RSP 15(1): 63-74.

OPTIMUM LOCATION AND THE THEORY OF PRODUCTION: AN EXTENSION

Abbas Mirakhor and A. Khalili*

In a paper in 1974, Khalili, et al [4], determined the condition for cost minimizing optimum production location for the case of Weber's locational triangle and the effect of changes in the level of output on the optimum location under conditions of perfect competition.

The purpose of this paper is (1) to derive profit maximizing conditions for the case of Weber's locational triangle when the firm is operating in imperfect input and output markets and (2) to determine the effects of income, output transport rate, elasticities of input supply curves on the optimum production location using comparative static analysis.

The "location problem" of the firm can be posed as follows: Assume a one-plant firm which is buying its inputs and selling its output in imperfectly competitive markets and which is interested in finding the optimum production location, uses two transportable inputs, M_1 and M_2 and supplies its single final product to a consumption center M_3. The triangle in Figure 1 depicts this problem.

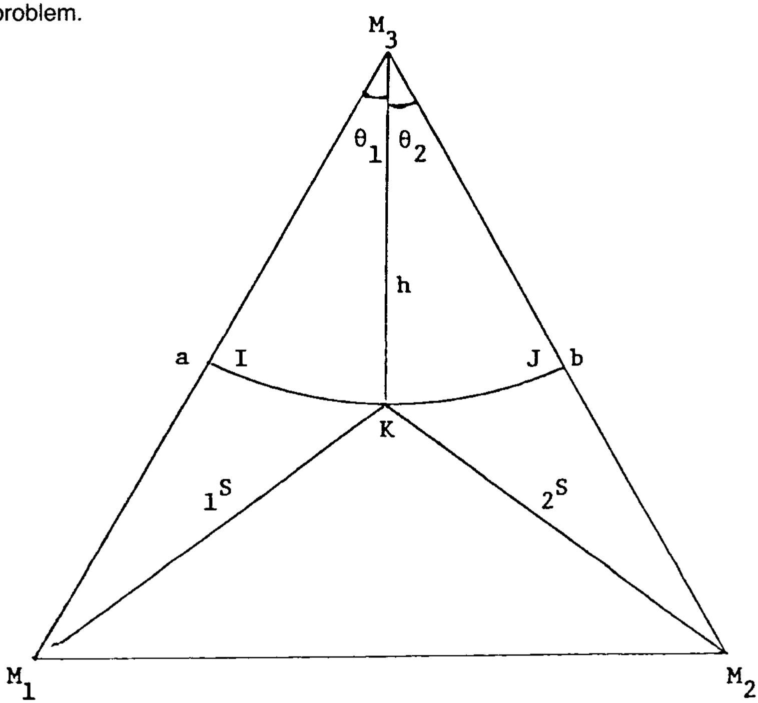

Figure 1.

* Florida Institute of Technology.

Mathematically the problem is to Maximize

$$\text{Maximize}\ \pi = R - C = P_oF(M_1,M_2) - ({}_1P + r_1\,{}_1S)M_1 - ({}_2P + r_2\,{}_2S)M_2 - r_ohF(M_1,M_2)$$

where, $P_o = g(F,Y)$ is the price of the product, ${}_iP = {}_iP(M_i), i = 1,2$ is the price of input i and $F(M_1,M_2)$ is the production function. The distance from the production location to the sources M_1 and M_2 and to the market are ${}_1S$, ${}_2S$ and h respectively. The distances ${}_1S$ and ${}_2S$ could be expressed as

$${}_1S = \sqrt{a^2 + h^2 - 2ah\text{Cos}\theta}$$

$${}_2S = \sqrt{b^2 + h^2 - 2bh\text{Cos}\,(\theta - \theta_1)}$$

where a, b, and $\theta = \theta_1 + \theta_2$ are given. θ_1 and h are the polar coordinates of the production location point. r_o is the transport rate of the final product to M_3, r_1 and r_2 are transport rates of M_1 and M_2 respectively. The first order conditions are:

