POPULATION DENSITY, "POTENTIAL," AND POSSIBLE PROXIES

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Introduction

Ever since the empirical discovery of Colin Clark [2], those engaged in urban studies have been fascinated by the tendency for urban population and employment densities to decline exponentially with distance from the center of the city according to the negative exponential density:

$$(1) \quad f(x,y) = c_o e^{-cr}$$

where f(x,y) is density at point (x,y) in region and r is distance of point from center or core of city. The parameters c_o, density at core, and c, the density gradient, have been estimated for many cities, years, and spatial variables and the differences in estimates obtained have been analyzed by researchers in various disciplines.[1] Estimation has also been performed using alternative density specifications[2] but the negative exponential remains the most widely accepted because of the theoretical economic models which have been developed that yield negative exponential rent, as well as population and employment density, functions in competitive equilibrium provided specific assumptions are made.[3]

Another concept which traditionally has been of interest to urban and regional specialists is the "gravity law" of spatial interaction which originated as a physics analogue. Recently, Niedercorn and Bechdolt [19] have attempted to provide a theoretical economic basis for the "gravity law" and Wilson [24] has offered an explanation based on entropy theory, but, nonetheless, empirical studies employing this concept have been numerous, especially in the analysis of travel demand.[4] In this paper we will be

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[1]Berry, Simmons and Tennant [1], Guest [5], Harrison and Kain [6], Kemper and Schmenner [9], Mills [13, 14], Muth [15], and Newling [16].

[2]Newling [17] suggests a quadratic exponential density as a means of analyzing the development of a city through time and this formulation was later used for Toronto by Latham and Yeates [11].

[3]Mills [14], Muth [15], Niedercorn [18], and Pines [20].

[4]Quandt and Baumol [21], and Long [12].

concerned with the related concept of "potential," which is also a physics analogue that has been used in urban and regional studies to measure accessibility to some spatial variable (e.g., income, employment, or population) defined over a bounded region or urban area.[5] To be specific, define for an urban area made up of K zones, the potential value of variable Z in zone k to be

$$(2)\quad \overline{Z} = \sum_{k=k'} \frac{Z_{k'}}{d_{kk'}}$$

where $d_{kk'}$ is the distance between k and k'.[6] In a recent article Richardson [22] has attempted, using the Lancaster [10] approach, to generalize potential so as to make it consistent with economic theory.

While the theoretical underpinnings of both the negative exponential density and "potential" are subject to debate, the use of these concepts remains popular and relevant in empirical urban and regional studies. An important consideration in empirical studies which employ these concepts is the "cost" of obtaining parameter estimates of equation (1) and performing calculations of equation (2) for each zone. The former estimation is relatively simple and, in fact, estimates are available (e.g., Mills [14]) for most cities in the United States. On the other hand, evaluating equation (2) requires values for variable Z in each zone, k, and also a distance matrix with typical element, $d_{kk'}$. The determination of such a matrix is a cumbersome task whether one chooses to measure distance using a map or, given sufficient expertise, to write a computer program to determine such distance given coordinates for each zone relative to some origin or center. Confronted with the calculation of potential in equation (2) an alternative (e.g., Muth [15]) has been to use r, distance of area from center (or Central Business District), as a proxy for potential.

In this paper we will develop and evaluate alternative proxies for potential which are simpler, computationally, than equation (2) and are at the same time consistent, mathematically, with the negative exponential density specification. These proxies can be used whenever one has an estimate of parameters of equation (1) but they will not require the distance matrix, $d_{kk'}$. Before discussing these proxies we will first synthesize the negative exponential density and "potential" concepts by deriving mathematically the continuous potential function for a city which follows from a negative exponential specification of density.

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[5] Clark, Wilson, and Bradley [3], Dunn [4], Muth [15], and Steinnes and Fisher [23].

[6] Equation (2) is the general form used in empirical studies to calculate potential though $d_{kk'}$ may be raised to some power and/or be replaced by travel time between areas k and k'. In the next section we will define and consider the continuous form of potential which involves integration, rather than summation in equation (2).

The Nature of "Potential" for Negative Exponential Density

The potential,[7] U, of a plane region, R, of density f at a point (x,y) is given by

$$(3)\quad U(x,y) = \iint_R \frac{f(\varepsilon,\eta)\,d\varepsilon\, d\eta}{\sqrt{(x-\varepsilon)^2 + (y-\eta)^2}} \quad . \qquad [8]$$

The interest here is to evaluate (3) in the case where R is a sector (i.e., between 0° and 360°) of a unit circle centered at the origin, (x,y) is interior to R, and the density is

$$(4)\quad f(\varepsilon,\eta) = K(c)\; e^{\left[-c\sqrt{(\varepsilon 2 + \eta 2)}\right]}$$

with c a positive constant and K(c) chosen so that $\iint_R f(\varepsilon,\eta)\,d\varepsilon\, d\eta = \pi$.[9] In particular, if R is a sector of ϕ radians, then

$$(5)\quad K(c) = \frac{\pi\, c^2}{\phi(1 - ce^{-c} - e^{-c})}$$

We are particularly interested in the two extreme cases of $c = 0$ and $c = \infty$ since these correspond to special assumptions about the distribution of a spatial variable defined over the region or urban area. If $c \to 0$ in (4) then

$$(6)\quad f(\varepsilon,\eta) = 2\pi/\phi$$

and so the density becomes uniform on the sector ϕ. As $c \to \infty$ we approach the case in which all the "mass" (e.g., employment) is concentrated at the

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[7]For a more detailed mathematical treatment of potential than will be provided in this section, the interested reader is referred to Kellog [8].

