas function in log form.

$$\max_{u_1} J_1 = E\left[\int_0^\infty [a \log y_1 + b \log u_1] e^{-rt} dt \mid F_0\right] \quad \text{Player 1}$$

$$\max_{u_2} J_2 = E\left[\int_0^\infty [a \log y_2 + b \log u_2] e^{-rt} dt \mid F_0\right] \quad \text{Player 2}$$

where u_i is the offtake rate, y_i the size of each herd/flock, r_j the discount rate's. Thus pastoral households face a tradeoff between the herdsize and offtake required for subsistence (survival).

The dynamics of the system are represented generally by the following system of Ito differential equations.

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} f(x,y,z) \\ g_1(x,y,z,u_1) \\ g_2(x,y,z,u_2) \end{pmatrix} dt + \begin{pmatrix} \sigma(x) & cov(x,y) & cov(x,z) \\ cov(x,y) & \sigma(y) & cov(y,z) \\ cov(x,z) & cov(y,z) & \sigma(z) \end{pmatrix} \begin{pmatrix} dw_0 \\ dw_1 \\ dw_2 \end{pmatrix}$$

where x is the available pasture measured in livestock units[6].

The control problem is of the indirect type where the control variables (u_1, u_2) or offtake rates have an indirect impact on the state variable x (pasture) via the total change in stock numbers (dy, dz). Offtake rates are chosen simultaneously, so the game is of the Nash type[7].

It is important to note in choosing specific functional forms for $g_1(x,y,z,u_1)$ and $g_2(x,y,z,u_2)$ that the game structure of the problem is not purely economic but is also induced via the degree of "niche sharing" that occurs between the two (or more) herds.

[6] e.g. dry sheep equivalent, tropical livestock units, animal unit years or some equivalent measure

[7] The definition of nash equilibrium in dynamic games differs somewhat from the static literature. The distinction is however somewhat technical, the interested reader is referred to the literature. for a discussion of indirect control problems, see Lefschetz ()

It is assumed that the the current state of the state variable "pasture" is completely observable. The problem is therefore known as a *Markovian game* or a *game of perfect observation played by pure strategies*[8]. It should be noted that both Friedman and Davis and Varaiya considered the case where the state is partially observable. Partial observability of the state variable is analogous to Clark (1976)'s concept of metered models that are commonly used in fisheries problems. Both resource economists and ecologists have tended to treat the problem of partially observed state variables by using a discrete time framework. Strictly speaking this is not a correct or rigorous approach to the problem of observability of the system. The approach used by both Friedman and Davis and Varaiya maintains the continuous time framework but assumes only partial observability of the system. In this paper we ignore this more complex case by assuming that pastoralists know the current system state. This assumption seems plausible for traditional pastoralists who move with their herds and herd livestock more intensively than their Australian or North American counterparts, although it may not be appropriate for continuous stocking strategies most frequently used by Australian and North American pastoralists.

The model represents a continuous time analogue of Perrings model of optimal stocking in the Sahel, with the added feature of common property. Common property is incorporated by using MacArthur and May's model of niche overlap, the degree of niche overlap representing the degree of competition for pasture. In general, given n species the populations of the n species possess the following dynamics,

$$\frac{dN_i(t)}{dt} = N_i(t)\left[k_i - \sum_{j=1}^{n} \alpha_{ij} N_j(t)\right]$$

where α_{ij} are competition or niche overlap parameters. This model is a version of the Lotka-Volterra model.

It differs from the Lotka-Volterra model in the interpretation of the parameters. These are no longer interpreted as phenomenological constants but as integrals of resource utilization,

$$k_i = \int K(x) f_i(x) dx$$

[8]Friedman (1972): p. 86. The term Markovian game is attributed by Friedman to Fleming. Note the term Markovian game predates the term Markov game that has been popularized by Fudenberg and Tirole (1991). Concretely it is assumed that the drift and diffusion terms depend only on the current state of the system. Also note that a Markovian game differs from a Markov game in that the state dynamics follow a stochastic differential equation.

$$\alpha_{ij} = \int f_i(x) f_j(x) dx$$

where $K(x)$ is a one-dimensional resource continuum and $f_i(x)$ the resource utilization (density) function of the i-th species, respectively j-th species[9].

Note this system of differential equations becomes a system of Ito differential equations if one sets.

$$k_i(t) = \bar{k}_i + \gamma_i(t)$$

where $\gamma_i(t)$ is a white noise term and $\bar{k}_i$ the mean intrinsic growth rate of the i-th species.

By introducing a capacity term into this system of equations one obtains a system of coupled stochastic logistic equations with niche overlap.

