SHADOW PRICES IN LINEAR PROGRAMMING PROBLEMS

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ABSTRACT

Shadow prices in linear programming problems can be obtained from the coefficients of slack and surplus variables in the canonical form of the objective function at the optimal solution. However, most introductory textbooks present a method for extracting shadow prices from the canonical form of the objective function in conjunction with material on the dual. An introductory course in operations research would normally begin with the graphical solution to simple linear programming problems including shadow prices and then cover the simplex method and the related topics of artificial variables, the big M method and the two phase method followed by material on the dual. It is desirable to teach a method for determining shadow prices in conjunction with the simplex method because the continuity of concepts introduced at the beginning of the course is maintained and the interpretation of dual variables as shadow prices reinforced when the material on the dual is covered. The student can verify that the shadow prices calculated in conjunction with the solution to linear programming problems using the simplex method are numerically identical to the optimal values of the dual variables in each case. This article presents a simple method for calculating shadow prices from the optimal canonical form of the primal objective function. The method and its justification does not depend in any way on the dual.

Key words: Shadow prices, linear programming

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1. Introduction

In linear programming problems the shadow price of a constraint is the difference between the optimised value of the objective function and the value of the objective function, evaluated at the optimal basis, when the right hand side (RHS) of a constraint is increased by one unit. It may be shown, using the approach of Winston (1995, pp.250-255, 293-300) that the shadow prices corresponding to the constraints of a primal LP problem are equal to the optimal values of the dual variables.

The importance of shadow prices stems from the fact that over the range for which changes to the RHS coefficient of a constraint in a primal problem do not lead to a change in the variables which are in the optimal basis, the shadow price associated with the constraint gives the change in the maximised value of the primal objective function per unit of change in the RHS coefficient of the constraint. Thus, the shadow price of each constraint gives the marginal valuation of the fixed resource represented by the RHS coefficient of the constraint as determined by the optimal solution to the primal problem.

The optimal values of the dual variables may be obtained from the coefficients of slack and excess variables which are in the canonical form of the primal objective function at the optimal solution of the primal problem. However, it is an unfortunate fact from the pedagogic viewpoint, that most textbooks explain how to extract the shadow prices of the constraints from the canonical form of the objective function of the primal problem at the optimal solution after covering a considerable amount of material on the dual (see for example: Winston (1995), Daellenbach, George and McNickle (1983) and Bierman, Bonini and Hausman (1986)). An important exception is Wagner (1970, pp.113-114) who covers shadow prices in conjunction with material on finding the range over which the RHS coefficient of a primal constraint may be changed and the marginal valuation interpretation of the shadow price can be maintained, before covering material on the dual.[1]

A typical introductory course in operations research would start with the graphical solution to simple linear programming problems including the definition and calculation of shadow prices, proceed to the simplex method for solving standard linear programming problems and then cover artificial variables and the solution of non-standard linear programming problems using the big M and two phase methods. Material on the dual, mixed integer linear programming and other topics would then be covered.

The author has found it useful to incorporate material on how to extract shadow prices from the coefficients of slack and excess variables in the canonical form of the primal objective function at the optimal solution, when the simplex method is taught. This approach maintains the continuity of the ideas introduced when the graphical solution to simple linear programming problems is covered and prepares students for the interpretation of the solution to the dual as giving the marginal valuations of the resources represented by the RHS coefficients of the constraints of the primal problem. After material on the dual is covered, students can use the solutions to standard and non-standard linear programming problems they have solved and the complementary slackness theorem, to obtain the solutions to the corresponding dual problems. Thus they can verify that the optimal values of the dual variables correspond numerically to the shadow prices they have already calculated for each problem.

