



**AgEcon** SEARCH  
RESEARCH IN AGRICULTURAL & APPLIED ECONOMICS

*The World's Largest Open Access Agricultural & Applied Economics Digital Library*

**This document is discoverable and free to researchers across the globe due to the work of AgEcon Search.**

**Help ensure our sustainability.**

Give to AgEcon Search

AgEcon Search

<http://ageconsearch.umn.edu>

[aesearch@umn.edu](mailto:aesearch@umn.edu)

*Papers downloaded from **AgEcon Search** may be used for non-commercial purposes and personal study only. No other use, including posting to another Internet site, is permitted without permission from the copyright owner (not AgEcon Search), or as allowed under the provisions of Fair Use, U.S. Copyright Act, Title 17 U.S.C.*

## ON OPTIMAL DEVELOPMENT OVER SPACE AND TIME\*

Walter Isard and Panagis Liossatos\*\*

This paper was motivated by several interrelated objectives. First, we wished to develop preliminary definitions of Social Mass, Social Momentum, Social Velocity and Social Acceleration to complement our concept of Social Energy, already defined and found useful for analysis [7]. While we have been able to make some progress in this direction, it will be seen that much ground is yet to be covered, and that at least some of our definitions must be basically revised.

Second, we wish to take first steps toward the development of a parallel treatment of space and time, in keeping with the view previously set forth by the senior author that space-time constitutes one and only one general concept or entity [3].

Third, we wish to place some of our thinking in the framework of field theory, thereby to gain certain new insights and utilize more effectively basic notions which stem from it. This potential advance has already been suggested in a previous paper [4].<sup>1</sup> We shall also see that use of field theory facilitates the parallel treatment of space and time.

Fourth, we wish explicitly to treat space as a continuous variable, partly to begin to meet certain criticisms levelled at regional science because its theories and techniques pertain primarily to discrete sets of points. Treatment of space as a continuous variable is in one sense a natural outgrowth of the introduction of field theory into our framework, since in field theory a change at one point in the field affects the neighboring points which in turn affect their neighboring points and so on in unending fashion. Thus, system interrelationships are expressed in terms of local relationships, that is in terms of partial differential equations which together with appropriate

---

\*Research supported by grant A. P. 00842-02, of the Environmental Protection Agency and grant P251018, National Science Foundation.

\*\*Department of Regional Science, University of Pennsylvania and Regional Science Research Training Program, Center for Urban Development Research, Cornell University.

<sup>1</sup> Since the writing of [4], our attention has been called to the pioneering work of Jutila [9, 10] in using the field theory approach in regional development analysis. He is not concerned with welfare maximization, as we are, but rather with the implications of certain behavioral assumptions pertaining to regional production and consumption activity.

initial and boundary conditions specify the behavior of a whole system over space and time. Our treatment of space as a continuous variable will represent a much more adequate treatment than was accorded space by the senior author in [2], and in certain directions permit the use of more powerful analytical techniques.<sup>2</sup>

Fifth, we wish to take yet another step or two in developing a more adequate dynamic space theory, or general space-time theory for social science.<sup>3</sup>

Lastly, we wish to put to practical use this more general space-time theory by forging new spatial-temporal pollution models more relevant for the problem of environmental management. Such models will represent a further improvement upon those we have already constructed for preliminary consideration [6, 8] and will be developed as a sequel to this paper.

### A Space-Time Model with Capital as a Single Production Factor

To begin, imagine the development of a primitive agriculture in an isolated region wherein there is, loosely speaking, a continuous distribution of population (and labor) eking out subsistence from hunting and gathering of wild fruits. The introduction of this primitive agriculture is sparked by some simple technologic advance, and occurs initially at some point in space and time  $(x, t)$  which we designate  $(0, 0)$ . As a consequence of the successful application of this advance, we have as initial conditions for our model a spatial pattern of consumption of a new agricultural good which falls off very sharply from  $x = 0$ , and a spatial pattern of capital which is also highly concentrated. This primitive agriculture catches on and spreads out in space and grows in time. To facilitate analysis, we posit that the spatial spread is along, and only along, a line, the  $x$ -dimension over some large, but finite, interval  $[0, B]$ . Later we shall generalize and permit spread up to infinity and along the  $y$  and  $z$  dimensions as well. We confine our analysis to the production, investment and consumption of this single new good,  $y$ . Such activities can take place at all points of time within the time interval  $[0, t_1]$ . We assume that only capital is required for production (labor and land being free goods, available in unlimited quantities at all points). Specifically we take production  $Y$  at any point of space-time to be a non-linear function of capital,  $K$ , at that point of space-time:

$$(1) \quad Y(x, t) = F(K(x, t))$$

---

<sup>2</sup>On the other hand we shall return, in later manuscript, to the further development of interregional trade theory with reference to a multi-region system, each region being represented as a point.

<sup>3</sup>For comments on the need for such theory, see [3, 5].

with  $\frac{dF}{dK} > 0$  (positive marginal product) and  $\frac{d^2F}{dK^2} < 0$  (diminishing marginal product), for all  $K > 0$ . For example,

$$Y(x, t) = a \sqrt{K(x, t)}$$

Because we shall be dealing with continuous distribution of production, capital and consumption, all quantities are magnitudes per unit length, or per unit time, or both. Thus in equation  $K(x, t)$  is the capital per unit length at point  $x$  at time  $t$ ,  $Y(x, t)$  is output per unit length per unit time, and  $a$  is a numerical coefficient with dimensions of  $\sqrt{K}/t$ , that is dimensions of  $\sqrt{K}$  per unit of time.

We further posit that output,  $Y$ , at any point in space-time may be allocated to consumption  $C$  at that point, investment  $(\partial K/\partial t) \equiv \dot{K}$  at that point, or net exports  $(\partial U/\partial x) \equiv \dot{U}$ . Here  $U$  represents the flow of goods through a point for use elsewhere for consumption, or investment or both, per unit of time so that, for any unit length  $dx$ ,  $\dot{U}$  measures  $U$  at  $x + dx$  minus  $U$  at  $x$ . It is clear that net exports may be either negative or positive. Accordingly we have

$$(2) \quad C = Y - \dot{K} - \dot{U}$$

Further we postulate that welfare  $\omega$  at any space-time point, per unit of time and length, is a function of consumption  $C$  at that point. Most generally, we have

$$(3) \quad \omega = \omega(C)$$

where we take  $(\partial \omega/\partial C) = \omega_C > 0$ , an assumption of positive marginal utility, and  $(\partial^2 \omega/\partial C^2) = \omega_{CC} < 0$ , an assumption of diminishing marginal utility, for all  $C > 0$ . Over the entire region  $[0, B]$  and for the finite time period from 0 to  $t_1$ , total welfare  $W$  is:

$$(4) \quad W = \int_0^B \int_0^{t_1} \omega dx dt$$

---

<sup>4</sup>In this model we expressly avoid the discounting of utility over time. Although the rationale for discounting utility over time may be defended in terms of a behaving individual, it is not at all clear that it is proper to discount utility for social welfare planning analysis. We have examined discounting in [8]. Because we choose not to discount utility over time, it becomes necessary to consider a finite time interval in order to avoid infinities.