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} nx(1 - \frac{x}{K}) - cx(y+z) \\ my(1 - \beta\frac{y}{x} - (1-\beta)\frac{z}{x}) - u_1 \\ lz(1 - \beta\frac{y}{x} - (1-\beta)\frac{z}{x}) - u_2 \end{pmatrix} dt + \begin{pmatrix} \sigma x & cov\,(x,y) & cov\,(x,z) \\ cov\,(x,y) & \sigma y & cov\,(y,z) \\ cov\,(x,z) & cov\,(y,z) & \sigma z \end{pmatrix} \begin{pmatrix} dw_0 \\ dw_1 \\ dw_2 \end{pmatrix}$$

This model combines features of both Perrings optimal stocking model and May's niche overlap model in such a way as to allow the incorporation of common property into a (stochastic) differential game of optimal stocking.

The Ito approach provides for a more general version of the state and transition approach to range dynamics in that it does not necessarily assume stationarity and by allowing stock numbers to be subject to random fluctuations takes into considerations fluctuations in available drinking water that are, in addition to fluctuations in available forage, common in many arid and semi-arid regions.

In order to solve the above stochastic differential game of grazing some use of measure theory is required. In particular, the usual technique of solving the Hamilton-Jacobi-Bellman equation is not employed. Instead use is made of the Girsanov measure transform technique (see Appendix). Two approaches to solving stochastic control problems have been suggested in

[9] For the mathematically inclined it may be of interest to note that $K(x)$ is in fact a Green's function. See Adomian (1983) for a discussion of Green's functions and operator theory as applied to stochastic differential equations.

the literature that make use of the Girsanov measure transform technique. The first approach known as the method of Davis and Varaiya (1973) involves transforming the problem into a Hamiltonian. The second approach attributed too Karatzas, Lehoczky and Shreve and Cox and Huang[12] is commonly used in financial economics. this approach involves transforming the state equations into Martingales thereby allowing the solution of a stochastic control problem using standard Lagrangian techniques. Both approaches avoid the requirement of solving a system of partial differential equations (system of Hamilton-Jacobi-Bellman equations).

We employ the Martingale technique of Cox and Huang to the problem of solving a stochastic differential game of common property grazing.

To the best of our knowledge this approach has not previously been applied to the study of stochastic differential games or resource economic problems.

3 Solution

Consider the following stochastic differential game

$$max_{u_1} E\left[\int_0^\infty U_1(y,u_1)e^{-\rho t}dt \mid F_0\right]$$

and.

$$max_{u_2} E\left[\int_0^\infty U_2(z,u_2)e^{-\rho t}dt \mid F_0\right]$$

subject to.

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} nx(1-\frac{x}{K}) - ex(y+z) \\ my(1-\beta\frac{y}{x}-(1-\beta)\frac{z}{x}) - u_1 \\ kz(1-\beta\frac{y}{x}-(1-\beta)\frac{z}{x}) - u_2 \end{pmatrix} dt + \begin{pmatrix} \sigma x & cov(x,y) & cov(x,z) \\ cov(x,y) & \sigma y & cov(y,z) \\ cov(x,z) & cov(y,z) & \sigma z \end{pmatrix} \begin{pmatrix} dw_0 \\ dw_1 \\ dw_2 \end{pmatrix}$$

In order to solve this game we attempt to transform the drift terms in such a way that the state equations become Martingales. this then allows one to apply standard Lagrangian techniques to determine the optimal control rules of each player[12, Cox and Huang]. Inserting these optimal control rules back into the state equation and the objective function gives a solution to the game.

Because the first state equation (pasture dynamics) is independent of the the control variable (offtake rate) the resultant Lagrangian need not contain a pasture constraint (the shadow price of pasture is zero). Pasture is therefore undervalued and one would expect overgrazing to result. Whether or not this occurs depends on the dynamics of the controlled system. Thus, although pasture dynamics does not constrain the the herders optimal offtake decision it does have an impact on livestock dynamics and vice versa. The optimal stocking game being played between herders is therefore rather indirect and somewhat subtle.

In the Martingale approach used here we therefore only need to consider two state equations in the following.