The remainder of the paper is as follows. In the next section a simple method for computing shadow prices is outlined. Two examples are provided and in each case it is shown that the shadow prices are identical to the optimal values of the corresponding dual variables. In section 3 the relationship between the shadow price of a constraint of a linear programming problem and the coefficient of the corresponding slack or surplus variable in the canonical form of the objective function of a linear programming problem at the optimal solution is rigorously derived. It is also shown that the method for computing shadow prices which is outlined in section 2 gives correct answers. The final section contains some concluding remarks.

2. Computing Shadow Prices

In this section we shall outline a simple method for computing shadow prices which can be taught in conjunction with material on the simplex method. The method outlined below for computing shadow prices is based on the following two propositions:

(1) The coefficient of a non-basic variable in the canonical form of the objective function at the optimal solution to the linear programming problem gives the

change in the value of the objective function per unit increase in the value of the non-basic variable when all other non-basic variables are maintained at a zero level and the basic variables are allowed to change. (Hadley, 1962, p.91)

(2) The optimal solution to a linear programming problem must satisfy the constraints of the problem. Increasing the RHS of a constraint (expressed in standard form) which has a slack or excess variable which is not in the optimal basis, can be shown to be equivalent to inducing a change of +1 to the optimal value of the excess variable (if applicable) or a change of -1 to the optimal value of the slack variable (if applicable) which is in that equation. Using proposition (1), the change in the value of the excess or slack variable can be used in conjunction with canonical form of the objective function at the optimal solution to find the change in value of the objective function. This change in the value of the objective function is the shadow price of the constraint which had its RHS coefficient increased.

If a constraint has a slack or excess variable which is in the optimal basis, increasing the RHS of the constraint by 1 unit will change the value of the slack or excess variable but will not change the value of the objective function because the slack or excess variable has a coefficient of zero in the primal objective function. The shadow price of this constraint is zero.

We shall now give two examples of the method of computing shadow prices outlined in proposition (2) above.

Example 1

Choose x_1, x_2

to maximise: $z = 5x_1 + x_2$

subject to: $4x_1 + x_2 \leq 20$

$\frac{3}{4} x_1 + x_2 \leq 15$

$x_1 + x_2 \geq 10$

$x_1 \geq 0, x_2 \geq 0$

This problem may be written in standard form:

Choose x_1, x_2

to maximise: $z = 5x_1 + x_2$ (1)

subject to: $4x_1 + x_2 + x_3 = 20$ (2)

$\frac{3}{4} x_1 + x_2 + x_4 = 15$ (3)

$x_1 + x_2 - x_5 = 10$ (4)

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$

This problem may be solved by adding an artificial variable to (4) and using either the big M method or the two phase method. The optimal basic solution is $\left(x_1^* = 10/3, x_2^* = 20/3, x_3^* = 0, x_4^* = 35/6, x_5^* = 0\right)$ and the canonical form of the objective function corresponding to the optimal basic solution is:

$$z^* + 4/3 x_3^* + 1/3 x_5^* = 70/3 .$$

To obtain the shadow price of the first constraint, we note that:

$$4x_1^* + x_2^* + x_3^* = 20$$

Increasing the RHS by 1:

$$4x_1 + x_2 + x_3^* = 21 \,. \tag{5}$$

The basis variables x_1 and x_2 are not starred in (5) because they cannot both be equal to their optimal values when x_3 is held at its optimal value and the RHS of the constraint equation is increased by 1 unit.

Rearrange (5) to obtain:

$$4x_1 + x_2 + \left(x_3^* - 1\right) = 20. \tag{6}$$

Equation (6) shows that increasing the RHS of the first constraint by 1 unit is equivalent to keeping the RHS of the first constraint constant, changing x_3^* to (x_3^* - 1) and letting the basis variables change to satisfy the original constraint equation.

Let the value of the objective function when x_3 is changed from x_3^* to $\left(x_3^* - 1\right)$ be $\bar{z}_1$, then:

$\bar{z}_1 = 70/3 - 4/3\left(x_3^* - 1\right) - 1/3\,x_5^*$ and: $\Delta z_1 = \bar{z}_1 - z^* = -4/3\left(x_3^* - 1 - x_3^*\right) = 4/3$, is the shadow price of the first constraint.