$$(1)\quad \frac{\partial\pi}{\partial M_1} = g_1F_1F + F_1g - {}_1P_1M_1 - ({}_1P + r_1\,{}_1S) - r_ohF_1 = \{g\,(1 + \frac{F}{g}\,g_1) - r_oh\}\,F_1 - {}_1P'(1 + \frac{M_1}{{}_1P'}\,{}_1P_1') = 0$$

$$(2)\quad \frac{\partial\pi}{\partial M_2} = g_1F_2F + F_2g - {}_2P_2M_1 - ({}_2P + r_2\,{}_2S) - r_ohF_2 = \{g\,(1 + \frac{F}{g}\,g_1) - r_oh\}\,F_2 - {}_2P'(1 + \frac{M_2}{{}_2P'}\,{}_2P_2') = 0$$

$$(3)\quad \frac{\partial\pi}{\partial\theta_1} = -r_1\,S_{\theta_1}\,M_1 - r_2\,{}_2S_{\theta_1}M_2 = 0$$

$$(4)\quad \frac{\partial\pi}{\partial h} = -r_1\,{}_1S_hM_1 - r_2\,{}_2S_hM_2 - r_oF = 0$$

where ${}_iP_j, i = 1,2, j = 1,2, F_1, F_2, {}_1S_{\theta_1}, {}_2S_{\theta_1}, {}_1S_h, {}_2S_h$ are first partial derivatives, and $P_1' = P_1 + r_1\,{}_1S;\ P_2' = P_2 + r_2\,{}_2S$.

Conditions (1) and (2) state that: net marginal revenue product (net of per unit output transport cost) for each of the two inputs must equal their respective marginal expense.

The total differential of the first order conditions are:

$$(5)\quad (F_{11}B_1 + F_12\,\frac{\partial B_1}{\partial F} - {}_1P_{11}M_1 - 2\,{}_1P_1)dM_1 + (F_{12}B + F_1F_2\,\frac{\partial B_1}{\partial F})dM_2 - r_1\,{}_1S\theta_1d\theta_1 - (r_1\,{}_1S_h + r_oF_1)dh = -B_2F_1dY + hF_1dr_o + {}_1Sdr_1 - 2g_1F_1dF$$

(6) $(F_{12}B_1 + F_1F_2\frac{\partial B_1}{\partial F}dM_1 + (F_{22}B_1 + F_22\frac{\partial B_1}{\partial F} - {}_2P_{22}M_2 - 2_2P_2)dM_2$
$- r_2\,{}_2S_{\theta_1}d\theta_1 - (r_2\,{}_2S_h + r_oF_2)dh = -B_2F_2dY + hF_2dr_o + {}_2Sdr_2 - 2g_1F_2dF$

(7) $-r_1\,{}_1S_{\theta_1}dM_1 - r_2\,{}_2S_{\theta_1}dM_2 - C_{\theta_1\theta_1}d\theta_1 - C_{\theta_1h}dh = {}_1S_{\theta_1}M_1dr_1 + {}_2S_{\theta_1}M_2dr_2$

(8) $-(r_{1}\,{}_1S_h + r_oF_1)dM_1 - (r_2\,{}_2S_h + r_oF_2)dM_2 - C_{\theta_1h}d\theta_1 - C_{hh}dh =$
$Fdr_o + {}_1S_hM_1dr_1 + {}_2S_hM_2dr_2$

Let: $A_{11} = F_{11}B_1 + F_12\frac{\partial B_1}{\partial F} - {}_1P_{11}M_1 - 2_1P_1$

$A_{12} = F_{12}B_1 + F_1F2\frac{\partial B_1}{\partial F}$

$A_{22} = F_{22}B_1 + F_22\frac{\partial B_1}{\partial F} - {}_2P_{22}M_2 - 2_2P_2$

where $B_1 = g + gF - r_oh$, $B_2 = g_{12}F + g_2$, $B_3 = 2g_1 + Fg_{11}$, $g < 0$ and ${}_1P_1, {}_2P_2$ are positive.

