DYNAMIC REGIONAL INPUT-OUTPUT MODELS WITH TIME-VARYING COEFFICIENTS: A THEORETICAL FRAMEWORK

Se-Hark Park*

Recently, attention has been directed toward the theoretical formulation of dynamic input-output systems. As a result a variety of models have been posited [1, 2, 3, 5, 6, 11, 12, 13]. In most studies, input-output and stock-flow coefficients were assumed to be constant in the dynamic Leontief system. The objective of the paper is to characterize more general time-varying regional systems which include a dynamic model with constant coefficients as a special case and to obtain solutions to these systems.

A State Variable Characterization of a Dynamic Regional Input-Output Model

Suppose a region has two possible sources of supply for each commodity, import sources and local production, then the usual input-output equation** is given by

$$(1)\quad \underline{x}_t = A_t \underline{x}_t + B_{(t+1)} (\underline{x}_{(t+1)} - \underline{x}_t) + \underline{y}_t - \underline{x}^m_t$$

where

A_t = an (nxn) interindustry input coefficients matrix

B_t = an (nxn) capital coefficients

$\underline{x}_t$ = an (nx1) output vector

$\underline{y}_t$ = an (nx1) final demand vector, and

$\underline{x}^m_t$ = an (nx1) import vector,

*Professor of Economics, Marquette University.

**Unless otherwise specified, matrices are denoted by capital letters, vectors are written as lower case letters with a bar attached below them, and scalars represented by lower-case letters without a bar.

all in period t,* and a lower case letter t in bracket denotes the time-varying nature of coefficient matrix or vector. Capital goods produced in year t are assumed to be installed and put into operation in the next year, t+1. Eq. (1) states that each period's output plus imports must exactly satisfy three uses: Current consumption, interindustry flows of intermediate goods and the increment to capacity to provide the next period's output demand. Regional exports are treated as a part of final demand.

Now, the direct interdependence between the outputs of all the sectors of a given regional economy in two successive years can be described in different form of balance equations, corresponding to the different assumptions made regarding the import function. If we assume imports as exogenous, we obtain:

$$(2) \quad B_{(t+1)}\, \underline{x}_{(t+1)} = [I - A_t + B_{(t+1)}]\, \underline{x}_t - [\underline{y}_t - \underline{x}^m_t]$$

and if $B_{(t+1)}$ is nonsingular,** we may write

$$(3) \quad \underline{x}_{(t+1)} = [I + B^{-1}_{(t+1)}\, (I - A_t)]\, \underline{x}_t - B^{-1}_{(t+1)}\, (\underline{y}_t - \underline{x}^m_t)$$

or in a simple form

$$(4) \quad \underline{x}_{(t+1)} = G_t\, \underline{x}_t + H_t\, [\underline{y}_t - \underline{x}^m_t]$$

where

$$G_t = I + B^{-1}_{(t+1)}\, (I - A_t) \text{ and } H_t = -B^{-1}_{(t+1)}$$

An (nxn) matrix G_t corresponds to the state transition matrix in the state variable analysis [7].

If we consider $\underline{x}^m$ as a vector of competing imports, i.e.,

$$\underline{x}^m_t = M_t\, \underline{x}_t$$

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*As usual, we assume that A_t is indecomposable. That is, there exists no permutation of like rows and colums for which A_t may be written in the form

$$\left[\begin{array}{c|c} A_1 & A_2 \\ \hline 0 & A_3 \end{array}\right]$$

with square submatrices A_1 and A_3 on the diagonal (see [9, .254ff.] for an economic interpretation of indecomposability). Also we assume that A_t satisfies the Hawkins-Simon condition [10], i.e., basically the condition that there can be no negative entries in the table of direct and indirect requirements.