The second constraint has a slack variable which is in the optimal basis $\left(x_4^* = 35/6\right)$ and hence a shadow price of zero, that is $\Delta z_2 = 0$.

To find the shadow price of the third constraint, note that:

$$x_1^* + x_2^* - x_5^* = 10 \,,$$

increasing the RHS by 1:

$$x_1 + x_2 - x_5^* = 11$$

This is equivalent to:

$$x_1 + x_2 - \left(x_5^* + 1\right) = 10 \,.$$

Let the value of the objective function when x_5 is changed from x_5^* to x_5^*+1 be $\bar{z}_3$, then

$\bar{z}_3 = 70/3 - 4/3\,x_3^* - 1/3(x_5^*+1)$ and: $\Delta z_3 = \bar{z}_3 - z^* = -1/3(x_5^* + 1 - x_5^*) = -1/3$,

is the shadow price of the third constraint.

The dual of this problem is:

choose w_1, w_2, w_3

to minimise $W = 20\,w_1 + 15\,w_2 + 10\,w_3$

subject to: $4\,w_1 + ¾\,w_2 + w_3 \geq 5$

$w_1 + w_2 + w_3 \geq 1$

$w_1 \geq 0,\ w_2 \geq 0,\ w_3 \leq 0$.

Let w_1^*, w_2^* and w_3^* solve the dual problem. The complementary slackness theorem gives:

$w_1^*(4x_1^* + x_2^* - 20) = 0,\ w_2^*(3/4\,x_1^* + x_2^* - 15) = 0,\ w_3^*(x_1^* + x_2^* - 10) = 0$,

$x_1^*(4w_1^* + 3/4\,w_2^* + w_3^* - 5) = 0,\ x_2^*(w_1^* + w_2^* + w_3^* - 1) = 0$.

Since $x_1^* = 10/3$ and $x_2^* = 20/3$, $w_2^* = 0$, $4w_1^* + w_3^* = 5$ and $w_1^* + w_3^* = 1$, so $w_3^* = 1 - w_1^*$ and $4w_1^* + 1 - w_1^* = 5$, so that $w_1^* = 4/3$ and $w_3^* = -1/3$.

Thus we find that: $w_1^* = \Delta z_1 = 4/3$, $w_2^* = \Delta z_2 = 0$ and $w_3^* = \Delta z_3 = -1/3$. The shadow price of each constraint is equal numerically to optimal value of the dual variable associated with that constraint.

Example 2

Choose x_1, x_2

to minimise: $Z = 5x_1 + x_2$

subject to: $x_1 \geq 1$

$3x_1 + x_2 \geq 4$

$x_1 \geq 0, x_2 \geq 0$

It is well known that if $z = -Z$, then maximising z is equivalent to minimising Z (Hadley, 1962, pp.131-132). This problem may be written as a maximisation problem in standard form:

Choose x_1, x_2

to maximise: $z = -5x_1 - x_2$ (7)

subject to: $x_1 - x_3 = 1$ (8)

$3x_1 + x_2 - x_5 = 4$ (9)

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$

The problem may be solved using the big M method or the two phase method after adding an artificial variable to the first constraint (the second constraint is already in canonical form for a basis which includes x_2).

The optimal basic feasible solution to this problem is $\left(x_1^* = 1, x_2^* = 1, x_3^* = 0, x_4^* = 0\right)$ and the canonical form of the objective function evaluated at the optimal basic solution is $z^* + 2x_3^* + x_4^* = -6$ so that the canonical form of the objective function corresponding to the objective function of the minimisation problem is (because $Z^* = -z^*$):

$$Z^* - 2x_3^* - x_4^* = 6 .$$

To obtain the shadow price of the first constraint, note that:

$$x_1^* - x_3^* = 1 .$$

Increasing the RHS of this constraint by 1 yields:

$$x_1 - x_3^* = 2 .$$

This is equivalent to:

$$x_1 - (x_3^* + 1) = 1.$$

Let the value of the objective function when x_3 is changed from x_3^* to $(x_3^* + 1)$ be $\overline{Z}_1$, then:

