Exact Measures of Income
in Two Capital-Resource Economies

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Abstract.   Exact optimal paths are calculated for two closed, continuous-time economies with explicit functional forms for utility from consumption, and for production from human-made capital and a non-renewable resource. Features of the first economy are non-linear utility, hyperbolic utility discounting and (possibly) hyperbolic technical progress. In it: (a) welfare-equivalent income > wealth-equivalent income > Sefton-Weale income > Net National Product, confirming that even if income is viewed only as a measure of prosperity, there is no point in trying to define it uniquely; (b) the Solow (1974) constant consumption path is a special case for a particular discount rate; (c) for a low enough discount rate, sustained growth is optimal even when technical progress is zero. The second economy has linear utility, a non-linear output split between consumption and investment, and exponential technical progress. In it, (a) Weitzman's (1997) technological progress premium works only if an upwards correction factor is first applied to the rate of progress in production, to convert it to a rate of progress in Net National Product; (b) Hartwick's rule has an unfamiliar form.

1. Introduction

This note derives exact formulae for the optimal development paths which maximise the present value of utility in two economies with explicit functional forms. Both economies are closed, deterministic, have constant population, and a representative agent. In both, there are three inputs to production: the stock of human-made capital, the depletion of a finite stock of a non-renewable resource, and time in the form of an exogenous technical progress factor (which may be constant as a special case). Distinguishing features in the first economy are that both the utility discount factor and the technical progress factor are hyperbolic rather than exponential functions of time, so it will be called the Hyperbolic economy below. In the second, the division of output between consumption and investment goods is non-linear, so it will be called the NLO (Non-Linear Output) economy. These economies are thus in the tradition of Dasgupta and Heal (1974), Solow (1974) and Stiglitz (1974), but with some new twists.

Naturally, because of their explicit functional forms, no new general theory can be derived from these economies. But their value is the way they illustrate yet reveal the often limited generality of existing theories, and suggest some new lines of enquiry. The Hyperbolic economy confirms that income is impossible to define uniquely. It also shows that the constant consumption path of Solow (1974) is a special case of a somewhat more general, present-value-maximising economy. Moreover, unlike in a Dasgupta-Heal economy, forever growing consumption can be optimal in the Hyperbolic economy, even when technical progress is absent, as long as the discount rate is low enough. The NLO economy shows that the technical progress premium (TPP), the excess of wealth-equivalent income over net national product, does not follow the formula in Weitzman (1997) unless the rate of technical progress is redefined to allow for resource depletion.

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The two economies may also prove useful in extending the range of computable economies which can be used to develop and check new theories – a range which appears to comprise only Solow's constant consumption solution, Stiglitz's asymptotic steady state, and Pezzey and Withagen's (1998) non-steady solution of a Dasgupta-Heal economy. In particular, the NLO economy appears to be the first exact solution of a capital-resource economy with the assumptions in Weitzman (1976) of a linear utility function, a constant interest rate and a non-linear trade-off between consumption and investment goods.

As a preliminary, Section 2 lists ten features of a capital-resource economy, some of which are always defined in a simplifying way in theoretical models, so that results almost never fully general. Section 3 defines the Hyperbolic economy and derives and interprets its results, while Section 4 does likewise for the NLO economy, with details of both calculations being given in Appendices. Section 5 concludes.

2. Ten sources of non-generality in theoretical results

Any new features in the Hyperbolic and NLO economies in Sections 3 and 4 spring from the inevitable lack of full generality found in theoretical models of capital-resource economies, even when these are confined to the common basic form of representative-agent models with constant population where consumption is the sole determinant of utility. For example, two of the best known results of the mid-1970s use significantly different assumptions, which conceals their interrelationship within a more general overarching theory. Weitzman's (1976) result, on the annuity-equivalent properties of net national product, is set in a context of non-linear production, non-constant consumption, a linear utility function and a constant

interest rate. Hartwick's (1977) rule, on constant consumption resulting from zero net investment, is set in a context of linear production, constant consumption, and (implicitly) a non-linear utility function and a declining interest rate. As a reminder of the simplifying assumptions that have to be made before almost any results can be found, Table 1 lists ten key features about production functions, utility functions, intertemporal objectives and trade, and typical simplifying assumptions which can be made about them. (Notation is standard, but is fully defined in the next section.) Our two exact economies in Sections 3 and 4 make quite different assumptions about features 1, 6, 7 and 8.