**If not, we may find a reduced order of transformed variables by a linear combination of the old variables in such a way that the transformed matrix becomes nonsingular. Obviously the calculations are involved. We ignore this complication.

where M_t is an (nxn) diagonal matrix whose diagonal elements are sectoral import coefficients. Then Eq. (2) can be rewritten as:

$$(5) \quad \underline{x}_{(t+1)} = [I+B^{-1}_{(t+1)} (I-A_t + M_t)] \, \underline{x}_t - B^{-1}_{(t+1)} \, \underline{y}_t$$

and the state transition matrix becomes now

$$(5') \quad G_t = I+B^{-1}_{(t+1)} (I-A_t + M_t)$$

Finally assume that imports provide a part of final demand, intermediate goods, and investment, i.e.,

$$(6) \quad \underline{x}^m_t = A^m_t \, \underline{x}_t + B^m_{(t+1)} \, [\underline{x}_{(t+1)} - \underline{x}_t] + \underline{y}^m_t$$

where A^m_t is an (nxn) imported input coefficient matrix, B^m_t is an (nxn) imported capital coefficient matrix and y^m_t is a vector of imported final goods.

Substituting (6) into (1) and rearranging will give

$$(7) \quad \underline{x}_{(t+1)} = [I+(B_{(t+1)}-B^m_{(t+1)})^{-1} (I-(A_t-A^m_t))] \, \underline{x}_t$$

$$-(B_{(t+1)}-B^m_{(t+1)})^{-1} (\underline{y}_t - y^m_t)$$

or

$$(7') \quad \underline{x}_{(t+1)} = [I+\bar{B}^{-1}_{(t+1)} (I-\bar{A}_t)] \, \underline{x}_t - \bar{B}^{-1}_{(t+1)} \, \bar{\underline{y}}_t$$

where a bar above a symbol refers to local production. Let the input-output balance equation be represented for simplicity by

$$\underline{x}_{(t+1)} = G_t \, \underline{x}_t + H_t \, \underline{y}_t$$

where

$$G_t = I+B^{-1}_{(t+1)} [I-A_t] \text{ or } I+B^{-1}_{(t+1)} [I-A_t + M_t];$$

$H_t = -B^{-1}_{(t+1)}$; and A_t, B_t and $\underline{y}_t$

related only to local production or local production plus imports depending upon the type of import functions assumed. Then if the initial conditions $\underline{x}_{t_o}$ are known, we obtain

$$(8) \quad \underline{x}_{(t_o+1)} = G_{t_o} \, \underline{x}_{t_o} + H_{t_o} \, \underline{y}_{t_o}$$

and similarly

$$(9)\quad \underline{x}_{t_o2} = G_{(t_o+1)}\, \underline{x}_{(t_o+1)} + H_{(t_o+1)}\, \underline{y}_{(t_o+1)}$$

$$= G_{(t_o+1)} [G_{t_o}\, \underline{x}_{t_o} + H_{t_o}\, \underline{y}_{t_o}] + H_{(t_o+1)}\, \underline{y}_{(t_o+1)}$$

$$= G_{(t_o+1)}\, G_{t_o}\, \underline{x}_{t_o} + G_{(t_o+1)}\, H_{t_o}\, \underline{y}_{t_o} + H_{(t_o+1)}\, \underline{y}_{(t_o+1)}$$

Let us define the state transition matrix as

$$(10)\quad \phi_{(t,t_o)} = \prod_{n=t_o}^{t-1} G_n \quad (t>t_o)$$

$$= I \qquad (t=t_o)$$

which describes the process of the movement of output over time from its initial position $\underline{x}_{t_o}$. To be more specific, the state transition matrix $\phi_{(t,t_o)}$ maps the state $\underline{x}_{t_o}$ at t_o into the state $\underline{x}_t$ at time t, and hence the "time path" of $\underline{x}_t$ for any future time t if at t_o, $\underline{x}_{t_o}$ are known. By a process of iteration, we obtain the solution to the open model

$$(11)\quad \underline{x}_t = \phi_{(t,t_o)}\, \underline{x}_{t_o} + \sum_{m=t_o}^{t-1} \phi_{(t,m+1)}\, H_m\, \underline{y}_m$$

where

$$\phi_{(t,t_o)} = \prod_{n=t_o}^{t-1} G_n \quad \text{for } t>t_o$$

$$= I \qquad \text{for } t=t_o$$

The first term on the right hand side of Eq. (11) represents the initial condition response of the input-output systems, while the second term shows a superposition summation of the effects of final demand.