$\overline{Z}_1 = 2(x_3^* + 1) + x_4^* + 6$ and: $\Delta Z_1 = \overline{Z}_1 - Z_1^* = 2(x_3^* + 1 - x_3^*) = 2$ is the shadow price of the first constraint. Similarly, the shadow price of the second constraint is: $\Delta Z_2 = \overline{Z}_2 - Z^* = 1$.

The dual of this problem is:

choose w_1, w_2,

to minimise $\quad W = w_1 + 4\ w_2$

subject to:

$$w_1 + 3\ w_2 \leq 5$$

$$w_2 \leq 1$$

$$w_1 \geq 0,\ w_2 \geq 0$$

Let w_1^* and w_2^* represent the optimal values of the dual variables, then the complementary slackness theorem gives: $x_1^*(w_1^* + 3w_2^* - 5) = 0$, $x_2^*(w_2^* - 1) = 0$, $w_1^*(x_1^* - 1) = 0$ and $w_2^*(3x_1^* + x_2^* - 4) = 0$. Since $x_1^* = 1$ and $x_2^* = 1$, , $w_1^* + 3w_2^* = 5$ and $w_2^* = 1$ so that $w_1^* = 2$.

Thus we find that: $w_1^* = \Delta Z_1 = 2$ and $w_2^* = \Delta Z_2 = 1$. The shadow price of each constraint is numerically equal to the optimal value of the dual variable associated with each constraint.

We conclude this section by noting that the shadow prices of a non-standard linear programming problem containing equality constraints may be obtained by replacing each equality constraint with two weak inequalities whose intersection gives the equality constraint and proceeding as above.

3. Shadow Prices and the Canonical Form of the Objective Function at the Optimal Solution

Consider the following linear programming problem written in standard form:

Choose: x_1

To maximise: $z = c_1'x_1$

Subject to: $A_1x_1 + A_2x_2 = b$

$x_1 \geq 0, x_2 \geq 0$

where: x_1 is an $n_1\times1$ vector of choice variables with elements x_{1j}, $j=1,\ldots,n_1$;

x_2 is an $m\times1$ vector of slack and/or excess variables with elements x_{2j}, $j=1,\ldots,m$;

c_1 is an $n_1\times1$ vector of objective function coefficients with elements c_{1j}, $j=1,\ldots,n_1$;

A_1 is an $m\times n_1$ matrix of coefficients with columns a_{1j}, $j=1,\ldots,n_1$;

A_2 is an m×m matrix with column j being 1_j or -1_j, depending on whether x_{2j} is a slack variable or an excess variable, respectively, 1_j is an m×1 vector with element j being equal to 1 and the remaining elements being zero, and

b is an m×1 vector of coefficients.

To simplify notation, we shall now re-label the vectors and matrices defined above so that we can easily obtain expressions for the optimal basic solution of this problem. Let y_1, $y_2,\ldots,y_m$ be the variables in the optimal basis, collected in the m×1 vector y; let q_1, $q_2,\ldots,q_{n1}$ be the variables which are not in the optimal basis, collected in the n_1×1 vector q; let c_1, $c_2,\ldots,c_m$ be the objective function coefficients corresponding to the basis variables q_1, $q_2,\ldots,q_m$. Note that $c_j=0$ if y_j represents a slack or excess variable, otherwise c_j is equal to the element of c_1 corresponding to the basis variable q_j (which is a re-labelling of one of the variables in x_1).