Then equations (5) - (8) become:

(9) $A_{11}dM_1 + A_{12}dM_2 - r_1\,{}_1S_{\theta_1}d\theta_1 - (r_1\,{}_1S_h + r_oF_1)dh =$
$-B_2F_1dY + hF_1dr_o + {}_1Sdr_1 - 2g_1F_1dF$

(10) $A_{12}dM_1 + A_{22}dM_2 - r_2\,{}_2S_{\theta_1}d\theta_1 - (r_2\,{}_2S_h + r_oF_2)dh =$
$-B_2F_2dY + hF_2dr_o + {}_2Sdr_2 - 2g_1F_2dF$

(11) $-r_1\,{}_1S_{\theta_1}dM_1 - r_2\,{}_2S_{\theta_1}dM_2 - C_{\theta_1\theta_1}d\theta_1 - C_{\theta_1h}dh =$
${}_1S_{\theta_1}M_1dr_1 + {}_2S_{\theta_1}M_2dr_2$

(12) $-(r_1\,{}_1S_h + r_oF_1)dM_1 - (r_2\,{}_2S_h + r_oF_2)dM_2 - C_{\theta_1h}d\theta_1 - C_{hh}dh =$
$Fdr_o + {}_1S_hM_1dr_1 + {}_2S_hM_2dr_2$

where:

$C_{\theta_1\theta_1} = r_1M_1\,{}_1S_{\theta_1\theta_1} + r_2M_2\,{}_2S_{\theta_1\theta_1}$
$C_{\theta_1h} = r_1M_1\,{}_1S_{\theta_1h} + r_2M_2\,{}_2S_{\theta_1h}$
$C_{hh} = r_1M_1\,{}_1S_{hh} + r_2M_2\,{}_2S_{hh}$

{see equations (15) - (17) of [4]}.

The second order condition requires that the principle minors of the relevant Hessian determinant alternate in sign.

Proposition 1: Assuming h is a positive constant, θ_1 is a variable ($\theta_1 < \theta$) and the firm's marginal revenue is an increasing function of income, then the firm's production location is independent of the level of income, and output transport rate, if and only if the expansion path is linear.

Proof: Using the system of equations (5)-(7) and holding h constant we have:

$$\frac{\partial\theta_1}{\partial Y} = \frac{1}{D^*}\begin{vmatrix} A_{11} & A_{12} & -B_2F_1 \\ A_{12} & A_{22} & -B_2F_2 \\ -r_1\,{}_1S_{\theta_1} & -r_2\,{}_2S_{\theta_1} & 0 \end{vmatrix}$$

$$= \frac{-B_2}{D^*M_1M_2}\begin{vmatrix} M_1A_{11} + M_2A_{12} & M_2A_{12} & F_1 \\ M_1A_{12} + M_2A_{22} & M_2A_{22} & F_2 \\ 0 & -r_2M_2\,{}_2S_{\theta_1} & 0 \end{vmatrix}$$

where D^* is the relevant bordered Hessian (see [4]).

Expanding and simplifying this determinant, we obtain:

$$\frac{\partial\theta_1}{\partial Y} = \frac{-r_2\,{}_2S_{\theta_1}B_2}{D^*M_1}\{F_2(M_1A_{11} + M_2A_{12}) - F_1(M_1A_{12} + M_2A_{22})\}$$

$$= \frac{-r_2\,{}_2S_{\theta_1}B_2}{D^*M_1}N''$$

Since $r_2\,{}_2S_{\theta_1} < 0$, $D^* < 0$, and $B_2 > 0$, $\frac{\partial\theta_1}{\partial Y} = 0$ if and only if N'' is zero. However, $N'' = 0$ if and only if the expansion path is linear through origin (see Appendix A).

Similarly

$$\frac{\partial\theta_1}{\partial r_o} = \frac{h}{D^*M_1M_2}\begin{vmatrix} M_1A_{12} + M_2A_{12} & M_2A_{12} & F_1 \\ M_1A_{12} + M_2A_{22} & M_2A_{22} & F_2 \\ 0 & -r_2M_2\,{}_2S_{\theta_1} & 0 \end{vmatrix} = 0$$

if and only if the expansion path is linear.