Table 1   Ten key features, some of which are simplified in almost all theoretical models of capital-resource economies

No.	General feature	Simplifying assumption
1	Non-linear consumption/ investment frontier (e.g. $F = (C^{\varepsilon}+\dot{K}^{\varepsilon})^{1/\varepsilon}$, $\varepsilon>1$	Linear consumption/ investment frontier (e.g. $F = C + \dot{K}$)
2	Resource extraction costs	No resource extraction costs
3	Capital depreciation	No capital depreciation
4	Unspecified returns to scale in production	Constant returns to scale in production
5	Exogenous technical progress	No exogenous technical progress
6	Non-linear utility function (e.g. $U = C^{1-\eta}/(1-\eta)$)	Linear utility function (e.g. $U = C$)
7	Non-constant utility discount rate, (e.g. $\phi(t) = (1+\theta t)^{-\rho}$)	Constant utility discount rate (i.e. $\phi(t) = e^{-\rho t}$, $\rho>0$ constant)
8	Non-constant interest rate $r(t)$	Constant interest rate r
9	No constant consumption goal	Constant consumption goal, $\dot{C} = 0$
10	Closed or large open economy (so prices are endogenous)	Small open economy (so prices are exogenous)

3. The Hyperbolic economy

3.1 Definition, and the optimal path

The economy is a special case of that described in the appendix of Asheim (1997). Population is constant; consumers are identical and have no age structure, with each generation represented as an instant in continuous time, which stretches from zero to infinity; and the economy is closed to trade. The variables below are non-negative quantities along any development path in the economy, using terminology similar to that in Asheim (2000). Less familiar terms, or ones which are often given different meanings in the literature, are highlighted in italics.

$K(t)$ is the non-depreciating, manmade capital stock, $K(0) = K_0 > 0$

$S(t)$ is the non-renewable, natural resource stock, $S(0) = S_0 > 0$

$C(t)$ is consumption of a single produced good

$R(t) = -\dot{S}(t)$ is the resource depletion flow, with zero extraction costs

$F(K,R,t)$ is output; $F = F(K,R)$ if technology is constant

$U(C)$ is instantaneous utility

$\phi(t)$ is the utility discount factor

$\Phi(t) := \phi(t)U_C(C)$ is the consumption discount factor

$W(t) := \int_t^\infty [\phi(s)/\phi(t)]U[C(s)]ds, \ t \geq 0$ is (current) *welfare*

$\Theta(t) := \int_t^\infty [\Phi(s)/\Phi(t)]C(s)ds, \ t \geq 0$ is (current) *wealth*

$\delta(t) := -\dot{\phi}(t)/\phi(t)$ is the *instant discount rate*

$\delta_\infty(t) := \phi(t) / \int_t^\infty [\phi(s)ds$ is the *infinite discount rate*

$r(t) := -\dot{\Phi}(t)/\Phi(t)$ is the *instant interest rate*

$r_\infty(t) := \Phi(t) / \int_t^\infty \Phi(s)ds$ is the *infinite interest rate*

$A(t) := U^{-1}(\delta_\infty W)$ is *welfare-equivalent income*

$Y_e(t) := r_\infty(t)\Theta(t)$ is *wealth-equivalent income*

$SW(t) := [\int_t^\infty r(s)\Phi(s)C(s)ds] / \Phi(t)$ is *Sefton-Weale income* after Sefton and Weale (1996)

$Y(t) := C(t) + \dot{K}(t) - F_R(t)R(t)$ is Net National Product

$M[K(t),S(t)]$ is *sustainable income,* i.e. the maximum sustainable consumption level. This is calculated only for the case where there is no technical progress. An analytic solution will not be available for M with technical progress, and as Solow (1974, p41) observed, it is not a very satisfactory concept then.

The representative agent acts to maximise welfare at all times, and the resulting path is called optimal. Existence and uniqueness are assumed.

The specific functional forms used in the Hyperbolic economy are:

Production:	$F = K^\alpha R^\beta (1+\theta t)^\nu = \dot{K} + C, \ \nu \geq 0$	)	
Instantaneous utility:	$U(C) = C^{1-\alpha}/(1-\alpha)$	)	[3.1]
Discount factor:	$\phi(t) = (1+\theta t)^{-\rho}$	)	

and restrictions on the parameters (all of which except ν are strictly positive) and algebraic abbreviations are:

$$\beta < \alpha < \alpha+\beta \leq 1 \tag{3.2}$$

$$\rho > 1+\alpha-\beta+\nu \tag{3.3}$$

$$\xi := (\rho-\alpha-\nu)/(1-\beta) \tag{3.4}$$

$$\sigma := (\alpha+\nu-\beta\rho)/(1-\alpha)(1-\beta) \tag{3.5}$$

$$\Rightarrow \xi+\sigma = \rho+\alpha\sigma = [\rho(1-\alpha-\beta)+\alpha(\alpha+\nu)] / (1-\alpha)(1-\beta)$$

$$\theta := [\alpha(\xi-1)^\beta S_0^\beta/(\xi+\sigma)K_0^{1-\alpha}]^{1/(1-\beta)} \tag{3.6}$$

Restriction [3.7] places the economy on a (hyperbolically) steady state path from time zero. Without it, only the asymptotically steady state can be computed analytically, much as in Stiglitz (1974).