Exoanding the drift term's gives,

$$my - \beta m\frac{y^2}{x} - (1-\beta)\frac{myz}{x} - u_1$$

$$lz - \beta\frac{lzy}{x} - (1-\beta)\frac{lz^2}{x} - u_2$$

Assuming the variance - covariance matrix is nonsingular and may therefore be inverted, the Girsanov transform (in vector form),

$$d\mathbf{w} = d\tilde{\mathbf{w}} - \Sigma^{-1}\mathbf{f}(\mathbf{x},\mathbf{y},\mathbf{z})dt$$

where $\mathbf{f}(\mathbf{x},\mathbf{y},\mathbf{z})$ is the drift vector, gives the follwing system of SDE's,

$$\begin{pmatrix} dy \\ dz \end{pmatrix} = \begin{pmatrix} my - u_1 \\ lz - u_2 \end{pmatrix} dt + \begin{pmatrix} \sigma y & cor(y,z) \\ cor(y,z) & \sigma z \end{pmatrix} \begin{pmatrix} d\tilde{w}_1 \\ d\tilde{w}_2 \end{pmatrix}$$

We now introduce the following riskless variables representing herd dynamics,

$$\bar{y} = y_0 e^{mt}$$

$$\bar{z} = z_0 e^{lt}$$

We then carry out the following change of variables ($\psi = \frac{y}{\bar{y}}, \phi = \frac{z}{\bar{z}}$) and apply the Ito chain rule

$$dF = \left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial y} f(y) + \frac{\partial F}{y} f(y) + \frac{1}{2}\frac{\partial^2 F}{\partial y^2}\Sigma\Sigma^T\right] dt + \Sigma\frac{\partial F}{\partial y} d\tilde{w}$$

to obtain a new system of equations.

$$\begin{pmatrix} d\psi \\ d\phi \end{pmatrix} = \begin{pmatrix} \frac{u_1}{y} \\ \frac{u_2}{z} \end{pmatrix} dt + \begin{pmatrix} \frac{1}{y} & \frac{1}{z} \end{pmatrix} \begin{pmatrix} \sigma y & cov(y,z) \\ cov(y,z) & \sigma z \end{pmatrix}^T \begin{pmatrix} d\hat{w}_1 \\ d\hat{w}_2 \end{pmatrix}$$

Integrating we obtain,

$$\psi(t) - \psi(0) = \int_0^t \frac{u_1}{y} ds + \int_0^t \left(\frac{1}{y}(\sigma y) + \frac{1}{z} cov(y,z) \right) d\hat{w}_1 + \int_0^t \left(\frac{1}{y}(\sigma y) + \frac{1}{z} cov(y,z) \right) d\hat{w}_2$$

$$\phi(t) - \phi(0) = \int_0^t \frac{u_2}{z} ds + \int_0^t \left(\frac{1}{z}(\sigma z) + \frac{1}{y} cov(y,z) \right) d\hat{w}_1 + \int_0^t \left(\frac{1}{z}(\sigma z) + \frac{1}{y} cov(y,z) \right) d\hat{w}_2$$

Rewriting gives,

$$\frac{y(t)}{y(t)} - \frac{y(0)}{y(0)} = \int_0^t \frac{u_1}{y} ds + \int_0^t \left(\frac{1}{y}(\sigma y) + \frac{1}{z} cov(y,z) \right) d\hat{w}_1 + \int_0^t \left(\frac{1}{y}(\sigma y) + \frac{1}{z} cov(y,z) \right) d\hat{w}_2$$

$$\frac{z(t)}{z(t)} - \frac{z(0)}{z(0)} = \int_0^t \frac{u_2}{z} ds + \int_0^t \left(\frac{1}{z}(\sigma z) + \frac{1}{y} cov(y,z) \right) d\hat{w}_1 + \int_0^t \left(\frac{1}{z}(\sigma z) + \frac{1}{y} cov(y,z) \right) d\hat{w}_2$$

Evaluating these in the limit as $t \to \infty$ and noting that $y = y_0 e^{mt}$ and $z = z_0 e^{lt}$ implies that the L.H.S is zero (as long as $y(\infty)$ and $z(\infty)$ are bounded).

$$0 = \frac{y(0)}{y(0)} - \int_0^\infty \frac{u_1}{y} ds + \int_0^t \left(\frac{1}{y}(\sigma y) + \frac{1}{z} cov(y,z) \right) d\hat{w}_1 + \int_0^t \left(\frac{1}{y}(\sigma y) + \frac{1}{z} cov(y,z) \right) d\hat{w}_2$$

$$0 = \frac{z(0)}{z(0)} - \int_0^\infty \frac{u_2}{z} ds + \int_0^t \left(\frac{1}{z}(\sigma z) + \frac{1}{y} cov(y,z) \right) d\hat{w}_1 + \int_0^t \left(\frac{1}{z}(\sigma z) + \frac{1}{y} cov(y,z) \right) d\hat{w}_2$$