Our objective is to determine the behavior of the system so that  $W$  is maximized. That is, we wish to determine in our defined space-time region the optimal paths of both  $K(x, t)$  and  $U(x, t)$  which we require to be twice continuously differentiable functions and consistent with appropriately specified values on the boundary of the space-time region.

The necessary conditions for maximizing  $W$  are given by the Euler-Lagrange equations [1]:

$$(5) \quad \frac{\partial \omega}{\partial U} = \frac{\partial}{\partial t} \left( \frac{\partial \omega}{\partial \dot{U}} \right) + \frac{\partial}{\partial x} \left( \frac{\partial \omega}{\partial \dot{U}^x} \right)$$

$$(6) \quad \frac{\partial \omega}{\partial K} = \frac{\partial}{\partial t} \left( \frac{\partial \omega}{\partial \dot{K}} \right) + \frac{\partial}{\partial x} \left( \frac{\partial \omega}{\partial \dot{K}^x} \right)$$

where  $\omega$  serves as the Lagrangian density,  $\mathcal{L}$ , and where the  $\dot{\phantom{x}}$  indicates the time derivative and the  $\dot{\phantom{x}}^x$  the spatial derivative. For the specified model, these equations become respectively:

$$(7) \quad 0 = \frac{\partial}{\partial x} \left( \frac{\partial \omega}{\partial C} \frac{\partial C}{\partial \dot{U}^x} \right) = \frac{\partial (-\omega_c)}{\partial x}$$

and

$$(8) \quad \frac{\partial \omega}{\partial C} \frac{\partial C}{\partial Y} \frac{\partial Y}{\partial K} = \frac{\partial}{\partial t} \left( \frac{\partial \omega}{\partial C} \frac{\partial C}{\partial \dot{K}} \right)$$

or

$$(9) \quad \omega_c \frac{\partial F}{\partial K} = \frac{\partial (-\omega_c)}{\partial t}$$

From (7) it follows that  $\omega_c$  is a function of time only; and therefore for any point of time is constant at all locations along the  $x$ -dimension. Likewise for  $C$ . Further, if  $\omega_c$  is a function of time only, so are  $(\partial \omega_c / \partial t)$  and  $(1/\omega_c)(\partial \omega_c / \partial t)$ . Hence from (9) we have

$$(10) \quad \frac{dF}{dK} = \frac{1}{\omega_c} \frac{\partial \omega_c}{\partial t} = G(t)$$

If the marginal product of capital is a function of time only, so must the capital

stock  $K$  be. Therefore, any point of time, both  $dF/dK$  and  $K$  have a constant value for all locations along the  $x$ -dimension. Likewise for  $Y$ .

Also observe that  $(\partial\omega/\partial\dot{U}) = (\partial\omega/\partial C)(\partial C/\partial\dot{U}) = -\omega_C$ , indicating that the loss of welfare at  $(x, t)$  from a unit increase in net exports from  $(x, t)$  equals the marginal utility of consumption at  $(x, t)$ .

Clearly, constant  $Y, K, C$  and  $\omega_C$  for all  $\underline{x}$  for any fixed point of time, including the initial point of time, is not a satisfactory result --- and cannot be said to characterize the realities of settlement dynamics. We therefore are motivated to alter the model so as to yield results more characteristic of reality.

In particular, we replace  $\dot{K}$  by  $(1+n)\dot{K}$  where  $n = n(x)$  is some rapidly increasing function of  $\underline{x}$ , and where  $n(0) = 0$ . We may take this function to reflect the fact that because of difficulties in transportation and communications, the cost (in terms of good  $y$ ) of putting a unit of capital in place rises sharply and increasingly so with distance from the initial point of agricultural development (and the location of know-how). Accordingly, we must replace equation (2) with:

$$(11) \quad C = Y - (1+n)\dot{K} - \dot{U}$$

This change in turn requires that we replace equation (9) with

$$(12) \quad \omega_C \frac{dF}{dK} = \frac{\partial}{\partial t} (-(1+n)\omega_C) = -(1+n) \frac{\partial\omega_C}{\partial t}$$

Note that if we multiply both sides of (12) by  $\underline{dt}$  and integrate from  $\underline{t}$  to  $\underline{t}_1$  we obtain for any fixed location  $\underline{x}$ ,

$$(1+n)\omega_C \Big|_{at \underline{t}} = \int_{\underline{t}}^{\underline{t}_1} \omega_C \frac{dF}{dK} dt + (1+n)\omega_C \Big|_{at \underline{t}_1}$$

Hence, the basic investment principle is revised to take into account the factor  $(1+n)$ . The principle now recognizes that because of the increase in the difficulty of putting capital in place with increase in  $\underline{x}$ , not the full unit but increasingly less than a full unit of a good is put into place when the consumption of a full unit is foregone. Thus to embody a full unit of good into the capital stock, it becomes necessary to take out of consumption, an increasingly greater amount than one unit, with increase in distance of site of investment from  $x = 0$  (as reflected in the factor  $(1+n)$ ). Thus we must equate, for optimality, the utility foregone at  $\underline{t}$  of this increasingly greater amount (i.e.,  $(1+n)\omega_C$ ) with the cumulative sum over time of the utility from one more unit of capital in place at  $\underline{x}$  (i.e.,  $\int_{\underline{t}}^{\underline{t}_1} \omega_C \frac{dF}{dK} dt$ )

plus the utility which is not foregone (and thus saved) at  $t_1$  if we desired to have existing at  $t_1$  that additional unit of capital and did not make the investment at  $t_1$ .<sup>5</sup>

Since by (7),  $\omega_c$  and consequently  $(1/\omega_c)(\partial\omega_c/\partial t)$  depend on time only and are constant over space, we have from (12):

$$(13) \quad \frac{dF}{dK} = (1+n) G(t)^6$$

Now we have posited that the marginal productivity of capital,  $dF/dK$ , and the marginal utility of consumption ( $\omega_c$ ) are positive. Hence it follows from (12) that  $\partial\omega_c/\partial t = \omega_c$  is negative, that is marginal utility at a fixed location declines over time. Since  $\partial\omega_c/\partial t = \omega_{cc} \dot{C}$ , and since we require that  $\omega_{cc} < 0$ , it follows that  $\dot{C} > 0$ , that is consumption at any given location  $x$  grows with time.

Also, since

$$(14) \quad \frac{\partial\omega_c}{\partial x} = \omega_{cc} \dot{C} = 0$$

and since  $\omega_{cc} < 0$ , it follows that  $\dot{C}$  is zero. Thus  $C$  is constant over all space at any fixed point of time, and so its time rate of change is the same for all points of space. By differentiating (13) with respect to  $x$ , we have

$$(15) \quad \frac{d^2F}{dK^2} \frac{x}{K} = \frac{dn(x)}{dx} G(t)$$

Since  $d^2F/dK^2 < 0$  by assumption, and since we have seen that the right hand side of (15) is positive, it follows that  $\frac{x}{K} < 0$ . That is the capital stock decreases with increase in  $x$ . From (1) it also follows that  $\dot{Y} < 0$ , that is that output decreases with increase in  $x$ .

Moreover, if we divide both sides of (12) by  $(1+n)\omega_c$ , we obtain

$$(16) \quad \frac{1}{\omega_c} \frac{\partial\omega_c}{\partial t} = - \frac{1}{(1+n)} \frac{dF}{dK}$$

<sup>5</sup>Note that  $(1+n)\omega_c$  at  $t_1$  has often been viewed as the scrap value of capital.