Appendix 1 then shows that optimal paths are as follows:

Consumption $C(t)$ $\qquad\qquad\qquad = [(\rho-\alpha)\theta K_0/\alpha] \ (1+\theta t)^\sigma$ $\qquad$ [3.7]

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Capital $K(t)$	$= K_0(1+\theta t)^{\sigma+1}$	[3.8]
Resource stock $S(t)$	$= S_0(1+\theta t)^{-(\xi-1)}$	[3.9]
Resource flow $R(t)$	$= (\xi-1)\theta S_0(1+\theta t)^{-\xi}$	
Instant interest rate $r(t)$	$= (\xi+\sigma)\theta/(1+\theta t)$	[3.10]
Infinite interest rate $r_\infty(t)$	$= (\xi+\sigma-1)\theta/(1+\theta t)$	

3.2 The five measures of income

From the above results, Appendix 1 also shows that the five measures of income on the optimal path of the Hyperbolic economy are at any time:

Welfare-equivalent income $A(t)$	$= [1+(1-\alpha)\sigma/(\xi-1)]^{1/(1-\alpha)}\, C(t)$	[3.11]
Wealth-equivalent income $Y_e(t)$	$= [1+\sigma/(\xi-1)]\, C(t)$	[3.12]
Sefton-Weale income $SW(t)$	$= (1+\sigma/\xi)\, C(t)$	[3.13]
Net national product $Y(t)$	$= [1-\nu/(\rho-\alpha)](1+\sigma/\xi)\, C(t)$	[3.14]
Sustainable income $M(t)$	$= (1-\beta)\{[K(t)]^{\alpha-\beta}[(\alpha-\beta)S(t)]^{\beta}\}^{1/(1-\beta)}$	[3.15]

$M(t)$ uses the Solow (1974) formula, and artificially assumes that technical progress can be positive from time 0 to t, but is then "switched off". So comparisons of M with other income measures are valid only if $\nu=0$ always.

The strict size order $A > Y_e > SW > Y$ follows, since all parameters are positive, as are $(1-\alpha)$, $(\xi-1)$ and $(\rho-\alpha)$ thanks to [3.2-3.4], and as consistent with the non-strict order in Asheim (2000). M bears little relation to the other income measures, because it is not defined in terms of quantities measured on the present-value-maximising path, and in fact grows at a slower rate than the other measures. However, all five measures of income are identical and equal to consumption if optimal consumption happens to be constant anyway. From [3.5], this happens if $\alpha+\nu-\beta\rho = 0$, and hence $\sigma = 0$ and $\xi = \rho$. The Solow (1974) constant consumption path is thus a special case of the optimal path of the Hyperbolic economy.

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One view this illustration reinforces is that *there is no single or "best" definition of income*. This case has been made by others, for example on the grounds that measuring income serves many different purposes, such as:

> "...charting business cycles, comparing prosperity among nations, observing industrial structure, measuring factor shares and so on. ...real income may be interpreted as a family of concepts, each member of which is best for some particular purpose." (Usher 1994, p124)

The results here remind us that even as a measure of prosperity, there is room for disagreement in defining income. Clearly, measuring current prosperity should take proper account of the future, and consumption alone is not a proper measure. But this leaves undefined what kind of future society may want. It can choose from an infinitude of intertemporal welfare objectives, and there is no shortage of unresolved arguments about which is the right one to maximise. Even when present value maximisation with a particular discount factor is chosen as the objective, there is still a difference, given diminishing marginal utility of consumption, between welfare-equivalence and wealth-equivalence methods of accounting for the future. Hicks (1946, Ch 14) himself emphasised many different definitions of income. Moreover, he used a framework (an individual facing exogenous prices, rather than a closed economy facing endogenous prices, as above) in which some different definitions would produce the same result. So the phrase "Hicksian income" is ambiguous, and has been deliberately avoided throughout this paper.

Highlighting five theoretically different measures of income does not mean that the differences would be significantly numerically if the hyperbolic model were in any way calibrated against reality. Empirical research, into what might be sensible parameter values to use above, would be useful, and would show what differences between definitions are worth debating for practical purposes.

3.3 Sustainable growth

Optimal consumption in our Hyperbolic economy is steadily rising over time if the discount rate is low enough ($\rho < (\alpha+\nu)/\beta$ so that $\sigma > 0$). Note that this condition can hold even with constant technology ($\nu = 0$). This reflects the way that a hyperbolic utility discount rate declines over time, in a way that can match the declining returns to capital investment that typically occur in a capital-resource economy with no technical progress. This hyperbolic result is in sharp contrast with the main Dasgupta and Heal (1974) economy, where no matter how low the (exponential) discount rate, it ultimately is greater than the declining return to capital, so that optimal consumption asymptotically falls toward zero.