Then inserting y and z, evaluating at $t = 0$ and taking expectations of both sides we obtain the following,

$$y(0) = E\left[\int_0^\infty \frac{u_1}{e^{mt}} p_y(t) dt \,|\, F_0 \right]$$

$$z(0) = E\left[\int_0^\infty \frac{u_2}{e^{lt}} p_z(t) dt \,|\, F_0 \right]$$

A change of Filtration gives the Martingales,

$$y(t) = E\left[\int_0^\infty \frac{u_1}{e^{mt}} p_y(t) dt \,|\, F_t \right]$$

$$z(t) = E\left[\int_0^\infty \frac{u_2}{e^{lt}} p_z(t) dt \,|\, F_t \right]$$

The game may now be formulated as a pair of Lagrangian's,

$$\max_{u_1} L(u_1, \lambda) = E\left[\int_0^{\infty} U_1(y, u_1)e^{-rt}dt|F_0\right] - \lambda\left[y(0) - E\left[\int_0^{\infty} \frac{u_1}{e^{mt}} p_y(t)dt|F_0\right]\right]$$

and

$$\max_{u_2} L(u_2, \mu) = E\left[\int_0^{\infty} U_2(z, u_2)e^{-rt}dt|F_0\right] - \mu\left[z(0) - E\left[\int_0^{\infty} \frac{u_2}{e^{lt}} p_z(t)dt|F_0\right]\right]$$

We represent herders preferences for livestock (services) and meat/money (offtake) using a Cobb-Douglas utility function in log form,

$$U_1(y, u_1) = a\log y + b\log u_1$$

$$U_2(z, u_2) = a\log z + b\log u_2$$

The first order conditions are,

$$\frac{\partial L}{\partial u_1} = E\left[\int_0^{\infty} \frac{b}{u_1} e^{-rt}dt|F_0\right] + \lambda E\left[\int_0^{\infty} \frac{p_y(t)}{e^{mt}}dt|F_0\right] = 0$$

$$\frac{\partial L}{\partial u_2} = E\left[\int_0^{\infty} \frac{b}{u_1} e^{-rt}dt|F_0\right] + \mu E\left[\int_0^{\infty} \frac{p_z(t)}{e^{lt}}d = 0t|F_0\right] = 0$$

Note the change in the filtration for the second first order conditions.

$$\frac{\partial L}{\partial \lambda}\Bigg|_t = y(t) - \frac{1}{p_y(t)} E\left[\int_0^{\infty} \frac{u_1}{e^{ms}} p_y(s)ds|F_s\right] = 0$$

$$\frac{\partial L}{\partial \mu}\Bigg|_t = z(t) - \frac{1}{p_z(t)} E\left[\int_0^{\infty} \frac{u_2}{e^{ls}} p_z(s)ds|F_s\right] = 0$$

This is necessary for the constraint to be a Martingale.
Cancelling the expectation and integration operators on both sides, gives,

$$\frac{b}{u_1} e^{-rt} = -\lambda \frac{p_y(t)}{e^{mt}}$$

$$\frac{b}{u_2} e^{-\beta t} = \mu \frac{p_z(t)}{e^{lt}}$$

Rearranging, we obtain the optimal controls for both players.

$$u_1^* = \frac{b e^{-\beta t} e^{mt}}{\lambda p_y(t)}$$

$$u_2^* = \frac{b e^{-\beta t} e^{lt}}{\mu p_z(t)}$$

We still need to eliminate the Lagrangian parameters in order to obtain a closed form solution.

Applying the continuous time version of Baye's theorem[25],

$$E\left[\int_0^\infty U(v, u_v) e^{-\beta t} dt \mid F_0\right] = \frac{1}{p_v(t)} E\left[\int_0^\infty U(v, u_v) e^{-\beta t} p_v(t) dt \mid F_0\right]$$

where in our case $v = y, z$, allows us to change the expectation back to the original measure, this gives (from the constraint),

$$y(t) = \frac{1}{p_y(t)} E\left[\int_t^\infty \frac{u_1}{e^{mt}} p_y(t) dt | F_s\right]$$

$$z(t) = \frac{1}{p_z(t)} E\left[\int_t^\infty \frac{u_2}{e^{lt}} p_z(t) dt | F_s\right]$$

Inserting u_1^* and u_2^* into these rearranging and evaluating the integrals and finally simplifying yields,

$$u_1^* = y(t) i e^{mt}$$

$$u_2^* = z(t) j e^{lt}$$

4 Interpretation of Results

This result suggests that herders base offtake decisions on the current total herd size that they would expect multiplied a growth factor for the herd in the absence of environmental noise adjusted by the discount rate. This expected stock level is an overestimate of actual livestock numbers as it fails to take into account the diminishing natural increase of the herd as it approaches the carrying capacity and it also fails to take into account any property rights externailities.

In the absence of discounting herders would therefore cull too many stock. Discounting serves to ameliorate the herders optimism by reducing their offtake rates. High discount rates close to one (1) tend to adjust the offtake rate downwards by very little. Low discount rates close to zero (0) tend to adjust the discount rate downwards by a great deal. In the latter case the offtake rate would be small.