[8]This equation is the continuous form of equation (2) and assumes accessibility is measured by distance between points. We might consider how (3) could be altered to reflect specific assumptions about the travel system. The simplest alteration would be to assume travel is slower near the center and this could be incorporated with an additional function in the numerator of integrand or by adjusting density specification. For example, using the negative exponential density function, equation (4), one could reduce the value of c. The authors are considering such alterations, which would increase the policy ramifications of the model, but in this paper only distance, not travel time, will be used.

[9]Equation (4) is the negative exponential density which was written with r, distance from the center or core, as equation (1) in the last section. We are going to assume for simplicity throughout this paper that total mass (e.g., total population or employment) of region R is π. To alter this assumption would require changing K(c) appropriately by equation (5) but this would not affect the integration part of the potential calculation in equation (3) since K(c) is a constant with respect to that integration.

origin (e.g., Central Business District or CBD) and the resulting potential, independent of ϕ is

$$(7) \quad U(x,y) = \pi/\sqrt{(\varepsilon^2 + \eta^2)} = \pi/r$$

where $r = \sqrt{(\varepsilon^2 + \eta^2)}$ = distance of point (x,y) from origin.

In attempting to evaluate (3) we observe that the integrand becomes singular when $\varepsilon = x$ and $\eta = y$. However, the singularity can be removed by transforming (3) using polar coordinates (ω,θ) centered at (x,y) (i.e., $\varepsilon = x - \omega\cos\theta$ and $\eta = y - \omega\sin\theta$). Equation (3) then becomes

$$(8) \quad U(x,y) = \iint_{R'} K(c)e^{-c\sqrt{(x - \omega\cos\theta)^2 + (y - \omega\sin\theta)^2}}\, d\omega\, d\theta$$

where R' is the region R shifted so that (x,y) is now located at the origin.

Since equation (8) is not easily solved in closed form, we will evaluate the potential using numerical integration. These calculations were performed for various values of c and for different sectors, ϕ, using FORTRAN programs developed by the authors and contained in the Appendix. These programs perform numerical integration of (3) using Simpson's Rule. While these calculations generated several points which were consequently used to develop alternative approximating functions of the next section, in this section we will use these points to construct various graphs of the potential for various values of c and ϕ.

Let us begin with Figure 1 which indicates potential for full city (i.e., sector of 360^o or 2π radians) along any ray[10] of unit circle for different values of c. From this figure we can see that potential is convex with respect to the origin as $c \to 0$, concave as $c \to \infty$, and that potential is most linear for case $c = 2$.[11] As suggested by this graph, we will find when approximating potential for full city that nonlinear fits are better than linear fit, especially when $c < 1$.

While the graphical presentation of potential for full city, as well as the underlying calculations (see Appendix) are relatively simple, the calculations and graphical presentation of potential for cities or regions, R,

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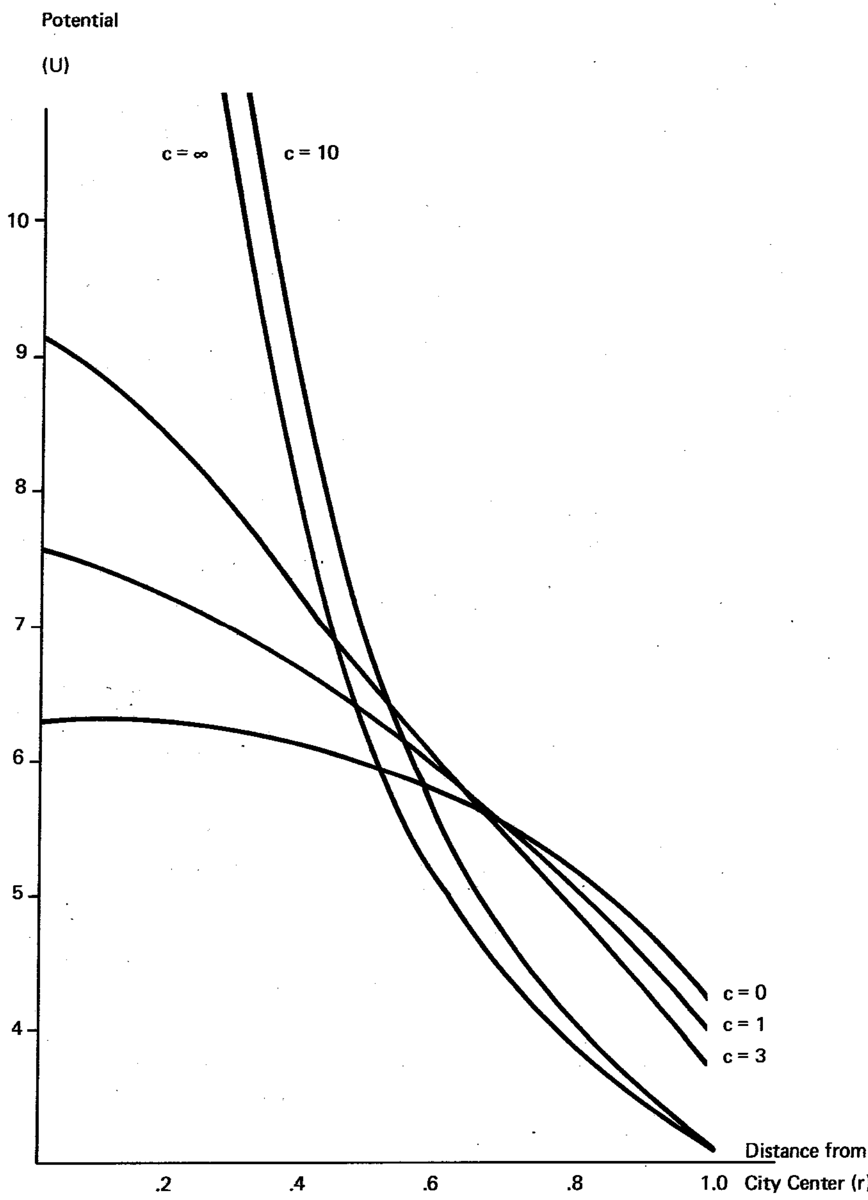
[10]The full city potential depends only on r, distance from origin, and not θ, angle of displacement (relative to the x-axis) of point (x,y) defined by polar coordinates (r,θ). It follows that equi-potential or contour lines would be concentric circles about the origin.