We shall assume that one of the basis variables, say y_r is a slack or excess variable corresponding to constraint s, and that one of the non-basis variables, say q_k is a slack or excess variable corresponding to constraint k. Let D be the m×m matrix with columns being the columns of A_1 or A_2 corresponding to the variables in y and let the columns of D be d_j, j=1,...,m. Let E be the m×n_1 matrix with columns being the columns of A_1 or A_2 corresponding to the variables in q and let the columns of E be e_j, j=1,...,n_1.

The optimal value of the objective function may be written:

$$z^* = c'y^* = \sum_{j=1}^{m} c_j y_j^* \tag{10}$$

where y* is the optimal value of the vector of variables which are in the optimal basis.

The constraint equations can be written:

$$Dy + Eq = b \tag{11}$$

Let D^{-1} be the inverse of D, then (11) can be solved for y*:

$$y^* = D^{-1}b - D^{-1}\,Eq^*, \tag{12}$$

where q^* is the optimal value (zero for each element) of the vector of non-basis variables.

Now solving for z*:

$$z^* = c'y^* = c'(D^{-1}b - D^{-1}\,Eq^*)$$

$$= c'D^{-1}b - c'D^{-1}Eq^*$$

or:

$$z^* + c'D^{-1}Eq^* = c'D^{-1}b, \tag{13}$$

which is the canonical form of the objective function evaluated at the optimal basis. Recalling that e_j represents the column j of E, (13) may also be written:

$$z^* + \sum_{j=1}^{n} c'D^{-1}e_j.q_j^* = c'D^{-1}b, \tag{14}$$

where $c'D^{-1}e_j$ is the coefficient of the non-basis variable q_j in the canonical form of the objective function evaluated at the optimal basis. Since c' is a $1 \times m$ vector, D^{-1} is an $m \times m$ matrix and e_j ... an $m \times 1$ vector, $c'D^{-1}e_j$ is a scalar, and since (14) is the canonical form of the objective function of a maximisation problem, $c'D^{-1}e_j$, $j = 1,\dots,m$ is a set of non-negative scalars.

To find the shadow price of a binding $\leq$ or $\geq$ constraint, consider constraint k which has a slack or excess variable q_k which is not in the optimal basis. To increase the RHS of constraint k by 1 unit, add the vector 1_k to b to obtain from (11):

$$Dy + Eq = b + 1_k, \tag{15}$$

let $\overline{y}_k$ be the value of the basis vector when 1_k is added to the RHS of (11):

$$\overline{y}_k = D^{-1}b + D^{-1}1_k - D^{-1}Eq^*, \tag{16}$$

and let the value of the objective function corresponding to this value of the basis vector be $\overline{z}_k$:

$$\overline{z}_k = c'\overline{y}_k = c'D^{-1}b + c'D^{-1}1_k - c'D^{-1}Eq^*. \tag{17}$$

The shadow price of constraint k is, from (13) and (17),

$$\Delta z_k = \overline{z}_k - z^* = c'D^{-1}1_k. \tag{18}$$

Now the coefficient of q_k^* in the canonical form of the objective function is, from (14):

$$c'D^{-1}e_k,$$

where e_k is column k of E. Now if q_k is a slack variable, $e_k = 1_k$ and the canonical form of the objective function coefficient of q_k may be written:

$$c'D^{-1}1_k,$$

which is identical to the shadow price.

If q_k is an excess variable, then $e_k = -1_k$ and the canonical form of the objective function coefficient of q_k becomes:

$$-c'D^{-1}1_k ,$$

which must be non-negative. The shadow price is equal to $-(-c'D^{-1}1_k) = -c'D^{-1}e_k$, or minus the canonical form of the objective function coefficient of q_k, and is therefore non positive.