<sup>6</sup>Note that  $G(t)$  is a function of time at most. Therefore  $dF/dK$  cannot be a constant because  $n(x)$  is a nontrivial function of  $x$ . Hence  $Y = F(K)$  is not permitted to be a linear production function, if the model is to exhibit optimal behavior.

Multiplying both sides of (16) by  $\frac{dt}{1+n}$  and integrating from time  $t_0 = 0$  to time  $t$ , we have:

$$(17) \quad \log \omega_c(t) - \log \omega_c(0) = -\frac{1}{1+n} \int_0^t \frac{dF}{dK} dt$$

or

$$(18) \quad \log \frac{\omega_c(t)}{\omega_c(0)} = -\frac{1}{1+n} \int_0^t \frac{dF}{dK} dt$$

or

$$(19) \quad \omega_c(t) = \omega_c(0) \exp \left( -\frac{1}{1+n} \int_0^t \frac{dF}{dK} dt \right)$$

Further, we can show, as we do in the Appendix, that  $\dot{K}(x, t) > 0$  when appropriate boundary and initial conditions are specified. Hence by equation (1),  $\dot{Y}(x, t) > 0$ .

In sum, we have a space-economy where flows of goods take place to equalize both consumption and marginal utility over all points in space, wherein consumption increases and marginal utility decreases over time at the same rate at all points of space, wherein capital stock and production decrease with increase in  $x$  for any fixed time point, but at all points of space increase over time.

Note however that our model still yields nonsensical results. In particular, we obtain consumption constant over all points of space, including all points up to  $B$ , some very large value short of infinity, for any fixed time point. This result is possible since we have implicitly assumed zero transport costs in our model, which allows the good  $y$  for consumption purposes to flow over all space. We now remove this very undesirable feature of the model by introducing transport cost  $T$  at any space time point, where  $T$  is defined in terms of output used up in transport per unit length per unit time. That is

$$(20) \quad T = \sigma U$$

where  $\sigma$  is a constant over our space-time region. (In later manuscript, we shall consider  $\sigma$  as a function of  $\underline{t}$ ,  $\underline{x}$  and  $U$ ). Accordingly equations (2) and (7) are replaced respectively by

$$(21) \quad c = Y - (1+n) \dot{K} - U - \sigma U$$

and

$$(22) \quad \sigma \omega_c = \frac{\partial \omega_c}{\partial x}$$



The K-equation (12) remains unchanged.

If we now multiply both sides of (22) by  $\frac{dx}{\omega_c}$  and integrate we have:

$$\int \sigma dx = \int \frac{\partial \omega_c}{\partial x} \frac{1}{\omega_c} dx = \int \frac{\partial}{\partial x} (\log \omega_c) dx$$

or

$$(23) \quad \sigma x = \log \omega_c \Big|_{at(x,t)} - \log \omega_c \Big|_{at 0,t}$$

$$\text{or} \quad \frac{\omega_c(x,t)}{\omega_c(0,t)} = \exp(\sigma x)$$

and

$$(24) \quad \omega_c(x,t) = \omega_c(0,t) \exp(\sigma x)$$

Equation (24) implies a marginal utility which increases with  $x$ , the increase being greater, the greater the value of  $\sigma$ . Also, equation (24) together with the assumptions on the welfare function (3) imply that  $C$  decline with increase in  $x$ , this decline being the greater, the greater the value of  $\sigma$ , the transport cost per unit flow.

Equation (22) may be interpreted as indicating that the difference in marginal utility (welfare price) of a unit of a good at two neighboring locations is equal to the transport cost incurred in shipping the good from one to the other. As already noted, this transport cost is associated with an  $\sigma$  fraction of a unit of good being used up in the transportation process per unit of distance traversed. Therefore the social cost of transportation of a unit of good per unit distance is the marginal utility lost,  $\sigma \omega_c$ , through the using up of the  $\sigma$  fraction of a unit of good. But we must bear in mind that the transport function is performed locally, that is the shipment of goods through any location is done by the people at that location. Hence the relevant social cost of transporting a unit of good through any location is  $\sigma$  times the  $\omega_c$  at that location. Accordingly if we now consider the transport cost on a unit of good from any location  $x_1$  to any other location  $x_2$  at some distance, the transport cost on that unit is

$$(25) \quad \int_{x_1}^{x_2} \sigma \omega_c dx$$

But from (22) after multiplying both sides by  $dx$  and integrating, we have:

$$(26) \quad \omega_c(x_2, t) - \omega_c(x_1, t) = \int_{x_1}^{x_2} \sigma \omega_c dx$$

That is the difference between the welfare prices at locations  $x_2$  and  $x_1$  is equal to the transport cost on the unit of good, a familiar spatial equilibrium principle.

Put in another way, equation (22) gives for any point of time the slope and the curvature of  $\omega_c$  consistent with optimization.<sup>7</sup>

Now we next wish to find  $\omega_c(0, t)$ . To do so we substitute in equation (12) the expression for  $\omega_c(x, t)$  from equation (24) to derive

$$(27) \quad \omega_c(0, t) \frac{dF}{dK} = - (1+n) \frac{d}{dt} (\omega_c(0, t))$$

Note that a solution of the form of equation (24) satisfies (12) only if

$$(28) \quad \frac{dF}{dK} \left( \frac{1}{1+n} \right) = J(t)$$

That is:

$$(29) \quad \frac{\partial}{\partial x} \left[ \frac{dF}{dK} \left( \frac{1}{1+n} \right) \right] = 0$$

This relation is a consequence of our differentiability assumptions which together with equations (12) and (24) imply that  $\omega_c$  and  $\dot{\omega}_c$  must be continuously differentiable and hence that

$$\frac{\partial^2 \omega_c}{\partial x \partial t} = \frac{\partial^2 \omega_c}{\partial t \partial x}$$

It follows from (29) that

---

<sup>7</sup>The curvature of the equation is obtained by differentiating equation (22)

with respect to  $x$ . We obtain  $\frac{\partial^2 \omega_c}{\partial x^2} = \sigma \frac{\partial \omega_c}{\partial x} = \sigma^2 \omega_c$

$$(30) \quad \frac{d^2 F}{dK^2} \times \left( \frac{1}{1+n} \right) - \frac{dF}{dK} \left( \frac{1}{1+n} \right)^2 \frac{dn}{dx} = 0$$

or

$$(31) \quad \frac{x}{K} = \frac{\left( \frac{dF}{dK} \right) \frac{dn}{dx}}{\frac{d^2 F}{dK^2} (1+n)}$$

Since  $\frac{dF}{dK} > 0$ ,  $\frac{dn}{dx} > 0$ ,  $n \geq 0$  and  $\frac{d^2 F}{dK^2} < 0$ ,

$$(32) \quad \frac{x}{K} < 0.$$

So capital stock and thus production at any fixed point of time decreases with increase in  $x$ .

Just as we obtained equation (19) as a solution to equation (12), we obtain as a solution to equation (27):

$$(33) \quad \omega_c(0, t) = \omega_c(0, 0) \exp \left( - \frac{1}{1+n} \int_0^t \frac{dF}{dK} dt \right)$$

Inserting the value of  $\omega_c(0, t)$  from (33) into (24), we obtain

$$(34) \quad \omega_c(x, t) = \omega_c(0, 0) \exp \left( \sigma x - \frac{1}{1+n} \int_0^t \frac{dF}{dK} dt \right)$$

From (34) we observe that for any fixed point of  $x$ ,  $\omega_c$  declines with time; and hence consumption increases with time, i.e.,  $\dot{C} > 0$ .