In the time-invariant case, where A and B are constant matrices, the solution becomes

$$(12)\quad \underline{x}_t = G_o^{(t-t_o)}\, \underline{x}_{t_o} - \sum_{m=t_o}^{t-1} G_o^{(t-m-1)}\, B_o\, \underline{y}_m.$$

where

$$G_o = [I+B_o^{-1}(I-A_o)]$$

Needless to say, a time varying input-output model is more useful in empirical applications than the systems with constant coefficients, but it is extremely difficult to measure input-output relationships over time, except for those periods for which tables are available. However, for the time-varying case where the state transition matrix can be written as the sum of a constant matrix and a time-varying perturbation matrix, a perturbation technique can be used to obtain the state transition matrix. This procedure may prove to be quite useful if the time-varying matrix represents a small perturbation upon the constant matrix. For this case

$$(13) \quad \underline{x}_{(t_o+t)} = [G_o + G_{1_t}]\, \underline{x}_{t_o} + [H_o + H_{1_t}]\, \underline{y}_{t_o}$$

$$= [G_o\, \underline{x}_{t_o} + H_o\, \underline{y}_{t_o}] + [G_{1_t}\, \underline{x}_{t_o} + H_{1_t}\, \underline{y}_{t_o}]$$

where t_o denotes the period for which the input-output table is prepared, and G_{1_t} and H_{1_t} are perturbation matrices. Eq. (13) states that output requirements at t_o+t is the sum of initial output requirements at t_o plus output changes from t to t_o+t. Therefore, terms in the second bracket represent the static projection error when the input-output relationship at t_o is used to project output requirements for the period t_o+t.

Then the question arises as to how this perturbation matrix can be expressed in terms of the original input-output and stock flow matrices. First, we take a simple case where a small perturbation occurs in the input-output coefficients matrix and the stock flow matrix remains constant, i.e.,

$$A_{(t_o+t)} = A_o + \delta A_t$$

and

$$B_{(t+1)} = B_o$$

hence

$$H_o = B_o^{-1} \text{ and } H_1 = 0$$

where δA_t is a small perturbation on the input-output matrix and B_o is a constant matrix. Then the state transition matrix G_t can be written as

$$(14) \quad G_t = I + B_o^{-1}[I - A_o - \delta A_t]$$

$$= [I + B_o^{-1}(I - A_o)] - B_o^{-1}\, \delta A_t$$

$$= G_o + G_{1_t}$$

where

$$G_o = I + B_o^{-1}(I - A_o)$$

and

$$G_{1_t} = -B_o^{-1}\, \delta A_t$$

However, if we assume a perturbation also in the stock flow matrix, the problem becomes much more complicated. In such a case we have

$$(15) \quad \underline{x}_{(t_o+t)} = [I+(B_o + \delta B_t)^{-1}(I-A_o - \delta A_t)]\ \underline{x}_{t_o}$$
$$-[B_o + \delta B_t]^{-1}\ \underline{y}_{t_o}$$

Using the Noble's theorem about the inversion of matrices [14, Chapter 5, Theorem 5.22], the inverse of the stock flow matrix can be written as

$$(16) \quad [B_o + \delta B_t]^{-1} = B_o^{-1} + B_o^{-1}[\delta B_t^{-1} - B_o^{-1}]^{-1} B_o^{-1} = B_o^{-1} + W$$

where $W = B_o^{-1}(\delta B_t^{-1} - B_o^{-1})^{-1} B_o^{-1}$ and δB_t is assumed to be non-singular. Then substituting (16) into (15) and rearranging terms will yield

$$(17) \quad \underline{x}_{(t_o+t)} = [I+B_o^{-1}(I-A_o)]\ \underline{x}_{t_o} - B_o^{-1}\ \underline{y}_{t_o} + \{W[I-A_o - \delta A_t]$$
$$- B_o^{-1}\ \delta A_t\}\ \underline{x}_{t_o} - W\ \underline{y}_{t_o}$$
$$= [G_o\ \underline{x}_{t_o} + H_o\ \underline{y}_{t_o}] + [G_{1_t}\ \underline{x}_{t_o} + H_{1_t}\ \underline{y}_{t_o}]$$

where

$$G_o = I+B_o^{-1}(I-A_o).$$
$$G_{1_t} = W[I-A_o - \delta A_t] - B_o^{-1}\ \delta A_t$$
$$H_o = -B_o^{-1}$$
$$H_1 = -W_t$$
$$W = B_o^{-1}\ (\delta B_t^{-1} - B_o^{-1})^{-1} B_o^{-1}$$