4. The Non-Linear Output (NLO) economy

4.1 Definition, and the optimal path

The definition of the NLO economy is as for the Hyperbolic economy except for the specific functional forms, which are:

Production:	$F = K^\alpha R^\beta e^{\nu t} = (\dot{K}^2 + C^2)^{1/2}; \; \dot{K} \geq 0$	)
Instantaneous utility:	$U(C) = C$	) [4.1]
Discount factor:	$\phi(t) = e^{-\rho t}$	)

All parameters are strictly positive, with other restrictions and algebraic abbreviations being:

$$\beta < \alpha < \alpha+\beta < 1 \qquad [4.2]$$

$$\beta\rho < \nu < (1-\alpha)\rho \qquad [4.3]$$

$$\psi := [(1-\alpha)\rho-\nu]/(1-\alpha-\beta) > 0 \qquad [4.4]$$

$$\gamma := (\nu-\beta\rho)/(1-\alpha-\beta) > 0 \qquad [4.5]$$

and $\psi + \gamma = \rho$

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$$(\rho\gamma)^{\frac{1}{2}}K_0^{1-\alpha} = \alpha^{\frac{1}{2}}(\psi S_0)^\beta \qquad\qquad [4.6]$$

[4.2], [4.3] $\Rightarrow \alpha\nu < (1-\alpha)(1-\beta)\rho \Rightarrow \alpha(\nu-\beta\rho) < (1-\alpha-\beta)\rho \Rightarrow$

$$\gamma < \rho/\alpha \qquad\qquad [4.7]$$

Restriction [4.6] is needed to put the NLO economy on a steady state, analytically soluble path from time zero. Appendix 2 then shows that optimal paths are as follows:

Consumption $C(t)$	$= (\rho/\alpha-\gamma)^{\frac{1}{2}}\gamma^{\frac{1}{2}}K_0 e^{\gamma t}$	[4.8]
Capital $K(t)$	$= K_0 e^{\gamma t}$	[4.9]
Resource stock $S(t)$	$= S_0 e^{-\psi t}$	[4.10]
Resource flow $R(t)$	$= \psi S_0 e^{-\psi t}$	
Instant interest rate $r(t)$	$= \rho$	

When $\nu = 0$, this is perhaps the first exact solution of an economy satisfying the conditions of Weitzman (1976), though both consumption and capital would then be declining. The condition $\nu > \beta\rho$ for $\gamma > 0$, and hence optimal consumption to be forever rising, is identical to that in Stiglitz (1974, p136).

4.2 The technical progress premium

Using the above results, Appendix 2 also shows that the five measures of income in the Hyperbolic economy are

Welfare-equivalent income $A(t) = \rho C(t)/(\rho-\gamma)$	[4.11]
Wealth-equivalent income $Y_e(t) = \rho C(t)/(\rho-\gamma)$	[4.12]
Sefton-Weale income $SW(t) = \rho C(t)/(\rho-\gamma)$	[4.13]
Net national product $Y(t) = (1-\beta)\rho C(t)/(\rho-\alpha\gamma)$	[4.14]
Sustainable income $M(t) = [\beta^{\beta/(1-\beta)}-\beta^{1/(1-\beta)}]^{\frac{1}{2}}Q$ where	
$\qquad\qquad Q := \{[K(t)]^{\alpha-\beta}[(\alpha-\beta)S(t)/\beta]^\beta\}^{1/(1-\beta)}$	[4.15]

with $M(t)$ defined only if technical progress $\nu = 0$, as in Section 3; but here, one can show that M grows at the same rate as the other income measures.

Thanks to the linear utility function and the constant interest rate here, the first three income measures are identical. The interest lies in the "technical progress premium" (TPP) defined by Weitzman (1997) as:

$$\text{TPP} := (Y_e/Y) - 1 \quad = \nu/(\rho - \gamma)(1-\beta) \text{ for the NLO economy.}^{1} \qquad [4.16]$$

Weitzman's own formula for the TPP:

$$\text{TPP} = \lambda/(r-g), \text{ where}$$

r, the interest rate, $= \rho$ here;

g, the growth rate of (inclusive or green) NNP, $= \dot{C}/C = \gamma$ here;

$\lambda = \int_t^\infty (\partial Y/\partial s)e^{-rs}ds \; / \int_t^\infty Y(s)e^{-rs}ds$, or the "average future growth rate of the...pure effect of time alone on enhancement of *productive capacity* not otherwise attributable to capital accumulation" (Weitzman 1997, p7, italics added);

fits with [4.16] only if

$$\lambda = \nu/(1-\beta) \qquad\qquad\qquad\qquad [4.17]$$

Result [4.17] may be surprising at first. From Weitzman's *verbal* definition of λ and the production function $F = K^\alpha R^\beta e^{\nu t}$, one might think that $\lambda = \nu$ in the NLO economy, rather than $\nu/(1-\beta)$. But on reflection, one needs to distinguish between technical progress *in production F* (here $(C^2 + \dot{K}^2)^{1/2}$, but more commonly $C + \dot{K}$), and technical progress *in net national product Y*. Technical progress does not occur in the resource rents $F_R R$. So the rate of progress in Y, the difference $C + \dot{K} - F_R R$, is *higher* (by a factor $1/(1-\beta)$ here, as it turns out) than that in $C + \dot{K}$.