Proposition 2: If h is constant and greater than zero, $\theta_1 < \theta$, and the firm's marginal revenue is an increasing function of income the firm's optimum location would swing along the arc IJ (see Figure 1) towards $M_1(M_2)$ if and only if $M_1(M_2)$ increases relative to $M_2(M_1)$ along the expansion path as the level of income increases.

Proof: From the last proposition, one determines the sign of $\frac{\partial\theta_1}{\partial Y}$ to be opposite that of N''. Therefore, when $N'' > 0$ (<0), $\frac{\partial\theta_1}{\partial Y} < 0$ (> 0) and the firm's optimal location would move towards $M_1(M_2)$. But, $N'' > 0$ (< 0) if and only if $M_1(M_2)$ is used more relative to $M_1(M_2)$ along the expansion path (see Appendix B) as the level of income increases.

Proposition 3: Assuming h is constant and greater than zero, $\theta_1 < \theta$, the firm's optimum production location would swing along the arc IJ toward $M_1(M_2)$

if and only if $M_1(M_2)$ increases relative to $M_2(M_1)$ along the expansion path, as the output transport rate decreases.

Proof: Using Cramer's rule, from the system of equations (9) - (11), we find:

$$\frac{\partial\theta_1}{\partial r_o} = \frac{1}{D^*}\begin{vmatrix} A_{11} & A_{12} & hF_1 \\ A_{12} & A_{22} & hF_2 \\ -r_1\,{}_1S_{\theta_1} & -r_2\,{}_2S_{\theta_1} & 0 \end{vmatrix}$$

$$= \frac{h}{D^*M_1M_2}\begin{vmatrix} M_1A_{12} + M_2A_{12} & M_2A_{12} & F_1 \\ M_1A_{12} + M_2A_{22} & M_2A_{11} & F_2 \\ 0 & -r_2M_2\,{}_2S_{\theta_1} & 0 \end{vmatrix}$$

$$= \frac{r_2h_2S_{\theta_1}}{D^*M_1}\{F_2(M_1A_{11} + M_2A_{12}) - F_1(M_1A_{12} + M_2A_{22})\} = \frac{r_2h_2S_{\theta_1}}{D^*M_1}N''$$

Since $r_2\,{}_2S_{\theta_1} < 0$ and $D^* < 0$, the sign of $\frac{\partial\theta_1}{\partial r_o}$ is the same as that of N''.

Therefore, when $N'' > 0$ (< 0), $\frac{\partial\theta_1}{\partial r_o} > 0$ (< 0) and the firm's optimal location would move towards $M_1(M_2)$ as r_o decreases. But $N'' > 0$ (< 0) if and only if $M_1(M_2)$ is used more relative to $M_2(M_1)$ along the expansion path (see Appendix B).

Proposition 4: If both θ_1 and h are variables, and the firm's marginal revenue is an increasing function of income, then the production function is independent of the level of income if the production function is linearly homogeneous and the marginal expense elasticities of the input supply curves are equal.

Proof: Using the system of equations (5) - (8), we obtain:

$$\frac{\partial h}{\partial Y} = \frac{1}{D}\begin{vmatrix} A_{11} & A_{12} & -r_1\,{}_1S_{\theta_1} & -B_2F_1 \\ A_{12} & A_{22} & -r_2\,{}_2S_{\theta_1} & -B_2F_2 \\ -r_1\,{}_1S_{\theta_1} & -r_2\,{}_2S_{\theta_1} & -C_{\theta_1\theta_1} & 0 \\ -(r_{11}S_h + r_oF_1) & -(r_2\,{}_2S_h + r_oF_2) & -C_{\theta,h} & 0 \end{vmatrix}$$

$$\frac{\partial h}{\partial Y} = \frac{1}{DM_1^2M_2^2}\begin{vmatrix} M_1^2A_{11} & M_1M_2A_{12} & -r_1M_1\,{}_1S_{\theta_1} & -B_2M_1F_1 \\ M_1M_2A_{12} & M_2^2A_{22} & -r_2M_2\,{}_2S_{\theta_1} & -B_2M_2F_2 \\ -r_1M_1\,{}_1S_{\theta_1} & -r_2M_2\,{}_2S_{\theta_1} & -C_{\theta_1\theta_1} & 0 \\ -(r_1M_1\,{}_1S_h + r_oM_1F_1) & -(r_2M_2\,{}_2S_h + r_oM_2F_2) & -C_{\theta,h} & 0 \end{vmatrix}$$