However, the assumption of nonsingularity is unacceptable in an economic sense, for it is quite possible for δB_t to contain whole rows of zeros since there may be no change in the stock flow ratios for some sectors. However, we can bypass this problem by using the same rationalization applied to the problem of non-singularity in the original stock flow matrix B_t. Namely, if the rank of δB_t is less than n, one works with a reduced order of transformed variables and converts back to the full set at the very end. Finally there remains an empirical question of how to measure a perturbation matrix. Given the paucity of direct short run measurements at the regional level, one may have to use indirect methods of estimating coefficient changes based on their actual past variability, if such data is available, or based on the past variability of national coefficients with proper adjustments for peculiar regional differences. Furthermore, recent works by Reifler [15] and Beyers [14] suggest that at the small area level trading relationships may change quite appreciably in the short run, whereas technological relationships may remain fairly constant. Therefore, special attention must be given to the measurement of the import components of the perturbation matrix. Needless to say, this is not easily implementable empirically.

An Expectation Model

Consider a time invariant case of Eq. (1)

$$(18)\quad \underline{x}_t = A\underline{x}_t + B[\underline{x}_{(t+1)} - \underline{x}_t] + \underline{y}_t$$

where A and B are constant matrices. Obviously the term $B\,[\underline{x}_{(t+1)} - \underline{x}_t]$ in Eq. (18) represents the investment that must take place during period t+1. Thus Eq. (18) represents a planning model for dynamic allocation of resources rather than a descriptive model of actual economy. In order to make the system (18) as a descriptive model, let us replace $\underline{x}_{(t+1)}$ by anticipated output $\underline{x}^*_{(t+1)}$ and postulate further that expectations are a distributed lag function of past actual changes in output, i.e.,

$$(19)\quad \underline{x}^*_{(t+1)} - \underline{x}_t = \sum_{i=1}^{n} E_i[\underline{x}_{(t+1-i)} - \underline{x}_{(t-i)}]$$

where E_i is an expectation matrix. Substituting (19) into (18) will give a descriptive model

$$(20)\quad \underline{x}_t = A\underline{x}_t + \sum_{i=1}^{n} B_i[\underline{x}_{(t+1-i)} - \underline{x}_{(t-i)}] + \underline{y}_t$$

where

$$B_i = BE_i$$

Then, by expanding the summation and rearranging terms we obtain

$$(21)\quad (I-A-B_1)\,\underline{x}_t + (B_1-B_2)\,\underline{x}_{(t-1)} + \ldots\ldots$$

$$+ (B_{n-1}-B_n)\,\underline{x}_{(t+1-n)} + B_n\,\underline{x}_{(t-n)} = \underline{y}_t$$

$$(22)\quad \underline{x}_t + D_1\,\underline{x}_{(t-1)} + D_2\,\underline{x}_{(t-2)} + \ldots + D_{n-1}\,\underline{x}_{(t+1-n)} + D_n\,\underline{x}_{(t-n)}$$

$$= V\underline{y}_t$$

where

$$V = (I-A-B_1)^{-1},$$

$$D_i = V(B_i-B_{i+1}) \quad \text{for } i=1,\ 2\ldots.\ n-1$$

$$D_n = VB_n \qquad \text{for } i=n$$

Eq. (22) can be alternatively written as, letting t=k+n,

$$\underline{x}_{(k+n)} + D_1 \underline{x}_{(k+n-1)} + D_2 \underline{x}_{(k+n-2)} + \cdots + D_n \underline{x}_k$$

$$= P_o \underline{y}_{(k+n)} + P_1 \underline{y}_{(k+n-1)} + \cdots + P_n \underline{y}_k = P_o \underline{y}_{(k+n)}$$

where $P_i = [0]$ for $i \neq 0$ and $P_o = V = (I-A-B_1)^{-1}$

If the initial data $\underline{x}(0)$, $\underline{x}(1)$, . . . , $\underline{x}_{(n-1)}$ are known,