[11]The highest correlation between r and potential (U) was found for $c = 2$ when values of $c = .1, .2, \ldots, 3.0$ were considered.

FIGURE 1: Potential for Full City (360° Sector ϕ)



of sector less than 2π radians is more complex. We will concentrate attention on the most common sector, π radians or a semi-circle city, which coincides with many cities on a lake, river, or ocean. Less detailed discussion will be given to cities of other sector size although the first program in the Appendix can calculate potential values for any city of sector less than π radians.[12]

In Figures 2 and 3 potential values are presented using two graphical techniques. Figure 2 indicates equi-potential or contour lines when c = 1 for all points interior to region R. If such a graph had been constructed for full city it would have been set of concentric circles about the origin. It may be seen in Figure 2 that center of gravity[13] (i.e., point of highest potential) is on the x-axis rather than at the origin as it always is for full city, independent of c. This center of gravity will approach origin as $c \to \infty$ and/or $\phi \to 2\pi$ but it will always lie on the x-axis provided sector is defined symmetrically about x-axis.

Figure 3 presents potential for c = .1 and 1.0 along various rays ($\theta = 0^\circ$, 40°, 80°) as a function of r. Examination of this figure suggests the nature of approximating functions to be presented in the next section. We can see potential is inversely related to θ and nonlinearly related to r for each c. Further, we observe potential is higher for c = 1 than c = .1 except for values of r near 1. Each of these tendencies in Figure 3 will be manifested in approximating functions.

The purpose of Figure 4 is to illustrate potential along x-axis (i.e., 0° ray θ) for various sectors, ϕ, although in the next section we will concentrate on full (360°) and half (180°) city sectors. The potential along x-axis is inversely, and nearly linearly, related to r for full city but becomes less so as sector, ϕ, decreases. This figure graphically demonstrates an earlier statement that center of gravity approaches r = 0 (i.e., origin) as ϕ increases. A final observation is that potential at origin is the same for any sector, ϕ, for a specific c although this value rises with c as seen in Figure 3.

Having graphically presented the potential values generated by numerical integration of equation (8) so as to illustrate the "nature" of potential we will now attempt to capture this "nature" in various approximating functions which will eliminate the need for, and associated cost of, numerical integration.

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[12]The authors have a FORTRAN program to calculate potential for cities of sectors greater than π and less than 2π which is available on request.

[13]The center of gravity, $(\overline{X},\overline{Y})$, is given by

$$\overline{X} = \frac{\iint_R x f(x,y)\,dx\,dy}{M} \qquad \overline{Y} = \frac{\iint_R y f(x,y)\,dx\,dy}{M}$$

where $M = \iint_R f(x,y)\,dx\,dy$ is mass of R as explained in footnote 9.

FIGURE 2: Equi-potential Lines for Half City (180° Sector ϕ), c = 1.0

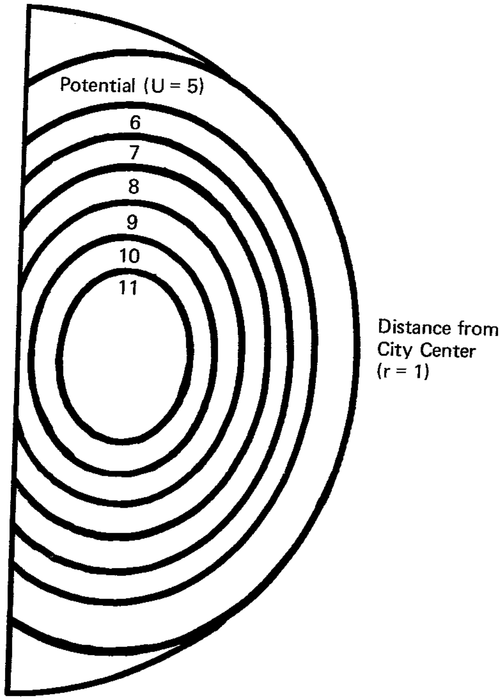


FIGURE 3: Potential for Half City (180° Sector ϕ), c = .1 and c = 1.