Let: $E_{11} = A_{11}M_1^2 + 2A_{12}M_1M_2 + M_2^2A_{22}$

$E_{12} = M_1M_2A_{12} + M_2^2A_{22}$

$E_{22} = M_2^2A_{22}$

Then:

$$\frac{\partial h}{\partial Y} = \frac{1}{DM_1^2\,M_2^2}\begin{vmatrix} E_{11} & E_{12} & 0 & -B_2nF \\ E_{12} & E_{22} & -r_2M_{2\,2}S_{\theta_1} & 0 \\ 0 & -r_2M_{2\,2}S_{\theta_1} & -C_{\theta_1\theta_1} & 0 \\ -r_oF(n-1) & -(r_2M_{2\,2}S_h + r_oM_2F_2) & -C_{\theta_1h} & 0 \end{vmatrix}$$

Letting n = 1, multiplying the first row by $\frac{-M_2F_2}{F}$ and adding it to the second we get:

$$\frac{\partial h}{\partial Y} = \frac{1}{DM_1^2M_2^2}\begin{vmatrix} E_{11} & E_{12} & 0 & B_2F \\ E_{12} - \frac{M_2F_2}{F}E_{11} & E_{12} - \frac{M_2F_2}{F}E_{12} & -r_2M_{2\,2}S_{\theta_1} & 0 \\ 0 & -r_2M_{2\,2}S_{\theta_1} & -C_{\theta_1\theta_1} & 0 \\ 0 & -(r_2M_{2\,2}S_h + r_oM_2F_2) & -C_{\theta_1h} & 0 \end{vmatrix}$$

$$\frac{\partial h}{\partial Y} = \frac{-2B_2(M_1F_{2\,1}P_1 - M_2F_{1\,2}P_2}{DM_1M_2}\{r_2M_{2\,2}S_{\theta_1}C_{\theta_1h} - (r_2M_{2\,2}S_h + r_oM_2F_2)C_{\theta_1\theta_1}\}$$

However, $E_{12}' = FE_{12} - M_2F_2E_{11} = M_1M_2(-M_2F_1B_2' + M_1F_2B_1')$ for n = 1, and if $\varepsilon_2 = \frac{M_2/dM_2}{C_2'/dC_2'} = \varepsilon_1 = \frac{M_1/dM_1}{C_1'/dC_1'}$, then $E_{12}' = 0$ (see Appendix C).

Therefore, $\frac{\partial h}{\partial Y} = 0$ for n = 1, and $\varepsilon_1 = \varepsilon_2$.

Similarly

$$\frac{\partial \theta_1}{\partial Y} = \frac{-B(M_1F_{2\,1}P_1 - M_2F_{1\,2}P_2)}{DM_1^2\,M_2}\{r_2M_{2\,2}S_{\theta_1}C_{hh} + (r_2M_{2\,2}S_h + r_oM_2F_2)C_{\theta_1h}'\} = 0$$