Because herders value current utility higher by discounting they will tend to cull stock in order to thereby gain additional utility. At lower discount rates they value the stream of benefits obtained from holding livestock more highly.

To determine the exact impact of this behaviour on pasture we need to analyze the controlled system by inserting the optimal controls into the system of stochastic differential equations. This requires numerical solution of the controlled system of differential equations. The results of this analysis are presented in sections 6.

5 Sustainability and stability

Whether or not one favours enclosure or common property from a policy perspective would require some sort of normative statement concerning the performance of common property in the face of environmental uncertainty.

The problem is how does one define some criteria of sustainability in a stochastic setting. Sustainable yield concepts (e.g. maximum sustainable yield) are based on steady-state assumptions which have a different interpretation in a stochastic setting. Bionomic equilibria also have little meaning in this setting as apart from the fact that we use utility and not profit functions, they are an essentially a static criteria.

A number of approaches are possible such as analysing the mean exit time from a sustainable region, in our case, a natural choice is $\Re^3_+$ as $(x, y, z) \in \Re^3_+$.

Alternatively, one might (ignoring noise) analyze the stability of (x, y, z) around the origin. Instability would then imply sustainability and stability would imply that the system is unsustainable. The impact of enclosure may then be analyzed by setting $\theta = 1$ and anlyzing the dynamics of the controlled system of stochastic differential equations either numerically or by carrying out a stability analysis.

Other criteria that might be used include the determination of whether the elliptic function of the corresponding Hamilton-Jacobi-Bellman equation is a Lyapunov function or not. (See Siljak (1978) for a treatment of this case.).

These examples all amount to criteria of the sustainability (in the sense) of the stability of the system. but not the resilience (Holling) of the system.

As one of the authors (Beard) has argued elsewhere [5], the Holling resilience of a stochastic dynamic system is best analyzed from the perspective of mean exit time from a defined region (basin of attraction). We will not address the issue of Holling resilience in this paper.

In the absence of an appropriate criteria of sustainability, numerical analysis is likely to shed some light on how sustainable the system is. To this end we analyzed the controlled deterministic and stochastic systems, in order to gain some insight into the impact of different property rights regimes on the dynamics of both pasture and herds.

6 Numerical Analysis

Numerical analysis of the controlled system was carried out in order to determine what impact the optimal strategies of the two herders is likely to have on the dynamics of a common property pasture[10]. In order to facilitate the analysis of systematic relationships we initially restrict the analysis to deterministic dynamics.

Fig.1 shows pasture and herd dynamics of a common pasture with a 50% pasture sharing rule between herders (pure common property) and an intrinsic pasture growth rate of 2.7, corresponding to an annual pasture.

Herders discount the future at a rate which corresponds approximately to cull rate equal (or slightly below) the intrinsic growth rate of the herd ($i = \frac{m}{pmx}$). this serves to maintain discount rates in a relatively realistic relationship to growth rates, so that the herd does not collapse through overculling. Figure 1 shows the herd and pasture dynamics under common property, comparing this with Figure 2., one sees that

although enclosure leads to a higher steady-state pasture level, the variation in both pasture and herd biomass has increased. Thus enclosure exposes herders to greater risks than common property. This result is consistent with and in fact confirms one of the explanations for common property that has been at the center of the debate in the static literature[4].

Interesting is also the case illustrated by Figure 3, in which the intrinsic growth rate of pasture has been increased from 2.7 to 3 in order induce greater fluctuations in pasture biomass, in order to represent a disequilibrium

[10] A simple first order Euler recursion scheme was used, which although not the most accurate of techniques has the virtue of simplicity.