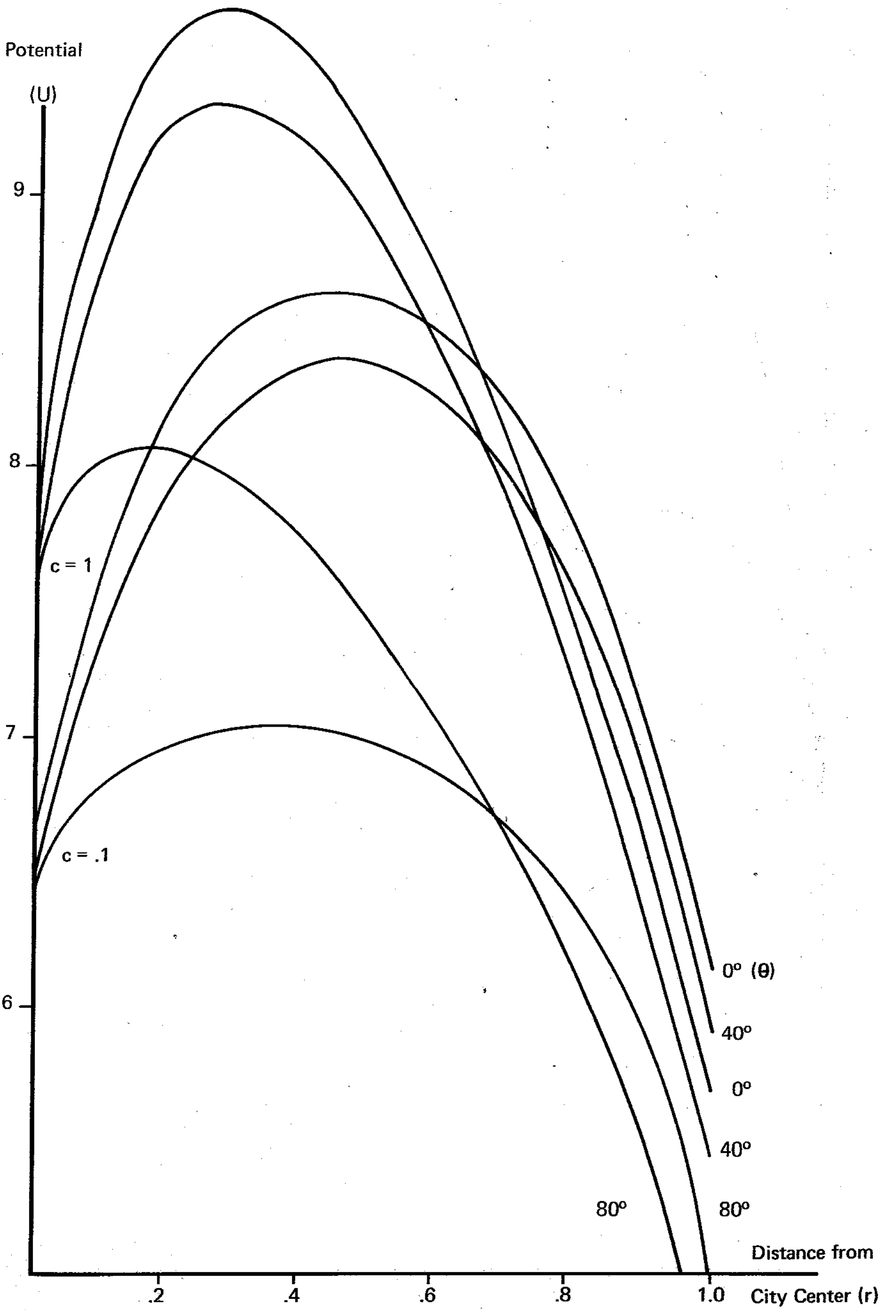
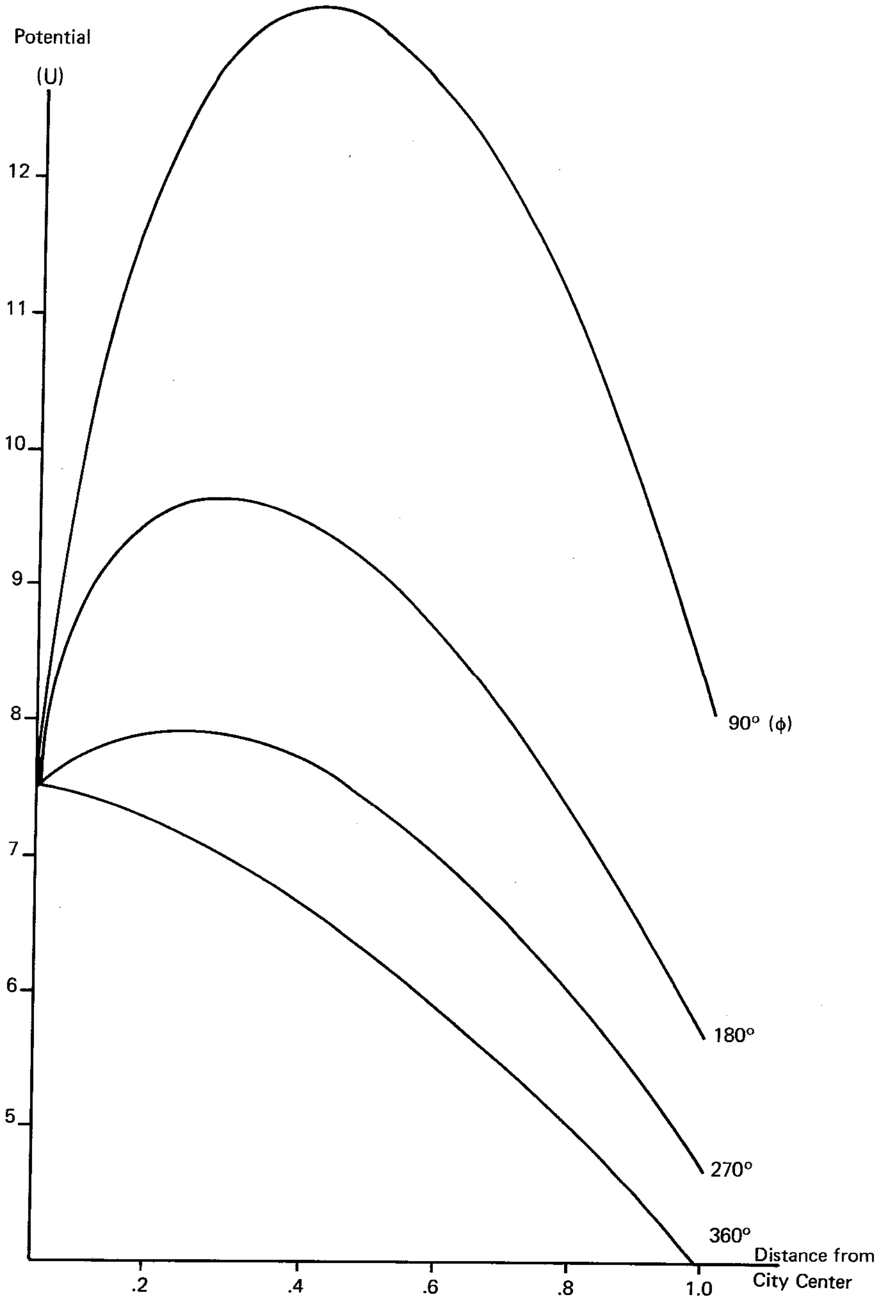