We shall now find the shadow price of a slack constraint or a binding constraint which has a slack or excess variable which is in the optimal basis (at zero value). If a constraint is slack, its corresponding excess or slack variable is positive and therefore in the optimal basis. We have already assumed that y_r represents a slack or excess basic variable associated with constraint s. Increasing the RHS of constraint s by one unit is equivalent to adding 1_s to the RHS of (11). The shadow price associated with this may be obtained by substituting 1_s for 1_k in (18) to obtain:

$$\Delta z_s = c'D^{-1}1_s . \tag{19}$$

Noting that 1_s is column r of D, if q_r is a slack variable corresponding to constraint s, we have from the definition of the matrix inverse:

$$D^{-1}1_s = 1_r,$$

so that $\Delta z_s = c'1_r = 0$, because the element of c corresponding to q_r is zero. So the shadow price of a non-binding ≤ constraint or a binding ≤ constraint which has an excess variable which is in the optimal basis is zero.

Now, if q_r is an excess variable corresponding to constraint s, column r of D is -1_s, and writing $-\Delta z_s = c'D^{-1} \cdot (-1_s)$, from the definition of the matrix inverse,

$$D^{-1}(-\mathbf{i}_s) = \mathbf{1}_r,$$

so that $-\Delta z_s = c' \cdot \mathbf{1}_r = 0$, since the element of c corresponding to q_r is zero. Thus, the shadow price of a non binding ≥ constraint or a binding ≥ constraint which has an excess variable which is in the optimal basis is also zero.

Using the convention that a slack or excess variable which is in the optimal basis has a coefficient of zero in the canonical form of the objective function at the optimal basis, these results may be summarised as follows. In a maximisation problem, the shadow price of:

(i) A non binding ≥ or ≤ constraint is zero.

(ii) A binding ≤ constraint is the coefficient of the slack variable corresponding to the ≤ constraint in the canonical form of the objective function (evaluated at the optimal basis) and is therefore non-negative.

(iii) A binding ≥ constraint is minus the coefficient of the slack variable corresponding to the ≥ constraint in the canonical form of the objective function (evaluated at the optimal basis) and is therefore non-positive.

Similar results hold for a minimisation problem except that the sign of the shadow price associated with a binding ≤ constraint is non-positive and the sign of the shadow price associated with a binding ≥ constraint is non-negative. This follows because the coefficients of the non-basis variables in the canonical form of the objective function (evaluated at the optimal basis) are non-positive.

These results are identical to the results obtained by Winston (1995, pp.296-297) on how to obtain the optimal values of the dual variables from the coefficients of slack and excess variables in the canonical form of the objective function, evaluated at the optimal solution of non-standard primal maximisation and minimisation problems.

We shall now verify that the method for obtaining shadow prices outlined in Section 2 is correct. We begin with a constraint which has a slack or surplus variable which is not in the optimal basis. Note that from (14) the canonical form of the objective function evaluated at the optimal basis may be written:

$$z^* = cD^{-1}b - \left(cD^{-1}e_1q_1^* + cD^{-1}e_2q_2^* + \ldots + cD^{-1}e_iq_i^* + \ldots + cD^{-1}e_{n_1}q_{n_1}^*\right). \tag{20}$$

If the value of q_i^* is changed to q_i', then the value of the objective function evaluated at the optimal basis may be calculated as:

$$\bar{z}_i = cD^{-1}b - \left(cD^{-1}e_1q_1^* + cD^{-1}e_2q_2^* + \ldots + cD^{-1}e_iq_i' + \ldots + cD^{-1}e_{n_1}q_{n_1}^*\right), \tag{21}$$

and

$$\Delta z_i = \bar{z}_i - z^* = -cD^{-1}e_i\left(q_i' - q_i^*\right) = -cD^{-1}e_i\Delta q_i, \tag{22}$$

where $\Delta q_i = q_i' - q_i^*$.

Now changing the element of b corresponding to constraint i, to $b_i + 1$, will be shown to be equivalent to changing q_i^* holding b_i constant. Writing the constraint equations (11) using the notation that e_j is column j of E:

$$Dy^{*}+e_1q_1^{*}+e_2q_2^{*}+...+e_iq_i^{*}+...+e_{n_1}q_{n_1}^{*}=b\ , \tag{23}$$

and changing b_i to b_i+1 yields:

$$Dy+e_1q_1^{*}+e_2q_2^{*}+...+e_iq_i^{*}+...+e_{n_1}q_{n_1}^{*}=b+1_i\ . \tag{24}$$

Now $e_i = 1_i$ if q_i is a slack variable, and $e_i = -1_i$ if q_i is an excess variable.