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1. We do not try to find the TPP for the Hyperbolic economy, since Weitzman's analysis does not apply if both technical progress and the utility discount factor are not exponential.

The same discrepancy can be shown to occur in a special case of the Stiglitz (1974) economy with a utility function $U = C^{1-\alpha}/(1-\alpha)$, and parameters are chosen so that it too starts and stays on a steady-state path. The production function there is as in the NLO economy, but the split between consumption and investment is linear, $F = K^{\alpha}R^{\beta}e^{\nu t} = C + \dot{K}$. One can show that the TPP in that economy is $\nu/[(1-\beta)\zeta]$, whereas the Weitzman formula would at face value give a TPP of ν/ζ, where $\zeta := (\rho-\nu)/(1-\beta)$.

More theoretical research is needed to improve this rather intuitive explanation of the $1/(1-\beta)$ adjustment factor needed here in the rate of technical progress. And the size of β – and indeed whether a Cobb-Douglas formula for production is appropriate for investigating very long-run sustainability questions – seems to merit further empirical research.

4.3 Hartwick's Rule

Hartwick's rule in the NLO economy has a slightly unusual reduced form here, which applies irrespective of the specific form of the production function $F(K,R)$. Fundamentally, the rule is of course unchanged. From (3.18a) of Aronsson et al (1997), $\dot{H} = \rho(H-C) = \rho\mathbf{P}.\dot{\mathbf{K}}$, where H is the Hamiltonian of the problem, $\mathbf{P}$ is the vector of shadow prices, and $\mathbf{K}$ is the vector of all capital stocks; so $\mathbf{P}.\dot{\mathbf{K}} = 0 \ \forall t \Rightarrow H = C \ \forall t \Rightarrow \dot{H} = 0 \ \forall t \Rightarrow \dot{C} = 0 \ \forall t$. But with $F = (\dot{K}^2+C^2)^{\frac{1}{2}}$, Appendix 2 shows that the shadow prices (co-state variables) are such that $\mathbf{P}.\dot{\mathbf{K}} = 0$ takes the form

$$\dot{K} = (F/\dot{K})F_R R \qquad\qquad [4.18]$$

rather than the form $\dot{K} = F_R R$ observed with a linear production function $F = \dot{K} + C$. The interpretation is that although capital investment still has to equal resource rents, the resource price in terms of investment goods is proportionately higher (by a factor $F/\dot{K}$) in the non-linear economy.

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5. Conclusions

Exact solutions have been presented for the optimal paths of two economies with accumulable capital, a non-renewable resource and specific functional forms. They illustrate some significant points in recent literature on income and sustainability accounting, and should prove useful as testbeds for future theoretical enquiry. In the first economy, a combination of hyperbolic discounting and hyperbolic technical progress makes five measures of income – welfare-equivalent income, wealth-equivalent income, Sefton-Weale income, net national product and sustainable income – all quite distinct, and the first four are in descending size order. The optimal (present-value-maximising) consumption path becomes the Solow constant consumption path for a specific discount rate. A lower discount rate leads to the optimal consumption level growing forever, and this happens even if there is no technical progress.

The second economy combines a non-linear trade-off between consumption and investment outputs, with a linear utility function. This gives the framework for Weitzman's (1997) result, that a technical progress premium (TPP) needs to be added to net national product to give a truer measure of an economy's productive potential. However, the rate of technical progress to use in Weitzman's formula for the TPP must be that for net national product, which is found to be larger than the rate of progress in production alone, because of the role of resource depletion in production. Further research would be interesting on the empirical significance of all these results, and on the theoretical foundation of the last result.

APPENDICES

In both Hyperbolic and NLO economies the maximisation problem is

$$\text{Max} \quad \int_0^\infty \phi(t) U[C(t)] dt \qquad \qquad)$$

$$C,R \qquad \qquad) \qquad \text{[A0.1]}$$

$$\text{s.t.} \ \dot{K} = J[F(K,R),C], \ \dot{S} = -R; \ K(0) = K_0, \ S(0) = S_0 \quad)$$

and the current value Hamiltonian is

$$H \ = U(C) + \omega\dot{K} + \mu\dot{S} = U(C) + \mathbf{P}.\dot{\mathbf{K}}; \ \ \mathbf{P} := (\omega,\mu), \ \mathbf{K} := (K,S) \quad \text{[A0.2]}$$

$$= U(C) + \omega J[F(K,R),C] - \mu R \qquad \qquad \text{[A0.3]}$$

The necessary first order conditions are

$$H_C = U_C + \omega J_C = 0 \qquad \Rightarrow \ \omega = -U_C/J_C \qquad \text{[A0.4]}$$

$$H_K = \omega J_F F_K = -(\dot{\phi}/\phi)\omega - \dot{\omega} \qquad \Rightarrow \ \dot{\omega}/\omega = -(\dot{\phi}/\phi) - J_F F_K \qquad \text{[A0.5]}$$

$$H_R = \omega J_F F_R - \mu = 0 \qquad \Rightarrow \ \mu = \omega J_F F_R \qquad \text{[A0.6]}$$

$$H_S = 0 = -(\dot{\phi}/\phi)\mu - \dot{\mu} \qquad \Rightarrow \ \dot{\mu}/\mu = -(\dot{\phi}/\phi) \qquad \text{[A0.7]}$$