FIGURE 4: Potential Along x-axis (0° ray θ) for Various Sector (φ) Cities, c = 1.0



Approximating the "Potential" Function

In the last section we indicated the definite mathematical relationship between a negative exponential density specification and the resulting "potential" function, equation (8). Given this connection it would be possible, using numerical integration programs in the Appendix, to generate potential values for all points in a city for which one had estimated values for c and c_0 of the negative exponential density, equation (1). Such estimated values of potential would be an alternative to obtaining potential values using equation (2) which as noted earlier requires the cumbersome determination of distance matrix, $d_{kk'}$. An additional advantage of estimating potential with equation (8) is that estimates will be mathematically consistent with negative exponential density (i.e., estimates of c and c_0).

Though potential values obtained by numerical integration of equation (8) are the "true" or best values in a mathematical sense, the "cost" in an applied study of performing numerical integration may be considered too high. Hence, we will attempt in this section to provide, using numerical analysis, alternative or approximating functions of equation (8) which will eliminate the need for numerical integration. These approximating or fitted functions will only require an estimate of c and the sector size, ϕ, and will clearly prove better than using r, distance from the core or origin, as a proxy.

Our procedure for obtaining the approximating functions is to consider the values generated by numerical integration in the last section as "true" values, U, and evaluate alternative or fitted equations on the basis of estimated or fitted potential values, $\hat{U}$, they generate. The criterion used will be mean squared error (MSE)[14] which is lower for better fits.

In order to develop fitted equations a decision had to be made as to how general a functional form should be considered. We have restricted ourselves to values of c between .1 and 1.0 since these are the values found for most cities in empirical studies (e.g., Mills [14] and Muth [15]) but instead of estimating fitted equation for each c we have made c a variable in fits. Values of r between .05 and .95 have been used in fits as a variable. The most difficult decision was whether to consider fitted equations with sector size, ϕ, as a variable. While this may be done in future work we will in this paper develop fits for full and half cities separately since these are the most predominant sector sizes.

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[14] $MSE = \Sigma (U - \hat{U})^2/n$. We will also present for each equation an approximate R^2, $R_A^2 = 1 - MSE/var(U)$, where var(U) is variance of "true" values, U. The approximation in R_A^2 is caused by using MSE, rather than MSE - $[\Sigma(U - \hat{U})/n]^2$. This mean of $(U - \hat{U})$ is small, but not zero as it would be in "least squares" polynomial, and consequently R_A^2 is slightly smaller than R^2 would be. R_A^2 is useful to compare fitted equations for different sets of points (e.g., different sector cities or values of c). For additional techniques for evaluating numerical analysis see Hildebrand [7].

Let us begin with an examination of fits for full city presented in Table 1. These polynomial fits use r and c as variables and were obtained using a set of 190 points (c from .1 to 1.0 at .1 intervals and r from .05 to .95 at .1 intervals which were subsequently used to form quadratic terms)[15] for which "true" potential values had been determined using second program in the Appendix. The simplest equation, (A), represents using r alone as a proxy and will be the bench mark by which other fitted equations are judged. Equation (A) has low MSE and high R_A^2 but, nonetheless, it can be improved upon (e.g., MSE can be reduced from .058 to .0005) by considering additional quadratic terms. In equation (C) r^2 has a high coefficient suggesting the nonlinear relationship between potential and r for full city which was observed in Figure 1. In equations (B and C) c has positive coefficient as expected but c^2 is unimportant. To summarize, using r as a proxy for potential of full city is acceptable but additional terms (i.e., equation (D)) yields the "best" approximating or fitted equation.