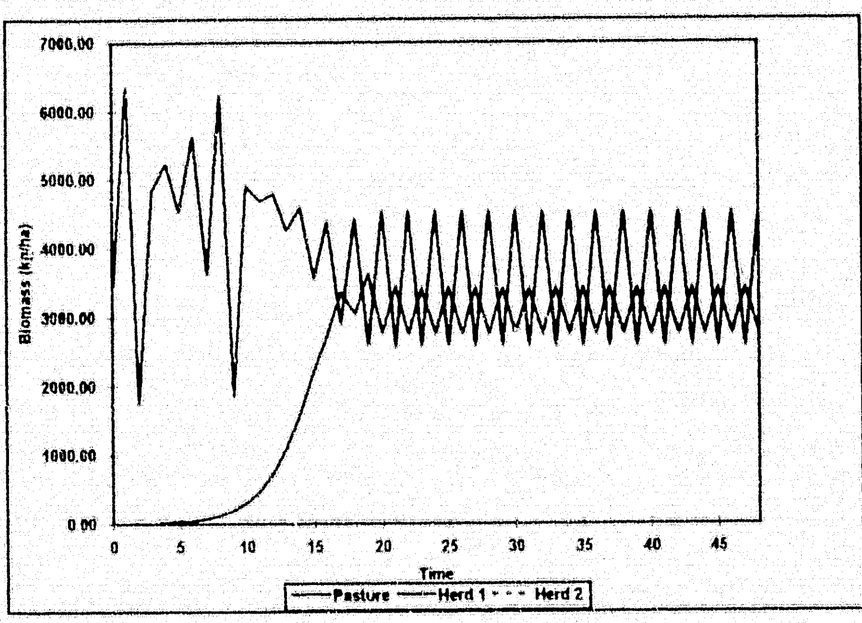

Figure 1: Herd and pasture dynamics under common property

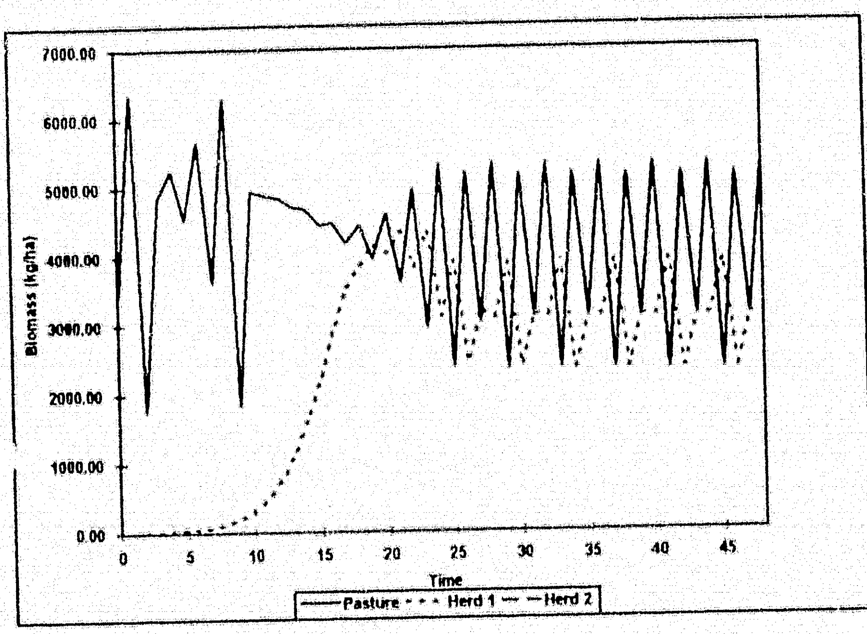

Figure 2: Herd and pasture dynamics under enclosure

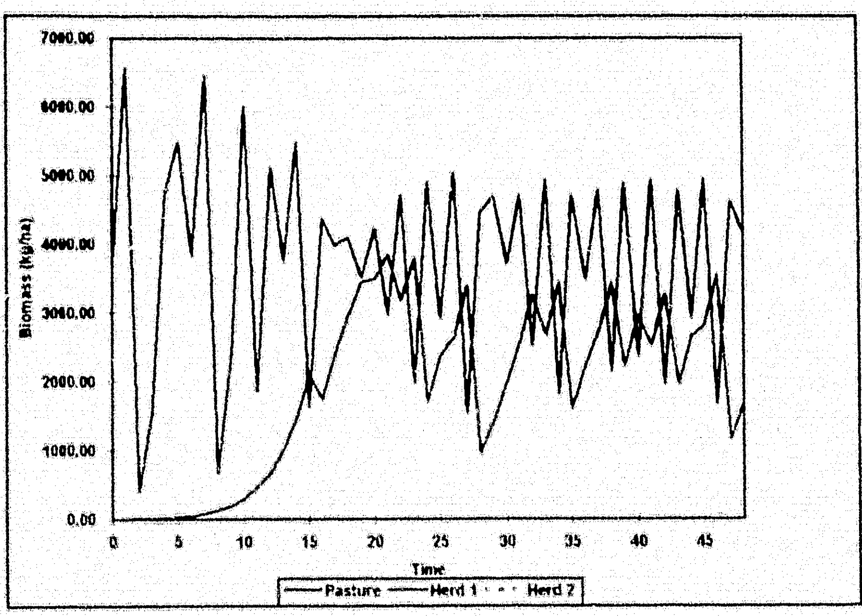

Figure 3: Sustainability of Common Property for a Rangeland in Disequilibrium

environment. Under these circumstances common property proves more sustainable than enclosure.