Rearranging (24):

$$Dy+e_1q_1^{*}+e_2q_2^{*}+...+e_iq_i^{*}-1_i+...+e_{n_1}q_{n_1}^{*}=b \tag{25}$$

or

$$Dy+e_1q_1^{*}+e_2q_2^{*}+...+e_iq_i'+...+e_{n_1}q_{n_1}^{*}=b\ , \tag{26}$$

where $q_i' = q_i^{*}-1$ if q_i is a slack variable and $q_i' = q_i^{*}+1$ if q_i is an excess variable.

Now the shadow price of constraint i may be obtained from (22). When q_i is a slack variable:

$$\Delta z_i = -c'D^{-1}e_i\Delta q_i = -c'D^{-1}e_i\left(q_i^{*}-1-q_i^{*}\right) = c'D^{-1}e_i$$

and when q_i is an excess variable:

$$\Delta z_i = -c'D^{-1}e_i\Delta q_i = -c'D^{-1}e_i\left(q_i^{*}+1-q_i^{*}\right) = -c'D^{-1}e_i\ .$$

These results agree with those obtained above.

If a constraint has a slack or excess variable which is in the optimal basis, then the constraint has (as shown above) a shadow price of zero.

4. Concluding Comments

The method for computing shadow prices presented in section 2 was taught by the author in conjunction with the simplex method and related topics for 5 years in a second year introductory operations research course. Students taking the course were completing degrees with majors in economics, econometrics, marketing, finance, accounting, applied science and textile science.

The minimum prerequisite for the course was a one semester course in quantitative methods covering basic calculus, linear algebra, mathematics of finance and some material on the graphical solution to linear programming problems. Students were generally happy to accept propositions (1) and (2) as justifying the method outlined in section 2 for computing shadow prices. While the justification of the method presented in section 3 was made available to students (on request), few asked for the material.

Endnote

[1] Wagner's method may briefly be described as follows. If the right hand side of a constraint equation (expressed in standard form) is altered by adding a scalar λ and the linear programming problem is solved for the new values of the variables which are in the optimal basic solution of the original problem, λ appears on the right hand side of each equation in which the slack or surplus variable of the modified constraint appears. If the modified constraint has a slack variable, then the coefficient of λ is the same as the coefficient of the slack variable in that equation. If the modified constraint has an excess variable, the coefficient of λ is minus the coefficient of the excess variable in that equation. The shadow price of the constraint is the coefficient of λ in the canonical form of the objective function evaluated at the new values of the basis variables. Thus, if a constraint is not binding, λ does not appear in the canonical form of the objective function at the new solution and the shadow price is zero. If a constraint is binding and the slack or excess variable of the modified constraint is not a basis variable, λ appears in the canonical form of the objective function and the shadow price is non-zero, being positive if the modified constraint has a slack variable or negative if the modified constraint has an excess variable. The range for λ which would not alter the variables in the optimal basis after the right hand side of a constraint is modified, is the range for λ which maintains the non-negativity of the new value of the slack or excess variable of the modified constraint. It should be noted that the coefficients of all variables appearing in the canonical form of the problem for the optimal basic solution remain unchanged in the canonical form of the problem for the solution to the modified problem. Thus, shadow prices may be obtained from the coefficients of slack and/or excess variables which appear in the canonical form of the objective function corresponding to the optimal solution of the original problem. The method for calculating shadow prices presented in section 2 is simpler than Wagner's method in that the range of the right hand side parameter of a constraint equation for which the shadow price of the constraint gives a marginal valuation of the resource represented by the constraint is not addressed.

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