While r is an acceptable proxy for potential for full city we find in Table 2 that using only r for half city yields a much less acceptable result (R_A^2 = .343 versus .915 in Table 1). The results in Table 2 were obtained using 900 points (c from .1 to 1.0 at .1 intervals, r from .05 to .95 at .1 intervals, θ from 0^o to 80^o at 10^o intervals[16]). Again we have restricted fitted equations to quadratic terms since third and fourth degree fits did not substantially improve fit and would only make use of fitted equations in an applied setting more complicated.

We find terms in Table 2 equations have, for the most part, the same sign as in Table 1 and these signs confirm observations made in discussing figures of the last section. In Table 2 terms involving θ are included but only θ and θ^2 are important indicating nonlinear relation between θ and potential as suggested by Figure 2. In Table 2 the "best" equation to use in an applied setting would be equation (E) since additional terms in equation (F) do not reduce MSE or raise R_A^2. The most important conclusion to be drawn from Table 2 is that r alone (equation (A)) yields a poor fit which can be dramatically improved with the addition of quadratic terms in c, r, and θ (i.e., equation (E)).

Having selected "best" equation in Tables 1 and 2 let us briefly delineate how these equations would be employed in an applied situation where one desired estimated values of potential for particular points, (r,θ), in a specific city. First, our "best" equations can only be used if one is considering a full or half city.[17] Assuming this is the case, one would also need an estimate of c

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[15] We present only up to quadratic fits since fitted equations of up to fourth degree did not yield substantially different or better results.

[16] Only points in first quadrant were used throughout since potential, as seen in Figure 2, is symmetric about x-axis.

[17] For other than half city best equation would have to be fitted with ϕ terms. This would require combining points used in Tables 1 and 2 and also points for other values of ϕ which would have to be generated.

TABLE 1: Approximate "Potential" Functions for Full City

	Approximate or Fitted Function	Mean Squared Error	$R_A^{2\,a}$
(A)	$U = 7.35 - 2.87r$	.058	.915
(B)	$U = 7.12 - 2.87r + .42c$	.044	.935
(C)	$U = 6.72 - .59r + .42c - 2.28r^2$	.020	.970
(D)	$U = 6.23 + .38r + 1.31c - 2.28r^2 - 1.77cr$	.0005	.999
(E)	$U = 6.24 + .38r + 1.27c - 2.28r^2 - 1.77cr + .031c^2$	.0005	.999

[a] $R_A^2 = 1 - \text{MSE}/\text{var}(U)$ where $\text{var}(U) = .678$

TABLE 2: Approximate "Potential" Functions for Half City

	Approximate or Fitted Function	Mean Squared Error	$R_A^{2\,a}$
(A)	$U = 8.78 - 1.96r$	.60	.343
(B)	$U = 9.13 - 1.96r + .53c - .93\theta$	.40	.563
(C)	$U = 7.86 + 5.65r + .53c - .93\theta - 7.61r^2$	.089	.904
(D)	$U = 7.61 + 5.65r + .53c + .32\theta - 7.61r^2 - .90\theta^2$	.063	.931
(E)	$U = 6.88 + 7.06r + 1.82c + .32\theta - 7.61r^2 - .90\theta^2 - 2.57cr$	.0175	.981
(F)	$U = 6.83 + 7.17r + 1.83c + .43\theta - 7.61r^2 - .90\theta^2 - 2.57cr - .15r\theta - .70c\theta + .04c^2$	.0175	.981

[a] $R_A^2 = 1 - \text{MSE}/\text{var}(U)$ where $\text{var}(U) = .914$

for the specific city being considered. One would then determine coordinates (r,θ),[18] of points for which potential estimates were desired. These values, for c, r, and θ, when inserted in "best equations will yield estimates, $\hat{U}$, which must be adjusted for the population of the city or region.[19]

Such estimated values, $\hat{U}$, can be quickly and easily obtained with the procedure just outlined for any points in any number of cities or years provided only that one has an estimate of c. Such an estimate of c may be easily calculated or possibly even obtained from empirical studies noted earlier which have been conducted for many U. S. cities.

Conclusion

In this paper we have mathematically synthesized two traditional concepts in urban and regional science--the negative exponential density and "potential" variables. This mathematical connection strongly suggests that any theoretical explanation of these two concepts will have to be general enough so as to explain the link between these concepts developed in this paper. To date, theoretical explanations have been offered for these concepts separately but no theoretical synthesis exists. The authors believe the mathematical synthesis in this paper may be the key to such a theoretical synthesis and intend to pursue this avenue in future work.

It is our hope that the synthesis we have provided will be useful in empirical and applied research where one has a model based on an assumption that population and/or employment are distributed according to the negative exponential density. In any such situation where a value for c is obtained or assumed this paper would provide the means of obtaining the corresponding potential values. Such values could be determined by programs in the Appendix and potential surfaces developed as in Figure 2, or if one only wanted potential for a discrete number of points in the region these would be most easily determined using approximating functions or proxies developed in the last section.