Further, we show in the Appendix that  $\dot{K} > 0$  which implies that  $\dot{Y} > 0$  and that for every point of space, capital stock and output increase with time.

At this point it is relevant to reiterate that appropriate boundary and initial conditions are required in order to specify a unique solution. We have taken as initial conditions: (i) capital stock at the initial space-time point  $(0, 0)$  is a given positive constant, i.e.,  $K(0, 0) = \tau$ ; (ii) addition to capital stock at all space points at the end of the time period (planning horizon)  $t_1$  is zero, i.e.,  $\dot{K}(x, t_1) = 0$ ; (iii) consumption at the initial space-time point  $(0, 0)$  is another given positive constant, i.e.,  $C(0, 0) = \bar{c}$ .

We also take as a boundary condition (iv) zero flows at the space boundaries, that is  $U(0, t) = U(\infty, t) = 0$  where our region is the infinite interval  $[0, \infty)$ . This condition implies that net exports for the system are zero, that is

$$\int_0^{\infty} U \, dx = 0$$

and consequently that net exports are positive for some interior points, and negative for others, and that  $U$  reaches a maximum at some interior point where net exports are zero. We intuitively suspect that  $U$  reaches a single maximum, but we have not yet proven this point. However since we cannot specify  $\dot{U} > 0$ , we can only guess at the shape of the  $U$  curve for different time points. We may now depict our space-economy growing over time with the following rough graphs (Figures 1 through 4).

### Social Mass: A Definition

As already indicated we have developed and found useful for analytical purposes an "exploratory" concept of Social Energy. We now try to develop the parallel concepts of Social Momentum, Social Acceleration and Social Mass, and seek relevant definitions, although with further research we may wish to discard our initial definitions for others which may prove to be more meaningful and precise.

The model of the previous section allows us to proceed some distance in this direction. The solution (34) depicts a wave-type phenomenon, as can be seen from Figure 1. Put otherwise, at time  $t = 0$ , and beginning at space-point  $x = 0$ , the curve describing the behavior of  $\omega_c$  is shifted horizontally to describe at  $t'$  the behavior of  $\omega_c$  beginning at space-point  $x'$ ; to describe at  $t''$  the behavior of  $\omega_c$  beginning at space-point  $x''$ ; etc. Thus it follows that

$$(35) \quad \omega_c(x, t) = \omega_c(x'', t'') = \omega_c(x + \Delta x, t + \Delta t)$$

where the limit of  $\Delta x/\Delta t$  (as  $\Delta t \rightarrow 0$ )  $\equiv dx/dt$  defined the speed of propagation of the wave with reference to  $\omega_c$ . That is, given a point  $(x, t)$ , we want appropriate  $\underline{dx}$  and  $\underline{dt}$  so that the value of  $\omega_c(x + dx, t + dt) = \omega_c(x, t)$ . Hence there must be no change in the exponent in (34) associated with the changes  $\underline{dx}$  and  $\underline{dt}$ .

That is

$$(36) \quad d \left( \sigma_x - \frac{1}{1+n} \int_0^t \frac{dF}{dK} dt \right) = 0$$

Recalling that  $\frac{1}{1+n} \int_0^t \frac{dF}{dK} dt$  is independent of  $\underline{x}$ , and that  $\sigma$  and  $n$  are independent of  $t$ , we have

$$(37) \quad \mathcal{L} dx - \frac{1}{1+n} \frac{\partial \int_0^t \frac{dF}{dK} dt}{\partial t} = 0$$

or

$$(38) \quad \frac{dx}{dt} = \frac{1}{\mathcal{L}(1+n)} \frac{dF}{dK}$$

Here  $dx/dt$  represents the speed of propagation of  $\omega_C$  (that is of any value of  $\omega_C$ ). As we would anticipate, this speed varies inversely with both  $\mathcal{L}$ , the transport rate, and  $n$ ,<sup>8</sup> the difficulty of capital construction at a distance, and varies directly with the marginal productivity of capital --- which we shall see to be the momentum of the system.

The rationale for this relation follows because, if there were zero transport cost, (i.e.,  $\mathcal{L} = 0$ ), and no differences among locations in the difficulty of capital construction, then to maximize welfare under conditions of diminishing marginal utility requires an even distribution over space of consumption up to some very large distance short of infinity. Such would require an infinite speed of propagation, which (38) yields when  $\mathcal{L} = 0$  and  $n = 0$ . However when  $\mathcal{L} > 0$ , such an even distribution is precluded.

Observe also that since  $\omega_C = \omega_C(C)$ , and since  $\omega_{CC} < 0$ , we can invert and consider  $C$  as a function of  $\omega_C$ . But if at  $(x', t')$  and  $(x'', t'')$ ,  $\omega_C$  takes the same value as at  $(x, t)$ , then so must  $C$ . Thus  $C$  propagates as a wave just as  $\omega_C$ , with the same speed of propagation. Likewise with  $\omega$ , social welfare, since  $\omega = \omega(C)$ .

We do not examine the speed of propagation of  $K$  and  $U$  here because they involve much more complex relationships. However we can state that  $K$  does follow a wave-like behavior since  $\ddot{K}$  and  $\dot{K}$  are nonzero. Further,  $K$  propagates to the right since  $\ddot{K}$  and  $\dot{K}$  have opposite signs such that we can find a positive speed

$$(39) \quad \frac{dx}{dt} = -\frac{\dot{K}}{\ddot{K}}$$

for which

$$(40) \quad \ddot{K} dx + \dot{K} dt \equiv dK(x, t) = 0$$

---

<sup>8</sup> In this particular case where  $n$  is independent of time, the speed of propagation is independent of  $n$  since  $\frac{1}{1+n} \frac{dF(K(x, t))}{dK} = \frac{1}{1+n(0)} \frac{dF(K(0, t))}{dK} = \frac{dF(K(0, t))}{dK}$

From this definition of speed of propagation of  $\omega_c$  we wish to move on to the definition of mass density at a point. In doing so, we conceive mass density at a point to measure the resistance at that point to change in the speed,  $dx/dt$  of the "wave". Hence we have in mind a concept of "inertial" mass --- the basic attribute of mass in the physical sciences --- and not a concept which necessarily describes a quantity of matter. Generally speaking we seek relations of the form:

$$(41) \quad m_a = F$$

where  $m$  = mass density,  $a$  = acceleration (the second partial with respect to time of the basic configuration or independent variables) and where  $F$  is the "driving force" per unit length.