Enclosure, leads to the collapse of the herd in such a situation because it results in both greater pasture and herd variability, thus increasing the risk of the system collapsing. Figure 4 portrays the latter case.

It is important to note that this numerical analysis ignores the impact of noise on the system dynamics. Noise can lead to the formation or destruction of limit cycles and other attractors, therefore it may have both a stabilizing (stochastic resonance) or destabilizing (e.g. noise induced chaos) influence depending on the size of the perturbation. In order to analyze these effects a numerical analysis of the controlled system of stochastic differential equations is necessary.

Figure 5 shows the results of such an analysis for the case of common property.

One of the interesting features of adding noise to the system is that the dynamics of the two herds may diverge as a result of environmental influences on the herd. Some herders may be luckier than others.

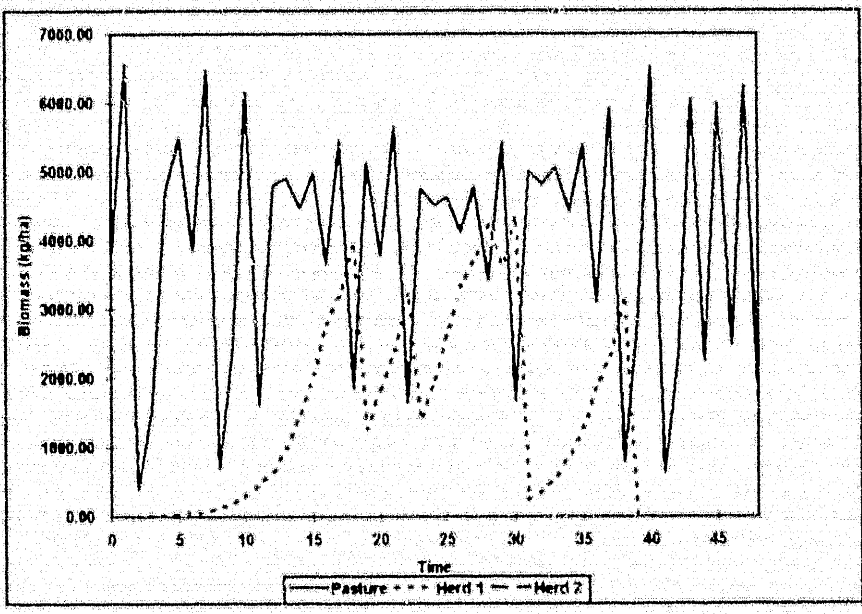

Figure 4: Collapse of the enclosed system for a rangeland in disequilibrium

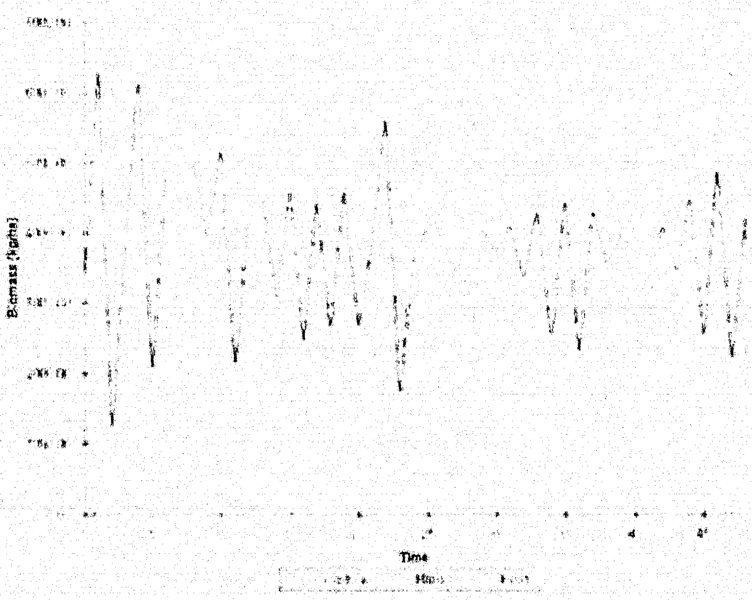

Figure 5: Stochastic dynamics under common property

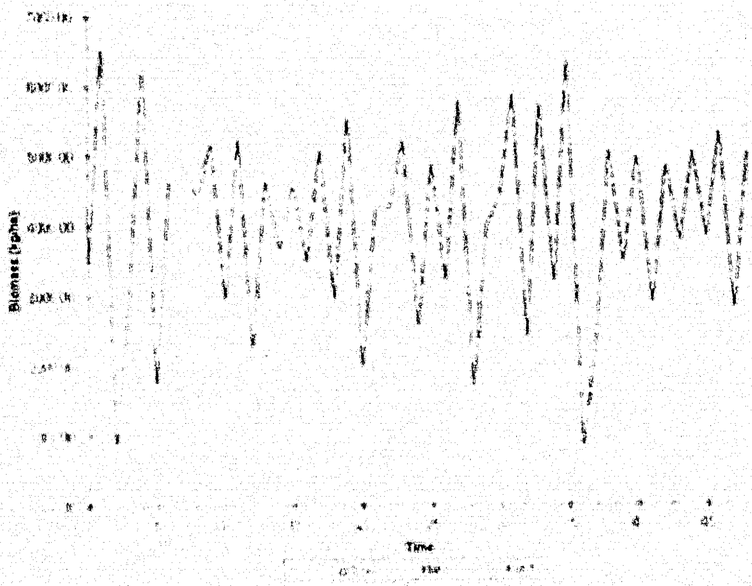

Figure 6: Stochastic dynamics under enclosure

In the stochastic setting enclosure is even less desirable than in the deterministic setting as Figure 6 illustrates. Enclosure leads to the collapse of the herd and to far more erratic pasture dynamics. Private property does not appear to handle risk very well at all.

7 Policy Implications

The policy implications seem clear. Attempts at enforced enclosure in many developing countries have been justified in terms of the tragedy of the commons. Considerable theoretical and empirical evidence has called the tragedy of the commons into question. The lack of bioeconomic models of common property grazing and the utilization of static and deterministic methods has meant that only particular aspects of the enclosure versus common property debate have been discussed, such as rent dissipation scattering or scale economies. In constructing a dynamic stochastic model of common property grazing, we have shown that common property has a number of advantages over private property when one takes risk into account. In a random environment such as an arid or semi-arid rangeland that may be in disequilibrium (nonstationary distributions) we find that private property is not a sustainable institution due to it's poor ability to handle environmental fluctuations.

It is the risk properties of common property systems that explain their robustness as a means of organizing grazing rights in many arid and semi-arid regions of the world.

8 Conclusion