A possible situation where the value of c might be assumed would be if one wanted to simulate urban growth. It has generally been found that for any city

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18 If the city were of radius n miles, rather than 1 mile as we have assumed, we would define a new unit of distance, 1 "league" = n miles, and density would have to be changed to $\eta^2 e^{-ncr}$ where r is now expressed in leagues. The program in the Appendix could be used to calculate "true" potential values using this new density or, alternatively, the fitted equations presented could be used to obtain estimated values, $\hat{U}$, for points (r expressed in leagues) by using nc in place of c and multiplying the result, $\hat{U}$, by n^2. It should be noted that c is for density of city with radius n or, in general, the value for c which might be obtained from previous empirical studies

19 That is, we convert K(c) to population of city by multiplying estimate, $\hat{U}$, by population/π since throughout K(c) has been for city of mass π (see footnote 9 for more details).

c declines through time and so one might simulate the growth of a city by systematically reducing c. The procedure in this paper would allow one to obtain consequent potential values or surfaces at each stage of the simulation (i.e., for each c assumed). In fact, one might consider using the potential surface obtained at a stage of the simulation as a means of modifying density at the next stage. That is, increase density or allocate population and/or employment to areas with highest potential which would be similar to lowry type "gravity" model. However, it is clear that it would be best to develop this latter type of simulation on the basis of the theoretical synthesis alluded to earlier.

It is realized that the procedure outlined can be expanded and refined by incorporating alternative assumptions about the transportation system, generalizing approximating functions to include sector size, ϕ, as a variable, and using mathematical synthesis as a means for estimating negative exponential density function from a set of potential values. Also, we have restricted our attention to negative exponential density because of its interdisciplinary popularity and acceptance but it would be possible to develop synthesis using an alternative density specification as, for example, the quadratic negative exponential which has been developed by geographers.

APPENDIX

In order to evaluate equation (8) using numerical integration we must define limits of integration more precisely

$$(9)\quad U(x,y) = K(c)\left[\int_{-\theta_1}^{\theta_o}\int_0^{g_1(\theta)} f(x,y,\omega,\theta)\,d\omega d\theta + \int_{\theta_o}^{\pi+\delta}\int_0^{g_2(\theta)} f(x,y,\omega,\theta)\,d\omega d\theta + \int_{\pi+\delta}^{2\pi-\theta_1}\int_0^{g_3(\theta)} f(x,y,\omega,\theta)\,d\omega d\theta\right]$$

where

$$\theta_o = \phi/2 + \mathrm{ARCSIN}\,\frac{\sqrt{x^2+y^2}\,\sin(\phi/2-\delta)}{C_o}$$

$$\theta_1 = \phi/2 + \mathrm{ARCSIN}\,\frac{\sqrt{x^2+y^2}\,\sin(\phi/2+\delta)}{C_1}$$

$$C_o = \sqrt{x^2+y^2+1-2\sqrt{x^2+y^2}\cos(\phi/2-\delta)}$$

$$C_1 = \sqrt{x^2+y^2+1-2\sqrt{x^2+y^2}\cos(\phi/2+\delta)}$$

$$\delta = \mathrm{ARCTAN}(y/x)$$

$$g_1(\theta) = -\sqrt{x^2+y^2}\cos(\theta-\delta) + \sqrt{1-(x^2+y^2)\sin^2(\theta-\delta)}$$

$$g_2(\theta) = (\sqrt{x^2+y^2}\sin(\theta/2-\delta))/(\sin(\theta-\phi/2))$$

$$g_3(\theta) = (\sqrt{x^2+y^2}\sin(\theta/2+\delta))/(-\sin(\theta+\phi/2))$$

$$f(x,y,\omega,\theta) = e^{-c\sqrt{x^2+y^2+2x\omega\cos\theta+2y\omega\sin\theta+\omega^2}}$$

Program A will perform integration in (9) for any sector ϕ less than π. U(IC,IR,IP,JA) is array of potential values determined by program for various IC, values of c in equation (4), IR, values of r, IP, sector sizes ϕ, and JA, rays θ. The version in Program A will do calculation for values of c from 1 to 2 at 1/10 intervals, values of r from .05 to .95 at .1 intervals, a π sector city, and values of θ from 0^o to 80^o at 10^o intervals. Increasing N and M will perform numerical integration more precisely but will also proportionally increase the cost of running the program.

For a full city equation (8) can be simplified to function of r

$$(10)\quad U(r) = 2K(c)\int_0^{\pi}\int_0^{g(\theta)} e^{-c\sqrt{r^2+\omega^2+2r\omega\cos\theta}}\,d\omega d\theta$$

where $g(\theta) = -r\cos\theta + \sqrt{1 - r^2 \sin^2\theta}$. Program B will numerically integrate equation (10) for various values of c and r.

Finally, it should be noted that WRITE and FORMAT statements must be appended to output U array from these programs.