for n = 1, and $\varepsilon_1 = \varepsilon_2$

APPENDIX A

(1) $N'' = F_2(M_1A_{11} + M_2A_{12}) - F_1(M_1A_{12} + M_2A_{22})$

$M_1A_{11} + M_2A_{12} = Q_4B_1 + Q_3F_1B_3 - {}_1P_{11}M_1^2 - 2M_{11}P_1$

$M_1A_{12} + M_2A_{22} = Q_2B_1 + Q_3F_2B_3 - {}_2P_{22}M_2^2 - 2M_{22}P_2$

$F_2(M_1A_{11} + M_2A_{12}) - F_1(M_1A_{12} + M_2A_{22}) = F_2Q_4B_1 + Q_3F_1F_2B_3 -$

$F_2({}_1P_{11}M_1^2 + 2M_{11}P_1) - F_1Q_2B_1 - Q_3F_1F_2B_3 + 2M_{22}P_2) = 0$

$= F_2B_1Q_4 - F_1B_1Q_2 - F_2M_1({}_1P_{11}M_1 + 2_1P_1)$

$+ F_1M_2({}_2P_{22}M_2 + 2_2P_2) = 0$

where $Q_2 = M_1F_{12} + M_2F_{22}$, $Q_3 = M_1F_1 + M_2F_2$, $Q_4 = M_1F_{11} + M_2F_{12}$, $B_1 = g + Fg_1 - r_oh$, and $B_3 = Fg_{11} + 2g_1$.

From the first order conditions, we have:

(2) $F_1B_1 = {}_1P' \left(1 + \frac{M_1}{{}_1P'} {}_1P_1'\right) = C_1'$

(3) $F_2B_1 = {}_2P' \left(1 + \frac{M_2}{{}_2P'} {}_2P_2'\right) = C_2$

Also, $\frac{dC_1'}{dM_1} = {}_1P_{11}M_1 + 2_1P_1$ and $\frac{dC_2'}{dM_2} = {}_2P_{22}M_2 + 2_2P_2$

Therefore, equation (1) becomes

$$C_2'(M_1F_{11} + M_2F_{12}) - C_1'(M_1F_{12} + M_2F_{22}) - F_2M_1 \frac{dC_1'}{dM_1} + F_1M_2\frac{dC_2'}{dM_2} = 0$$

or

$$M_1(C_2'F_{11} - C_1'F_{12} - F_2 \frac{dC_1'}{dM_1}) - M_2(C_1'F_{22} - C_1'F_{22} - F_1\frac{dC_2'}{dM_2}) = 0$$

or

$$(4) \quad \frac{M_1}{M_2} = \frac{C_1'F_{22} - C_2'F_{12} - F \dfrac{dC_2'}{dM_2}}{C_2'F_{11} - C_1'F_{12} - F \dfrac{dC_1'}{dM_1}}$$

From the first order conditions, we have:

(5) $\frac{F_1}{F_2} = \frac{C_1'}{C_2'}$ or $H(M_{11}M_2) = \frac{C_2'F_1}{C_1'F_2} = 1$

Differentiating H partially with respect to M_1 and M_2, we get:

$$(6)\quad \frac{\partial H}{\partial M_1} = \frac{C_2'F_{11} \cdot C_1'F_2 - (C_1'F_{21} + F_2 \frac{dC_1'}{dM_1})C_2'F_1}{(C_1'F_2)^2}$$

$$(7)\quad \frac{\partial H}{\partial M_2} = \frac{(C_2'F_{12} + F_1 \frac{dC_2'}{dM_2})C_1'F_2 - C_1'F_{22}C_2'F_1}{(C_1'F_2)^2}$$

From (6) and (7)

$$(8)\quad \frac{\frac{\partial H}{\partial M_2}}{\frac{\partial H}{\partial M_2}} = \frac{(C_2'F_{12} + F_1 \frac{dC_2'}{dM_2} C_1'F_2 - C_1'F_{22}C_2'F_1}{C_2'F_{11} - C_1'F_2 - (C_1'F_{21} + F_2 \frac{dC_1'}{dM_1})C_2'F_1}$$

From the implicit function theorem, we have:

$$(9)\quad \frac{\frac{\partial H}{\partial M_2}}{\frac{\partial H}{\partial M_1}} = -\frac{dM_1}{dM_2}$$

Therefore,

$$(10)\quad \frac{(C_2'F_{12} + F_1 \frac{dC_2'}{dM_2})C_1'F_2 - C_1'F_{22}C_2'F_1}{C_2'F_{11} - C_1'F_2 - (C_1'F_{21} + F_2 \frac{dC_1'}{dM_1} C_2'F_1} = -\frac{dM_1}{dM_2}$$

From the first order conditions, we have:

$C_2'F_1 = C_1'F_2$

Therefore, equation (10) become:

$$(11)\quad \frac{(C_2'F_{12} + F_1 \frac{dC_2'}{dM_2})C_1'F_{21} - C_1'F_{22}C_1'F_2}{C_2'F_{11} \cdot C_1'F_1 - (C_1'F_{21} + F_2 \frac{dC_1'}{dM_1})C_1'F_2} = -\frac{dM_1}{dM_2}$$

Simplifying (11), we get:

$$(12)\quad \frac{C_1'F_{22} = C_2'F_{12} - F_1\dfrac{dC_2'}{dM_2}}{C_2'F_{11} - C_1'F_{21} - F_2\dfrac{dC_1}{dM_1}}$$

From (4) and (12) we get:

$$(13)\ \frac{dM_1}{dM_2} = \frac{M_1}{M_2}$$

Equation (13) is that of a linear expansion path.

APPENDIX B

From Appendix A, we have

$$(14)\quad N'' = M_1(C_2'F_{11} - C_1'F_{12} - F_2\frac{dC_1'}{dM_1}) - M_2(C_1'F_{22} - C_1'F_{12} - F_1\frac{dC_2'}{dM_2})$$

For a firm (buying and selling in imperfect markets) using two inputs, only one of the inputs could be inferior, i.e., when M_2 is inferior, $C_2'F_{11} - C_1'F_{12} - F_2 \frac{dC_1'}{dM_1} > 0$, M_1 is superior and $C_1'F_{22} - C_2'F_{12} - F_1\frac{dC_2'}{dM_2} < 0$. Therefore, $N'' > 0$ if and only if

$$(15)\quad \frac{M_2}{M_1} > \frac{C_2'F_{11} - C_1'F_{12} - F_2dC_1'/dM_1}{C_1'F_{22} - C_2'F_{12} - F_1dC_2'/dM_2}$$

From equation (12) of Appendix A, it follows that

$$(16)\quad \frac{M_2}{M_1} > \frac{dM_2}{dM_1}$$

Equation (16) implies that M_1 increases relative to M_2 along the expansion path. Similarly, in the event M_2 is superior and M_1 is inferior, $N'' < 0$ if and only if

$$(17)\quad \frac{M_2}{M_1} < \frac{dM_2}{dM_1}$$

which implies that the firm's location will move towards M_2 if and only if M_2 is used more intensively along the expansion path. When both M_1 and M_2 are superior, $N' > 0$ if and only if the condition (16) holds. The converse is true when condition (17) holds.

(18) $E_{11} = M_1^2 A_{11} + 2M_1M_2A_{12} + M_2^2 A_{22}$

$M_1^2 A_{11} = M_1^2 gF_{11} + 2M_1^2 g_1F_1^2 + M_1^2 g_{11}F_1^2 F + M_1^2 F_{11}Fg_1 - r_oM_1^2 hF_{11}$

$2M_1M_2A_{12} = 2M_1M_2gF_{12} + 4M_1M_2g_1F_1F_2 + 2M_1M_2F_1F_2Fg_{11} +$
$\quad 2M_1M_2g_1FF_{12} - 2r_oM_1M_2hF_{12}$

$M_2^2 A_{22} = M_2^2 gF_{22} + 2M_2^2 g_1F_2^2 + M_2^2 g_{11}FF_2^2 + M_2^2 Fg_1F_{22} - r_oM_2^2 hF_{22}$

$E_{11} = g(M_1^2 F_{11} + 2M_1M_2F_{12} + M_2^2 F_{22}) + 2g_1(M_1^2 F_1^2 + 2M_1M_2F_1F_2 +$
$\quad M_2^2 F_2^2 + g_{11}F(M_1^2 F_1^2 + 2M_1M_2F_1F_2 + M_2^2 F_2^2) + g_1F(M_1^2 F_{11} +$
$\quad 2M_1M_2F_{12} + M_2^2 F_2^2) - r_oh(M_1^2 F_{11} + 2M_1M_2F_{12} + M_2^2 F_{22})$