Specifically, we consider (38), which differentiated with respect to time yields:

$$(42) \quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \frac{1}{(1+n)} \frac{d^2F}{dK^2} \frac{\partial K}{\partial t} \equiv \frac{1}{\sigma(1+n)} \frac{\partial}{\partial t} \left( \frac{dF}{dK} \right)$$

or

$$(43) \quad \sigma(1+n)a = \frac{\partial}{\partial t} \left( \frac{dF}{dK} \right)$$

Immediately, since we recognize  $\partial(dF/dK)/\partial t$  as the time rate of change of marginal productivity, and thus as the driving force, and since  $a = dv/dt$  is the time rate of change of motion of the system, equation (43) suggests that  $\sigma(1+n)$  is an appropriate indicator of mass density --- and of resistance to change in the motion of this system. This suggestion is fully consistent with the model since both the transport rate  $\sigma$  and the factor  $(1+n)$  reflecting the increasing difficulty of putting capital in place with increase in  $x$ , are the two elements, and the only two elements, which preclude the instantaneous and even spread of development (consumption and investment, the basic magnitudes of the system) along the real line up to  $\infty$ . (Recall that if  $n = 0$ ,  $K$  is <sup>9</sup> uniformly distributed; and if  $\sigma = 0$ , consumption is uniformly distributed).

---

<sup>9</sup>Alternatively we may rewrite equation (43) to read:  $\sigma a = \frac{\partial}{\partial t} \left( \frac{1}{1+n} \frac{dF}{dK} \right)$  and view  $\frac{\partial}{\partial t} \left( \frac{1}{1+n} \frac{dF}{dK} \right)$  as the driving force with  $\sigma$  as an appropriate mass density. Since  $\frac{\partial}{\partial t} \left( \frac{1}{1+n} \frac{dF(K(x,t))}{dK} \right) = \frac{\partial}{\partial t} \left( \frac{dF(K(0,t))}{dK} \right)$  such a formulation then emphasizes the marginal productivity of capital at the zero-space point ( $x = 0$ ), as the basic reference magnitude. However, this formulation just yields the propelling force at  $x = 0$ , and to obtain the propelling force at  $x = x'$ , we must multiply  $\frac{\partial}{\partial t} \left( \frac{dF(K(0,t))}{dK} \right)$  by  $(1+n)$ ; and accordingly  $\sigma$  must be multiplied by  $(1+n)$  to get the corresponding "inertial" measure at  $x'$ . Hence we prefer the direct formulations and definitions in the text.

Also from the natural concept of speed of propagation of  $\omega_c$ ,  $C$ , and  $\omega$ , we can proceed to a concept of momentum. From equation (38) we have

$$(44) \quad \sigma(1+n) \frac{dx}{dt} = \frac{dF}{dK}$$

Since  $\sigma(1+n)$  represent mass density  $\rho$  and  $\frac{dx}{dt}$  velocity  $v$ , we have the familiar definition of physics

$$(45) \quad \rho v = \text{momentum density} = \frac{dF}{dK}$$

Hence given our definition of mass density, momentum density corresponds to marginal productivity, a meaningful concept; and Social Momentum for any finite interval  $[0, B]$  may be defined as

$$\int_0^B \frac{dF}{dK} dx$$

the cumulative sum of marginal productivity from 0 to  $B$  at a given point of time. Correspondingly, from equation (43) we may define Social Acceleration as the acceleration

$$a \equiv \frac{d^2x}{dt^2}$$

of the "wave" motion. Further, the driving force for this acceleration (in our case deceleration) at a given space-time point is the rate of change over time of the marginal productivity at any fixed space-point. When marginal productivity is constant, Social Acceleration is zero, and momentum density and Social Momentum are constant.

We can also derive other insightful relations. If we differentiate the basic equations (12) with respect to time and (19) with respect to space we obtain respectively:

$$(45a) \quad \frac{\partial^2 \omega_c}{\partial t^2} = - \left( \frac{dF}{dK} \right) \frac{\partial \omega_c}{\partial t} - \frac{1}{1+n} \frac{d^2 F}{dK^2} \dot{K} \omega_c$$

and

$$(45b) \quad \frac{\partial^2 \omega_c}{\partial x^2} = \sigma \frac{\partial \omega_c}{\partial x}$$

Substituting in (45a) the value of  $\partial \omega_c / \partial t$  from (12), and in (45b) the value of  $\partial \omega_c / \partial x$  from (19), we obtain respectively:

$$(46) \quad \frac{\partial^2 \omega_c}{\partial t^2} = \left( \frac{dF}{dK} \right)^2 \omega_c - \frac{1}{1+n} \frac{d^2 F}{dK^2} \dot{K} \omega_c$$

and

$$(47) \quad \frac{\partial^2 \omega_c}{\partial x^2} = \sigma^2 \omega_c$$

Multiplying both sides of (46) by  $\sigma^2 (1+n)^2$  and both sides of (47) by  $\left(\frac{dF}{dK}\right)^2$  and combining yields:

$$(48) \quad [\sigma (1+n)]^2 \frac{\partial^2 \omega_c}{\partial t^2} = \left(\frac{dF}{dK}\right)^2 \frac{\partial^2 \omega_c}{\partial x^2} - (1+n) \sigma^2 \frac{d^2 F}{dK^2} \dot{K} \omega_c$$

In equation (48) the term  $\partial^2 \omega_c / \partial t^2$  may be viewed as acceleration of  $\omega_c$  (the marginal utility of a good). Accordingly then  $[\sigma(1+n)]^2$  might be considered to be a "quasi-mass" associated with the  $\omega_c$  movement, and to be a measure of the resistance to change of  $\omega_c$ . The two terms on the right hand side might then be viewed as constituting the "net" force, the first term being a gross force, the second being an adjustment due to diminishing marginal productivity of capital.

While  $\sigma$  and  $(1+n)$  constitute the factors or inertial properties which we use to define mass, we may note that there are still other properties of the system which may be considered to be inertial. For example, our assumption that  $\omega_{CC} < 0$ , implies that within the value system (welfare function) there is a "loss" of value per unit of commodity consumed, as consumption increases. This might be interpreted by some as an inertial property. And accordingly they might take the measure of mass to be  $-\sigma(1+n) \omega_{CC}$ . The relevance of this measure is clearly seen when we consider the behavior of consumption. We have from (11) and (22)

$$(49) \quad \frac{\dot{\omega}_c}{\omega_c} = - \frac{\frac{dF}{dK}}{\sigma(1+n)}$$

But since  $\omega_c = \omega_{CC} \dot{C}$  and  $\omega_c^x = \omega_{CC} \dot{C}^x$  and since  $\omega_{CC} < 0$ , we obtain



$$(50) \quad \dot{C} = \frac{\frac{dF}{dK}}{\sigma(1+n)} \dot{x} \sqrt{V}$$

If we assume that  $w_{ccc}$  exists and is continuous, then  $C$  is twice continuously differentiable and hence  $\dot{C}^x = \dot{C}^x$ . We can then differentiate equation (50) first with respect to time, and secondly with respect to  $x$ . Combining we obtain:

$$(53) \quad \sigma(1+n)(-w_{ccc}) \ddot{C} = -w_{cc} \left(\frac{dF}{dK}\right)^2 \frac{1}{\sigma(1+n)} \dot{C}^{xx} + \frac{d^2F}{dK^2} \dot{K} w_c$$

Equation (53) can be seen to have the form  $ma = F$  where the mass is  $\sigma(1+n)(-w_{ccc})$ , where the acceleration  $\ddot{C}$  relates to consumption, and where the Force  $F$  relates to the space curvature of  $C$ , namely  $\dot{C}^{xx}$ , in the first term, and to the diminishing marginal productivity of capital in the second term.