Appendix 1.   Optimal solution paths for the Hyperbolic economy

In [3.1], $U(C) = C^{1-\alpha}/(1-\alpha)$, $\qquad \phi(t) = (1+\theta t)^{-\rho} \qquad \text{[A1.1]}$

$$\dot{K} = J(F,C) = F - C \qquad F(K,R,t) = K^\alpha R^\beta (1+\theta t)^\nu \qquad \text{[A1.2]}$$

[A0.4-7] then give

$$-\alpha\dot{C}/C = -\dot{\phi}/\phi - F_K = \rho\theta/(1+\theta t) - \alpha K^{\alpha-1} R^\beta (1+\theta t)^\nu \qquad \text{[A1.3]}$$

$$\dot{F}_R/F_R = \alpha\dot{K}/K - (1-\beta)\dot{R}/R + \theta\nu/(1+\theta t) = F_K = \alpha F/K$$

$$\Rightarrow \ \theta\nu/(1+\theta t) - (1-\beta)\dot{R}/R = \alpha C/K \qquad \text{[A1.4]}$$

Also $\qquad \delta_\infty := 1 \ / \int_t^\infty [\phi(s)/\phi(t)] ds = (1+\theta t)^{-\rho} \ / \ [(1+\theta s)^{-\rho+1}/(1-\rho)\theta]_t^\infty$

$$= (\rho-1)\theta/(1+\theta t). \qquad \text{[A1.5]}$$

Try steady state solution

$$C = C_0(1+\theta t)^\sigma, \ K = K_0(1+\theta t)^{\sigma+1} \qquad \text{[A1.6]}$$

$$S = S_0(1+\theta t)^{-(\xi-1)}, \ R = (\xi-1)\theta S_0(1+\theta t)^{-\xi} \qquad \text{[A1.7]}$$

Comparing rates of growth and constant terms in [A1.3-4] then gives

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$$\sigma = (\sigma+1)\alpha - \xi\beta + \nu \qquad \text{[A1.8]}$$

$$C_0 = K_0^{\alpha}[(\xi-1)\theta S_0]^{\beta} - (\sigma+1)\theta K_0 \qquad \text{[A1.9]}$$

$$\alpha\sigma\theta = \alpha K_0^{\alpha-1}[(\xi-1)\theta S_0]^{\beta} - \rho\theta \qquad \text{[A1.10]}$$

and $\quad C_0 = (K_0/\alpha)[\theta\nu+(1-\beta)\xi\theta] \qquad \text{[A1.11]}$

[A1.9-11] $\Rightarrow \quad C_0/K_0 = [\nu+(1-\beta)\xi]\theta/\alpha \qquad \text{[A1.12]}$

$$= K_0^{\alpha-1}[(\xi-1)\theta S_0]^{\beta} - (\sigma+1)\theta = (\alpha\sigma+\rho)\theta/\alpha - (\sigma+1)\theta$$

$\Rightarrow \quad \nu+(1-\beta)\xi = \rho-\alpha \quad \Rightarrow \quad \xi = (\rho-\alpha-\nu)/(1-\beta) \qquad \text{[3.4]}$

[A1.8/12],[3.4] $\Rightarrow \sigma = (\alpha+\nu-\beta\xi)/(1-\alpha) = (\alpha+\nu-\beta\rho)/(1-\alpha)(1-\beta) \qquad \text{[3.5]}$

and $\quad \alpha\sigma+\rho = \xi+\sigma = [\rho(1-\alpha-\beta)+\alpha(\alpha+\nu)] / (1-\alpha)(1-\beta) \qquad \text{[A1.13]}$

Non-renewable resource stock requires $\xi-1 > 0$, hence $\rho > \alpha+\nu+1-\beta$.

[A1.10] $\Rightarrow \theta^{1-\beta} = \alpha K_0^{\alpha-1}[(\xi-1)S_0]^{\beta}/(\alpha\sigma+\rho)$. With [A1.13], [3.4] this gives

$$\theta = [\alpha(\xi-1)^{\beta}S_0^{\beta}/(\xi+\sigma)K_0^{1-\alpha}]^{1/(1-\beta)} \qquad \text{[3.6]}$$

Hence K, S, R are as in [3.8-9]; and [A1.12], [3.4] give

$$C = [(\rho-\alpha)\theta K_0/\alpha] (1+\theta t)^{\sigma}. \qquad \text{[3.7]}$$