$E_{11} = n(n-1)Fg + 2g_1n^2F^2 + g_{11}n^2F^3 + n(n-1)g_1F^2 - n(n-1)Fr_oh$

$\quad = n(n-1)Fg(1 + \frac{F}{P_o}g_1) + n^2F^2(2g_1 + g_{11}F) - n(n-1)Fr_oh$

$\quad = n(n-1)F\{g + g_1F - r_oh\} + n^2F^2\{2g_1 + Fg_{11}\}$

For $n = 1$, $E_{11} = F^2(2g_1 + Fg_{11}) = F^2B_3$

(19) $E_{12} = M_1M_2A_{12} + M_2^2 A_{22}$

$M_1M_2A_{12} = M_1M_2gF_{12} + 2M_1M_2g_1F_1F_2 + M_1M_2F_1F_2Fg_{11} + M_1M_2g_1FF_{12}$
$\quad - r_oM_1M_2hF_{12}$

$M_2^2 A_{22} = M_2^2 gF_{22} + 2M_2^2 g_1F_2^2 + M_2^2 g_{11}FF_2^2 + M_2^2 Fg_1F_{22} - r_oM_2^2 hF_{22}$

$E_{12} - (n-1)M_2F_2g + 2nM_2F_2Fg_1 + nM_2F_2F^2g_{11} + (n-1)M_2F_2Fg_1 -$
$\quad (n-1)r_oM_2hF_2$

For $n = 1$, $E_{12} = 2M_2F_2Fg_1 + M_2F_2F^2g_{11} = M_2F_2FB_3$

(20) $E_{12}F - E_{11}M_2F_2 = 2M_2F_2F^2g_1 + M_2F_2F^3g_{11} - 2M_2F_2F^2g_1 -$
$\quad M_2F_2F^3g_{11} = 0$

Since the input markets are imperfect, then for $n = 1$

(21) $E_{12}' = E_{12}F - E_{11}M_2F_2 = M_1M_2(-M_2F_1B_2' - M_1F_2B_1') = 0$

$$\frac{M_2}{M_1} = \left(\frac{F_2}{F_1}\right)\left(\frac{B_1'}{B_2'}\right) = \left(\frac{F_2}{F_1}\right)\left(\frac{dC_1'/dM_1}{dC_2'/dM_2}\right)$$

where $C_1' = {}_1P'(1 + \frac{M_1}{{}_1P'}\,{}_1P_1')$

$$C_2' = {}_2P'(1 + \frac{M_2}{{}_2P'}\,{}_2P_2')$$

But from the first order conditions, we have

$C_1' = F_1B_1$

$C_2' = F_2B_1$

or

$$\frac{C_2'}{C_1'} = \frac{F_2}{F_1}$$

Therefore, relation (21) becomes

$$\text{(22)} \quad \frac{M_2}{M_1} = \frac{C_2'}{C_1'}\left(\frac{dC_1'/dM_1}{dC_2'/dM_2}\right)$$

Relation (22) could be written as

$$\frac{M_2/dM_2}{C_2'/dC_2'} = \frac{M_1/dM_1}{C_1'/dC_1}$$

Therefore, if the marginal expense elasticities of input supply curves are equal, then $E_{12}' = 0$.

REFERENCES

1. Bradfield, M., "A Note on Location and the Theory of Production," *Journal of Regional Science,* 11 (1971), 263-266.

2. Emerson, D. L., "Optimum Firm Location and the Theory of Production," *Journal of Regional Science,* 13 (1973), 335-347.

3. Isard, W., *Location and Space — Economy,* New York: MIT Press and John Wiley, (1956), 77-90.

4. Khalili, Amir, et al, "Location and the Theory of Production: "A Generalization," *Journal of Economic Theory,* 9 (1974), 467-475.

5. Moses, L. N., "Location and the Theory of Production," *Quarterly Journal of Economics,* 72 (1958), 259-272.

6. Sakashita, N., "Production Function, Demand Function and Location Theory of Firm," *Papers, Regional Science Association,* 20 (1967), 109-122.

7. Weber, A., *A Theory of Location of Industries,* (C. J. Friedrick, trans.) Chicago: University of Chicago Press, 1962.