Further we might wish to introduce a factor in our model which relates to "loss" or "inertia" or "resistance to change" associated with changes in the rate of investment,  $\dot{K}$ . For example, we might imagine that at any point of space-time, the economy is "adjusted" to the absorption of new investment at the current rate  $\dot{K}$ . Any increase in this rate involves difficulties (associated with a cost or loss) --- difficulties of digestion and absorption into the economy; any decrease in the rate increases the efficiency with which the new investment is absorbed and assimilated into the system. So to equation (26) we might add the term  $-\gamma \dot{K}$  to obtain

<sup>10</sup>Note that this equation can be interpreted as the local conservation of consumption. We have seen that the value of  $C$  must propagate for optimal behavior with velocity  $v = (dF/dK)(1/\sigma(1+n))$ . Since  $C$  is really a consumption density (consumption per unit length),  $Cv$  represents the flow of consumption through the point  $x$  at time  $t$ . Further since  $\partial v/\partial x = 0$ , we may rewrite (50) as

$$(51) \quad \frac{\partial C}{\partial t} + \frac{\partial(Cv)}{\partial x} = 0,$$

or by considering a small interval  $(x, x + dx)$ , we have

$$(52) \quad \frac{\partial(Cdx)}{\partial t} = vC(x, t) - vC(x + dx, t)$$

which says that the time change of consumption within the length  $dx$  is the difference between the flow  $vC(x, t)$  which enters the interval and  $vC(x + dx, t)$  which leaves the interval.

$$(54) \quad C = Y - (1+n) \dot{K} - \dot{U} - \sigma U - \gamma \ddot{K}$$

Doing so would imply a still more complicated notion of mass involving the  $\sigma$  factor.

However, note that the assumption of existence of diminishing marginal productivity of capital must not be viewed as an inertial property. This point is seen immediately from equation (43) where it is clear that if  $dF/dK$  is constant, then  $\sigma(1+n) a = 0$  which implies for  $\sigma \neq 0$  a constant speed of wave propagation. On the other hand, if  $dF/dK \neq 0$ , with  $d^2F/dK^2 < 0$ , the diminishing marginal productivity of capital provides the basis for the deceleration of the wave propagation. As can be seen from (53) it is also a factor in the deceleration of consumption.

Finally, with respect to the energy of the system, it should be noted that our basic Lagrangian expression implied by equation (43) (which may be interpreted as a Lagrange equation) is

$$(55) \quad \mathcal{L} = \frac{1}{2} v^2 + \frac{1}{\sigma} \int_0^x \frac{1}{1+n} \frac{\partial}{\partial t} \left( \frac{dF}{dK} \right) dx$$

This equation pertains to a particle of unit mass moving with velocity  $v$  and subject to a force field reflected in the second term. As such it has no relevance to our social problem. On the other hand our concept of Social Energy density [7]

$$(56) \quad \mathcal{H} = \omega + (1+n) \omega_c \dot{K}$$

(which is the density of actual utility from consumption plus utility embodied in the investment at point  $(x, t)$ ) does not have any direct and meaningful relationship to our concept of mass,  $\sigma(1+n)$ . This finding is to be expected, because if we seek a parallel in physics, the interconnection between energy and mass results from more restrictive structural assumptions, parallels of which we have not yet incorporated in our model. More specifically, it is to be recalled that the well-known Einstein relationship  $E = mc^2$  follows from the postulate of Lorentz invariance of the physical theories as well as from the postulate that  $c$  is the upper bound for the speed of propagation of any action.

On the other hand, the  $\mathcal{H}(x, t)$  satisfies a useful continuity equation. As demonstrated in the Appendix (see equations A.1 to A.5),

$$(57) \quad \frac{\partial \mathcal{H}}{\partial t} + \frac{\partial (\omega_c \dot{U})}{\partial x} = 0$$

This equation states that at any space-time point the time rate of change in social energy (actual utility from consumption plus embodied utility from

investment) must equal the net utility value of the time rate of change of imports. Here  $\omega_C \dot{U}$  represents the flow of social energy through the point  $x$  at time  $t$ .

When we multiply both terms of (57) by  $dx$  and integrate, we obtain

$$(58) \quad \frac{\partial}{\partial t} \int_0^{\infty} \mathcal{H} dx = \omega_C (0, t) \dot{U} (0, t) - \omega_C (\infty, t) \dot{U} (\infty, t) = 0$$

since  $\dot{U} (0, t), \dot{U} (\infty, t) = 0$  by the boundary conditions  $U (0, t), U (\infty, t) = 0$ . This then implies that system social energy

$$H \equiv \int_0^{\infty} \mathcal{H} dx$$

is a constant; and the system is closed. Hence we may depict system social energy, and system welfare as in our articles dealing with a one-point space economy [7, 8].

### Conclusions

In bringing this paper to a close, we wish to emphasize that the elementary model which has been developed does set forth space-time as a single general concept, unifying the notions of both space and time. While it is still possible to reduce the model and the analysis to an economy developing over time at a single point in space, or to an economy at a single time point distributed over space, the basic contribution of the model is the integration of the development process over both time and space. Both space and time are treated on an equal basis, and concomitantly, so that we do not need to abstract from one or the other.

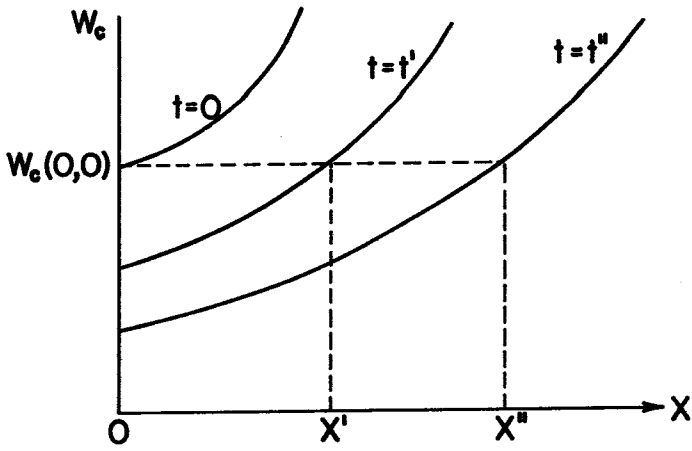
We make this statement, despite the fact that we have considered only the  $x$ -dimension in the positive quadrant, since the introduction of other spatial dimensions simply complicates the algebra at this stage.

Interconnected with our concomitant treatment of space and time has been field theory. That is, our system is not described, as many region and multi-region systems are, by a finite number of functions of time; rather it is described by a set of functions of both space and time. Thus our system has infinite degrees of freedom. Use of field theory opens the way for the introduction of relativistic considerations which will be considered in later manuscript.

More important we are able to treat space as a continuous variable, and thereby begin to consider welfare analysis and development processes in contexts which are not subject to the criticisms levelled at techniques treating discrete sets of one-point space economies. Already some basic principles have emerged. In connection with the K-equation, we have a new statement of the investment principle: At any space-time point goods should be invested as capital up to the point where the marginal utility foregone from not consuming the required units of goods in order to put a unit of capital in place at that space-time point just equals the cumulative sum (over the remaining points of time in the relevant planning horizon) of the utility from the additional products due to that unit of capital plus the utility that is saved by not having to use up consumption goods at the end of the planning horizon in order to have that unit of capital then. In connection with the U-equation, we have a new statement of the spatial flow (interregional trade) principle: At any and every time point, the difference in the marginal utility (welfare price) of a good at two locations in space must equal the social transport cost between those points, this cost being the cumulative sum of the utilities foregone in providing transport for a unit of good through each location in the space interval over which the good is transported. These principles plus a statement of inertial conditions yield a development process over space and time, as depicted by Figures 1 to 4, wherein the development process at any one space-time point is interconnected with the development process at all other points.