In this paper, we have employed a Martingale approach to solve a stochastic differential game of grazing a common pasture in a random environment. Numerical analysis of the controlled deterministic system was then used to analyze the impact of enclosure on the ecological and cultural sustainability of a common property susbsistence pastoral system. The results are consistent with the static literature on common property and the dynamic literature on optimal stocking under sole ownership. At the same time however, they further call into question. the appropriateness of the "tragedy of the commons" as a theoretical basis for policy decisions. Whilst enclosure leads to higher mean pasture biomass, because the tragedy of the commons theory involves a largely static analysis, it ignores variations in pasture biomass. As a means of managing risk common property has advantages over enclosure in terms of lower variability of stock numbers and pasture biomass.

A number of extensions to the work presented here are intended. These include analysis of mean exit times from basins of attraction. Extending the analysis to incorporate semi-subsistence economies rather than pure susbsistence economies, and the analysis of mixed systems of property rights such as transhumant grazing. The latter will require the introduction of seasonality and multiple ecological patches into the model.

A Girsanov's Measure Transform Technique

Girsanov's measure transform technique exploits both the Radon-Nikodym theorem from measure theory and the Cameron-Martin formula (translation theorem) from probability theory to transform one probability measure into another by way of a linear transform of the drift term of a stochastic differential equation.

Given a nonanticipating Brownian (Wiener) functional $\theta(t,\omega) = \sigma^{-1} f(x,t)$ then the following functional known as the Radon-Nikodym derivative may be defined,

$$\rho(.) = e^{\zeta(\theta(t,\omega))} = e^{\int_s^t \theta(t,\omega) dw(t) - \frac{1}{2}\int_s^t |\theta(t,\omega)|^2 dt}$$

such that $d\tilde{P} = e^{\zeta(\theta(t,\omega))} dP$ and $Ee^{\zeta(\theta(t,\omega))} = 1$.

The probability measure $\tilde{P}$ possesses the following properties,

i) $\tilde{P}$ is absolutely continuous with respect to P[11].

ii) Given a Wiener process w_t defined for the measure $\tilde{P}$ then,

$$d\tilde{w} = dw - \Sigma^{-1} f(t,x)dt]$$

Using this technique the existence of a *weak sense* optimal control may be shown[12].

The choice of $f(x,t)$ should be made in such a way as simplify the problem, e.g. linearize the differential equations.

[11] If (X, A) is a measurable space and $\tilde{P}$ and P positive measures defined on (X, A) then $\tilde{P}$ is absolutely continuous with respect to P if for each set A in A both $P(\tilde{A}) = 0$ and $P(A) = 0$. See Cohn (1980): p. 131.

[12] The term weak sense refers to an optimal solution of the problem existing in the sense of Schwartz distributions. See Rishel (1971).

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