Program A

```
DIMENSION U(10,10,6,9),FF(11),FS(11),FT(11)HF(11),HS(11),HT(11)
N=10
M=10
AN=N
AM=M
NM=N-1
MM=M-1
NP=N+1
MP=M+1
Z=0.0
DO 30 IC=1,10
C=IC
C=C/10.0
C=C+1.0
PDF=3.14159/36.0
IP=6
P=IP
JP=1.5*P
P=(P*3.14159)/12.0
Z=3.14159*C*C/((1.0-C*(EXP(-C))-EXP(-C))*2.0*P)
DO 30 IR=1.10
AR=IR
R=(AR/10.0)-.05
```

```
    DO 30 JA=1,JP
    SA=JA
    SA=(SA-1.0)*PDF*2.0
    XX=R*COS(SA)
    YY=R*SIN(SA)
    DELTA=ATAN2(YY,XX)
    SRS=XX*XX+YY*YY
    SR=SQRT(SRS)
    PMD=P-DELTA
    PPD=P+DELTA
    CO=SQRT(SRS+1.0-2.0*SR*COS(PMD))
    CI=SQRT(SRS+1.0-2.0*SR*COS(PPD))
    TON=SR*SIN(PMD)
    TIN=SR*SIN(PPD)
    TONCO=TON/CO
    TINCI=TIN/CI
    TO=P+ARSIN(TONCO)
    TI=P+ARSIN(TINCI)
    XF=(TO+TI)/AN
    DO 21 K=1,NP
    AK=K
    AI=(AK-1.0)*XF-TI+6.28318
    GF=-SR*COS(AI-DELTA)+SQRT(1.0-(SRS)*((SIN(AI-DELTA))**2))
    WF=GF/AM
    DO 22 J=1,MP
    AJ=J
    D=(AJ-1.0)*WF
    FF(J)=EXP(-C*(SQRT(SRS+2.0*XX*D*COS(AI)+2.0*YY*D*SIN(AI)+D*D)))
 22 CONTINUE
    FEVEN=0.0
    DO 41 J=2,M,2
 41 FEVEN=FEVEN+FF(J)
    FODD=0.0
    DO 42 J=3,MM,2
 42 FODD=FODD+FF(J)
    HF(K)=(WF/3.0)*(FF(1)+4.0*FEVEN+2.0*FODD+FF(MP))
 21 CONTINUE
    XS=(3.14159+DELTA-TO)/AN
    DO 23 K=1,NP
    AK=K
    AI=(AK-1.0)*XS+TO
    GS=TON/SIN(AI-P)
    WS=GS/AM
    DO 24 J=1,MP
    AJ=J
    D=(AJ-1.0)*WS
    FS(J)=EXP(-C*SQRT((XX+D*COS(AI))**2+(YY+D*SIN(AI))**2)))
 24 CONTINUE
```

```
      SEVEN=0.0
      DO 43 J=2,M,2
   43 SEVEN=SEVEN+FS(J)
      SODD=0.0
      DO 44 J=3,M,2
   44 SODD=SODD+FS(J)
      HS(K)=(WS/3.0)*(FS(1)+4.0*SEVEN+2.0*SODD+FS(MP))
   23 CONTINUE
      XT=(3.14159-TI-DELTA)/AN
      DO 25 K=1,NP
      AK=K
      AI=(AK-1.0)*XT+3.14159+DELTA
      GT=-TIN/SIN(AI+P)
      WT=GT/AM
      DO 26 J=1,MP
      AJ=J
      D=(AJ-1.0)*WT
      FT(J)=EXP(-C*(SQRT((XX+D*COS(AI))**2+(YY+D*SIN(AI))**2)))
   26 CONTINUE
      TEVEN=0.0
      DO 45 J=2,M,2
   45 TEVEN=TEVEN+FT(J)
      TODD=0.0
      DO 46 J=3,M,2
   46 TODD=TODD+FT(J)
      HT(J)=(WT/3.0)*(FT(1)+4.0*TEVEN+2.0*TODD+FT(MP))
   25 CONTINUE
      HFEVEN=0.0
      HSEVEN=0.0
      HTEVEN=0.0
      DO 52 K=2,N,2
      HFEVEN=HFEVEN+HF(K)
      HSEVEN=HSEVEN+HS(K)
   52 HTEVEN=HTEVEN+HT(K)
      HFODD=0.0
      HSODD=0.0
      HTODD=0.0
      DO 53 K=3,NM,2
      HFODD=HFODD+HF(K)
      HSODD=HSODD+HS(K)
   53 HTODD=HTODD+HT(K)
      BF=(XF/3.0)*(HF(1)+4.0*HFEVEN+2.0*HFODD+HF(NP))
      BS=(XS/3.0)*(HS(1)+4.0*HSEVEN+2.0*HSODD+HS(NP))
      BT=(XT/3.0)*(HT(1)+4.0*HTEVEN+2.0*HTODD+HT(NP))
   90 U(IC,IR,IP,JA)=(BF+BS+BT)*Z
```

```
      DIMENSION  U(10, 19), H(11), F(11)
      N=10
      M=10
      AN=N
      AM=M
      NM=N-1
      MM=M-1
      NP=N+1
      MP=M+1
      Z=0.0
      DO 30 IC=1,10
      C=IC
      C=C/10.0
      C=C+1.0
      Z=(C**2.0)/(1.0-C*(EXP(-C))-EXP(-C))
      DO 30 IR=1,19
      AR=IR
      R=0.0
      R=.05*AR
      X=3.14159/AN
      DO 25 K=1,NP
      AK=K
      AI=(AK-1.0)*X
      G=((-R)*COS(AI))+SQRT(1.0-(R**2.0)*((SIN(AI))**2.0))
      A=0.0
      W=G/AM
      DO 20 J=1,MP
      AJ=J
      D=(AJ-1.0)*W
      F(J)=EXP(-C*(SQRT((R**2.0)+(D**2.0)+(2.0*R*D*COS(AI)))))
   20 CONTINUE
      EVEN=0.0
      DO 40 J=2,M,2
   40 EVEN=EVEN+F(J)
      ODD=0.0
      DO 41 J=3,MM,2
   41 ODD=ODD+F(J)
      H(K)=(W/3.0)*(F(1)+4.0*EVEN+2.0*ODD+F(MP))
   25 CONTINUE
      HEVEN =0.0
      DO 42 K=2,N,2
   42 HEVEN=HEVEN+H(K)
      HODD=0.0
      DO 43 K=3,NM,2
   43 HODD=HODD+H(K)
      R=(X/3.0)*(H(1)+4.0*HEVEN+2.0*HODD+H(NP))
   30 U(IC,IR)=R*Z
```

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