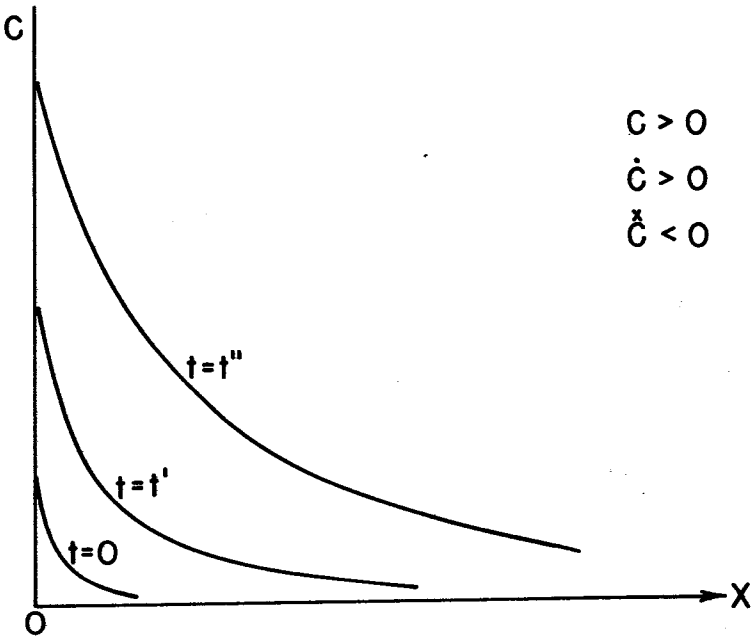
Finally, we must admit that we have not moved as far ahead as we would have liked in developing an integrated set of definitions of Social Energy, Social Mass, Social Momentum and Social Acceleration for reasons already stated. This integration is to be attempted once again in subsequent manuscript.

Against the background of this paper, a number of further advances and refinements are suggested. Labor and population should be introduced, as well as pollution and differential fertility. Additionally, a multi-commodity framework and variation of  $\sigma$  with space, time and volume of flow should be considered, as well as variation of  $n$  with time. We shall consider some of these factors in subsequent manuscript.



- $W_c > 0$
- $\dot{W}_c < 0$
- $\ddot{W}_c > 0$
- $W_{cc} < 0$
- $\ddot{W}_c > 0$
- $\ddot{W}_c > 0$

Figure 1



- $C > 0$
- $\dot{C} > 0$
- $\ddot{C} < 0$

Figure 2

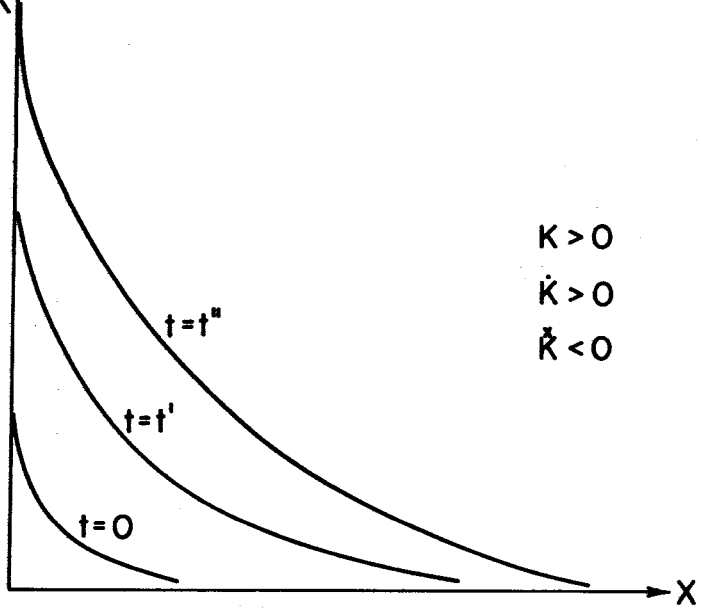


Figure 3

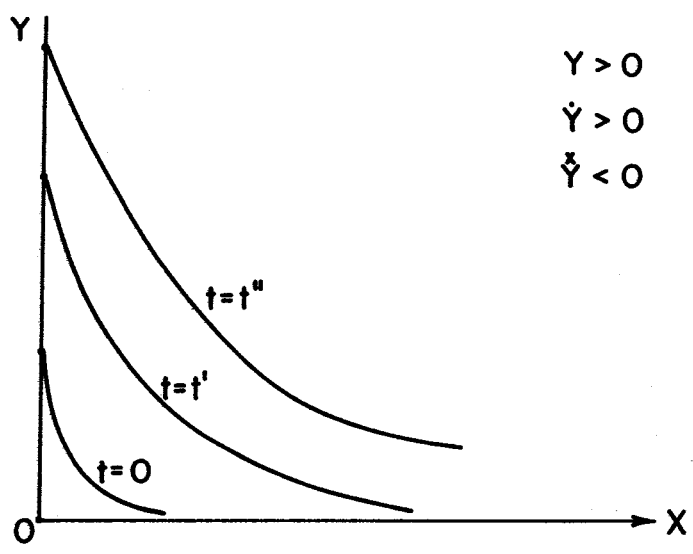


Figure 4

APPENDIX

We wish to prove that  $\dot{K}(x, t) \geq 0$ . Consider the Hamiltonian density:

$$(A.1) \quad \mathcal{H}(x, t) = \omega(C(x, t)) + (1+n)\omega_C(x, t)\dot{K}(x, t)$$

Differentiating with respect to time we obtain

$$(A.2) \quad \frac{\partial \mathcal{H}}{\partial t} = \omega_C \dot{C} + (1+n)\dot{\omega}_C \dot{K} + (1+n)\omega_C \ddot{K}$$

Substituting in values from (12) and using equations (1) and (21) after taking their time derivative, we have

$$(A.3) \quad \begin{aligned} \frac{\partial \mathcal{H}}{\partial t} &= \omega_C \left[ \frac{dF}{dK} \dot{K} - (1+n)\ddot{K} - \dot{X}\dot{U} - \sigma\dot{U} - \frac{dF}{dK} \dot{K} + (1+n)\ddot{K} \right] \\ &= -\omega_C [\sigma\dot{U} + \dot{X}\dot{U}] \end{aligned}$$

Substituting the value of  $\sigma\omega_C$  from (22) we obtain

$$(A.4) \quad \frac{\partial \mathcal{H}}{\partial t} = -\omega_C \dot{U} - \omega_C \dot{X}\dot{U} = -\frac{\partial}{\partial X}(\omega_C \dot{U})$$

or

$$(A.5) \quad \frac{\partial \mathcal{H}}{\partial t} + \frac{\partial}{\partial X}(\omega_C \dot{U}) = 0$$