Eq. (23) can be given in the following form:

$$(24) \quad \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \underline{x}_{n-1} \\ \underline{x}_n \end{bmatrix}_{(k+1)} = \begin{bmatrix} 0 & I & \ldots & 0 \\ 0 & 0 & I \ldots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 \ldots & & I \\ -D_n & -D_{n-1} & \ldots & -D_1 \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \cdot \\ \cdot \\ \underline{x}_{n-1} \\ \underline{x}_n \end{bmatrix}_k + \begin{bmatrix} C_1 \\ C_2 \\ \cdot \\ \cdot \\ \cdot \\ C_{n-1} \\ C_n \end{bmatrix} \underline{y}_k$$

or

$$(25) \quad \overline{\underline{X}}_{(k+1)} = \Omega \, \overline{\underline{X}}_k + C\underline{y}_k$$

where $\overline{X}$ is an $(n^2 x1)$ column vector, Ω is an $(n^2 xn^2)$ partioned state transition matrix whose elements are all (nxn) matrices, and C is an $(n^2 xn)$ partitioned matrix. The $\underline{x}_{i,(k+1)}$ (i=1, 2, ...,n) in Eq. (24) are defined by

$$\begin{aligned}
\underline{x}_k &= \underline{x}_{1,k} + C_o \underline{y}_k \\
\underline{x}_{1,(k+1)} &= \underline{x}_{2,k} + C_1 \underline{y}_k \\
\underline{x}_{2,(k+1)} &= \underline{x}_{3,k} + C_2 \underline{y}_k \\
\ldots \quad & \ldots \quad \ldots \\
\underline{x}_{n-1,(k+1)} &= \underline{x}_{n,k} + C_{n-1} \underline{y}_k \\
\underline{x}_{n,(k+1)} &= -D_n \underline{x}_{1,k} - D_{n-1} \underline{x}_{2,k} \cdots - D_1 \underline{x}_{n,k} + C_n \underline{y}_k
\end{aligned}$$

where the C_i are determined from

$$C_o = P_o = (I-A-B_1)^{-1}$$

$$C_1 = P_1 - D_1C_o = -D_1C_o$$

$$C_2 = P_2 - D_2C_o - D_1C_1 = -D_2C_o - D_1C_1$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$C_n = P_n - D_nC_o - \ldots - D_2C_{n-2} - D_1C_{n-1}$$

where

$$P_i = [0] \text{ for } i \neq 0 \text{ and } P_o = (I-A-B_1)^{-1}$$

The initial conditions are also given by

$$\underline{x}_{1,0} = \underline{x}_0 - C_o \underline{y}_0$$

$$\underline{x}_{2,0} = \underline{x}_1 - C_o \underline{y}_1 - C_1 \underline{y}_0$$

$$\underline{x}_{3,0} = \underline{x}_2 - C_o \underline{y}_2 - C_1 \underline{y}_1 - C_2 \underline{y}_0$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\underline{x}_{n,0} = \underline{x}_{(n-1)} - C_o \underline{y}_{(n-1)} - C_1 \underline{y}_{(n-2)} - \ldots - C_{n-2} \underline{y}_1 - C_{n-} \underline{y}_0$$

Using again the method of iteration and induction, the complete solution to the vector-matrix difference equation (25) can be stated as

$$(26) \quad \underline{\overline{X}}_k = \Omega^{(k-k_o)} \underline{\overline{X}}_{(k_o)} + \sum_{m=k}^{k-1} \Omega^{(k-m-1)} C\underline{y}_m$$

where $\overline{X}$, Ω, and C are same as defined in Eq. (25). Generalization of the expectation model to the time varying system remains to be done.

Summary and Concluding Remarks

This paper began with a state variable characterization of a regional input-output model with time varying coefficients in the input-output and stock flow matrices. The state variable approach was then applied to obtain solutions to open time varying Leontief systems which include a traditional time invariant model as a special case. A similar state variable characterization and solution was also obtained for a time invariant expectation model where the increment to capacity is postulated to be a function of the difference between the anticipated output and current output, and the expectations are in turn assumed to be a distributed lag function of past actual changes in output.

Advantages of the state variable approach to the dynamic input-output model is that the method enables us to determine the effects of final demand on output separate from those of initial conditions in additive form. Moreover, knowledge of output at any time t_o plus information on final demand subsequently observed is sufficient to determine completely the time profile of output movement at any time $t>t_o$. Also, the state variable representation of an open Leontief model renders itself readily to an optimal control problem when the final demand vector is replaced by a vector of control variables.

The study of controllability and relative stability of the models described above, and their empirical implementation is not pursued here but it is the subject matter of further investigation.

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