$F = K_0^{\alpha} [(\xi-1)\theta S_0]^{\beta} (1+\theta t)^{\sigma} = K_0^{\alpha} [(\xi-1)\theta S_0]^{\beta} (1+\theta t)^{\sigma}$, and using [3.6]

$$= K_0^{\alpha} [\theta(\xi+\sigma)K_0^{1-\alpha}/\alpha] (1+\theta t)^{\sigma} = [(\xi+\sigma)/(\rho-\alpha)]C$$

$F_R = \beta F/R = [\beta(\xi+\sigma)\theta K_0/\alpha] (1+\theta t)^{\sigma} / (\xi-1)\theta S_0(1+\theta t)^{-\xi}$

$= [\beta K_0(\xi+\sigma)/(\xi-1)\alpha S_0](1+\theta t)^{\xi+\sigma}$

$F_R R = \beta F = [\beta\xi/(\rho-\alpha)](1+\sigma/\xi)C$

$\Phi = \phi U_C = (1+\theta t)^{-\rho} [(\rho-\alpha)\theta K_0/\alpha]^{-\alpha} (1+\theta t)^{-\alpha\sigma}$

$\Rightarrow \quad r = -\dot{\Phi}/\Phi = (\rho+\alpha\sigma)\theta/(1+\theta t) = (\xi+\sigma)\theta/(1+\theta t) \qquad \text{[3.10]}$

$\quad r_{\infty} = (\xi+\sigma-1)\theta/(1+\theta t)$ follows analogously to δ_{∞}. $\qquad \text{[A1.14]}$

$U = C_0^{1-\alpha}(1+\theta t)^{\sigma(1-\alpha)}/(1-\alpha) = C_0^{1-\alpha}(1+\theta t)^{(\alpha+\nu-\beta\rho)/(1-\beta)}/(1-\alpha)$

$\phi U = (1+\theta t)^{-\rho}C_0^{1-\alpha}(1+\theta t)^{[(\alpha+\nu-\beta\rho)/(1-\beta)]}/(1-\alpha) = C_0^{1-\alpha}(1+\theta t)^{-\xi}/(1-\alpha)$

$W = (1+\theta t)^{\rho}C_0^{1-\alpha}(1+\theta t)^{-\xi+1}/(1-\alpha)(\xi-1)\theta = (1+\theta t)U/\theta(\xi-1)$

$\Rightarrow \delta_{\infty}W \quad = [(\rho-1)\theta/(1+\theta t)](1+\theta t)U/\theta(\xi-1) = (\rho-1)U/(\xi-1)$

$\quad = [1+(\rho-\xi)/(\xi-1)]U = [1+(1-\alpha)\sigma/(\xi-1)]U$

$$\dot{K} = (\sigma+1)\alpha C/(\rho-\alpha)$$

$$\mathbf{P.\dot{K}} = \dot{K} - F_R R = (\sigma+1)\alpha C/(\rho-\alpha) - \beta(\xi+\sigma)C/(\rho-\alpha)$$

$$= [(\sigma+1)\alpha\xi - \beta\xi(\xi+\sigma) + \nu(\xi+\sigma) - \nu(\xi+\sigma)] \, C/\xi(\rho-\alpha)$$

$$= \{ (\alpha+\nu-\beta\rho+1-\alpha-\beta+\alpha\beta)\alpha(\rho-\alpha-\nu) +$$

$$(\nu-\beta\nu-\beta\rho+\alpha\beta+\beta\nu)[\rho(1-\alpha-\beta)+\alpha(\alpha+\nu)] - \nu(\xi+\sigma)(1-\alpha)(1-\beta)^2 \}$$

$$\text{x } C/\xi(\rho-\alpha)(1-\alpha)(1-\beta)^2$$

$$= \{(\nu-\beta\rho+\alpha\beta)\rho(1-\beta) + (1-\beta)\alpha(\rho-\alpha-\nu) - \nu(\xi+\sigma)(1-\alpha)(1-\beta)^2\}$$

$$\text{x } C/\xi(\rho-\alpha)(1-\alpha)(1-\beta)^2$$

$$= [\sigma-\nu(\xi+\sigma)/(\rho-\alpha)]C/\xi$$

$$\Theta = \int_t^\infty [\Phi(s)/\Phi(t)]C(s)ds = (1+\theta t)^{(\xi+\sigma)} [\int_t^\infty (1+\theta s)^{-(\xi+\sigma)}C_0(1+\theta s)^\sigma ds$$

$$= (1+\theta t)C/\theta(\xi-1)$$

$$A = \{(1-\alpha)[1+(1-\alpha)\sigma/(\xi-1)]U\}^{1/(1-\alpha)} = [1+(1-\alpha)\sigma/(\xi-1)]^{1/(1-\alpha)}C \qquad [3.11]$$

$$Y_e = r_\infty\Theta = [(\xi+\sigma-1)\theta/(1+\theta t)](1+\theta t)C/\theta(\xi-1) = [1+\sigma/(\xi-1)]C \qquad [3.12]$$

$$\text{S-W income} = \int_t^\infty [r(s)\Phi(s)C(s)/\Phi(t)]ds$$

$$= (1+\theta t)^{\xi+\sigma} \int_t^\infty (\xi+\sigma)\theta(1+\theta s)^{-1-(\xi+\sigma)+\sigma}C_0 ds$$

$$= (1+\theta t)^{\xi+\sigma} \int_t^\infty (\xi+\sigma)\theta(1+\theta s)^{-1-\xi}C_0 ds = [(\xi+\sigma)/\xi]C \qquad [3.13]$$

$$Y = C + \mathbf{P.\dot{K}} = [\xi+\sigma-\nu(\xi+\sigma)/(\rho-\alpha)]C/\xi = [1-\nu/(\rho-\alpha)](1+\sigma/\xi)C \qquad [3.14]$$