Since we have shown in the text that

$$(A.6) \quad \begin{aligned} H &\equiv \int_0^{\infty} \mathcal{H} dx = \int_0^{\infty} \omega(C(x, t)) + \int_0^{\infty} (1+n)\omega_C(x, t)\dot{K}(x, t) \\ &= \text{a constant,} \end{aligned}$$

we have, using A.1,

$$(A.7) \quad \dot{H}(t) = \frac{d}{dt} \int_0^{\infty} \omega dx + \frac{d}{dt} \int_0^{\infty} (1+n)\omega_C \dot{K} dx = 0$$

Since  $\omega = \omega_C C > 0$ , it follows from A.7 that

$$(A.8) \quad \frac{d}{dt} \int_0^{\infty} (1+n) \omega_c K dx < 0$$

Imposing the end condition

$$\dot{K}(x, t_1) = 0 \text{ for all } 0 \leq x < \infty$$

equation (A.8) implies that

$$(A.9) \quad \int_0^{\infty} (1+n) \omega_c \dot{K} dx > 0 \text{ when } 0 \leq t < t_1$$

Now from (28) and (29) we have for any x:

$$(A.10) \quad \frac{1}{1+n(x)} \frac{dF(K(x,t))}{dK} = \frac{dF(K(0,t))}{dK}$$

Differentiating with respect to time yields:

$$(A.11) \quad \frac{1}{1+n(x)} \frac{d^2F(K(x,t))}{dK^2} \dot{K}(x, t) = \frac{d^2F(K(0,t))}{dK^2} \dot{K}(0, t)$$

Since  $d^2F/dK^2 < 0$  for all positive values of  $K$ , it follows that  $\dot{K}(x, t) > 0$  if and only if  $\dot{K}(0, t) > 0$ . But by replacing  $\dot{K}(x, t)$  in A.9 by its value in A.11, it must be that  $\dot{K}(0, t) > 0$  and hence  $K(x, t) > 0$  for  $t < t_1$ .

We wish to demonstrate that our initial and boundary conditions along with equations (12) and (22) are in principle sufficient to specify  $K(x, t)$  and  $U(x, t)$  over our space-time region.

Recalling equation (34) and bearing in mind A.10, we have:

$$(A.12) \quad \omega_c(x, t) = \omega_c(0, 0) \exp(\sigma x - \int_0^t \frac{dF(K(0,t'))}{dK} dt')$$

Since  $\omega_{cc} < 0$ , A.12 specifies  $C(x, t)$  within an arbitrary constant  $\omega_c(0, 0)$ . Hence we may write

$$(A.13) \quad C(x, t) = \phi \left( \omega_c(0, 0) \exp(\sigma x - \int_0^t \frac{dF(K(0,t'))}{dK} dt') \right)$$

Now from A.10, we can expect that

$$(A.14) \quad K(x, t) = h(x) K(0, t)$$



when  $h(x)$  is some specified function of  $1+n(x)$  and when  $h(0) = 1$ . Then from A.10, A.12 and A.13, we may view equation A.6 as a functional relation of the form

$$(A.15) \quad \mathcal{G} \left( K(0, t), \int_0^t \frac{dF(K(0, t'))}{dK} dt'; H, \omega_c(0, 0) \right) = 0$$

We can easily see through the transformation

$$(A.16) \quad \mathcal{X}(t) = \int_0^t \frac{dF(K(0, t'))}{dK} dt'$$

that (A.15) is a second-order differential equation with respect to  $\mathcal{X}$ . Thus (A.15) determines  $K(0, t)$ , in principle, within two additional arbitrary constants,  $Q_1$ , and  $Q_2$ . That is,

$$(A.17) \quad K(0, t) = \Psi(t; Q_1, Q_2, H, \omega_c(0, 0)) \text{ with a relation } G(Q_1, Q_2, H, \omega_c(0, 0)) = 0 \text{ since } \mathcal{X}(0) = 0.$$

Now from equation (21) we have

$$(A.18) \quad \ddot{U} + \sigma U = F(K) - (1+n) \dot{K} - C$$

It follows from (A.13), (A.14) and (A.17) that the right-hand side of (A.18) is a known function of  $x$  and  $t$ , and the four arbitrary constants  $Q_1, Q_2, H$  and  $\omega_c(0, 0)$ . Then the solution to (A.18) is

$$(A.19) \quad U(x, t) = \exp(-\sigma x) \left[ U(0, t) + \int_0^x [F(K) - (1+n) \dot{K} - C] \exp(\sigma x') dx' \right]$$

Since  $U(0, t)$  is specified in the text to be zero (A.19) yields

$$(A.20) \quad U(x, t) = g(x, t; Q_1, Q_2, H, \omega_c(0, 0))$$

Thus the solutions (A.17) and (A.20) are specified within four arbitrary constants. These four constants must satisfy the following:

$$(A.21) \quad \Psi(0; Q_1, Q_2, H, \omega_c(0, 0)) = K(0, 0)$$

$$(A.22) \quad \dot{\Psi}(t_1; Q_1, Q_2, H, \omega_c(0, 0)) = \dot{K}(0, t_1) = 0$$

---

<sup>11</sup> For example if we take  $F(K) = bK^\alpha$ , then  $h(x) = \frac{1}{(1+n(x))^{1-\alpha}}$

$$(A.23) \quad G(Q_1, Q_2, H, \omega_c(0, 0)) = 0$$

By prescribing  $K(0, 0) = \tau$ ;  $\dot{K}(0, t_1) = 0$ ; and  $C(0, 0) = \bar{\tau}$  (which determines  $\omega_c(0, 0)$ ) --- equations (A.21), (A.22) and (A.23) determine in principle  $Q_1$ ,  $Q_2$ , and  $H$ . The solutions (A.17) and (A.20) are then fully determined.

Finally selecting  $K(0, 0)$  and  $C(0, 0)$  positive, ensures that  $K(x, t)$  and  $C(x, t)$  are positive definite through our entire space-time region. We are not yet able to make a similar statement for  $U(x, t)$ .

## REFERENCES

1. Hildebrand, Francis B. Methods of Applied Mathematics. Englewood Cliffs, New Jersey: Prentice Hall, Inc., 1965.
2. Isard, W. Location and Space-Economy. Cambridge, Massachusetts: M.I.T. Press, 1956.
3. Isard, W. "On Notions and Models of Time," Papers, Regional Science Association, 25 (1970), 7-31.
4. Isard, W. "On Relativity Theory and Space-Time Models," Papers, Regional Science Association, 26 (1971).
5. Isard, W., et. al. General Theory: Social, Political, Economic and Regional. Cambridge, Massachusetts: The M.I.T. Press, 1969, Chapter 16.
6. Isard, W. and P. Liossatos. "On Location Analysis for Urban and Regional Growth Situations," Annals of Regional Science 4 (June 1972), 1-27.
7. Isard, W. and P. Liossatos. "Social Energy: A Relevant New Concept for Regional Science?" London Studies in Regional Science, 3 (1972).
8. Isard, W. and P. Liossatos. "Some Simple Air Pollution, Urban-Regional Growth Models," Papers, Regional Science Association, 29 (1972).
9. Jutila, S. T. "A Linear Model for Agglomeration, Diffusion and Growth of Regional Economic Activity," Regional Science Perspectives, 1 (1971).
10. Jutila, S. T. "Dynamics of Regional Economic Development," Regional Science Perspectives, 2 (1972).