Appendix 2. Optimal solution paths for the non-linear output (NLO) economy

In [4.1], $\quad U = C, \phi = e^{-\rho t}$ \hfill [A2.1],[A2.2]

$$\dot{K}^2 = F^2 - C^2, \text{ hence } \dot{K} = J(F,C) = (F^2-C^2)^{\frac{1}{2}} \qquad [A2.3]$$

$$\Rightarrow \qquad J_F = F/\dot{K}, \quad J_C = -C/\dot{K} \qquad [A2.4]$$

$$F(K,R) = K^\alpha R^\beta e^{\nu t} \qquad [A2.5]$$

Putting [A2.1-5] in [A0.4-7] then gives the optimal relationships

$$\omega = \dot{K}/C \qquad [A2.6]$$

$$\dot{\omega}/\omega = \rho - (F/\dot{K})F_K = \rho - \alpha F^2/\dot{K}K \qquad [A2.7]$$

$$\mu = \omega(F/\dot{K})F_R = \omega\beta F^2/\dot{K}R = \beta F^2/CR; \quad \mu/\omega = (F/\dot{K})F_R \qquad [A2.8]$$

$$\dot{\mu}/\mu = \rho \qquad\qquad\qquad\qquad\qquad\qquad\text{[A2.9]}$$

Look for steady state solution $\quad C/C_0 = K/K_0 = e^{\gamma t}$ $\qquad$ [A2.10]

$$\text{and } S/S_0 = e^{-\psi t} \quad\text{hence } R = \psi S_0 e^{-\psi t} \qquad\text{[A2.11]}$$

[A2.10],[A2.11],[A2.5] $\Rightarrow$ $\qquad F = F_0 e^{(\alpha\gamma-\beta\psi+\nu)t}$ $\qquad\qquad$ [A2.12]

Powers of e^t in [A2.1]: $\quad 2\alpha\gamma - 2\beta\gamma + 2\nu = 2\gamma$

$\Rightarrow$ $\qquad\qquad\qquad\qquad \nu = (1-\alpha)\gamma + \beta\psi$ $\qquad\qquad$ [A2.13]

[A2.6/10/7]] $\Rightarrow$ $\dot{\omega}/\omega = \gamma - \gamma = 0$ $\Rightarrow$ $\rho = \alpha F^2/\dot{K}K = \alpha F_0^2/\gamma K_0^2$ $\quad$ [A2.14]

[A2.9],[A2.8],[A2.10],[A2.11],[A2.12] $\Rightarrow$ $\quad \rho = 2\dot{F}/F - \dot{C}/C - \dot{R}/R$

$= 2(\alpha\gamma-\beta\psi+\nu)-\gamma+\psi = 2[\nu-(1-\alpha)\gamma-\beta\psi] + \gamma + \psi = \gamma + \psi$ $\qquad$ [A2.15]

[A2.13],[A2.15] $\Rightarrow$ $\gamma = \rho-\psi$ $\Rightarrow$ $\beta\psi = \nu - (1-\alpha)(\rho-\psi)$

$\Rightarrow (1-\alpha-\beta)\psi = (1-\alpha)\rho - \nu$ $\Rightarrow$ $\psi = [(1-\alpha)\rho-\nu]/(1-\alpha-\beta)$ $\qquad$ [4.4]

$\Rightarrow$ $\gamma = [\rho(1-\alpha-\beta)-(1-\alpha)\rho+\nu]/(1-\alpha-\beta) = (\nu-\beta\rho)/(1-\alpha-\beta)$ $\qquad$ [4.5]

[A2.14/5/10/11] $\Rightarrow \rho = \alpha K_0^{2\alpha}(\psi S_0)^{2\beta}/\gamma K_0^2 \Rightarrow \rho\gamma K_0^{1-2\alpha} = \alpha(\psi S_0)^{2\beta}$ $\qquad$ [4.6]

[A2.10] $\Rightarrow \int_0^\infty Ce^{-\rho t}dt = C_0/(\rho-\gamma) = Y_e(0)\int_0^\infty e^{-\rho t}dt = Y_e(0)/\rho$

$$\Rightarrow Y_e = C/(1-\gamma/\rho) = C\rho/\psi \qquad\qquad\text{[4.12]}$$

[A2.6/8/3] $\Rightarrow Y = C + \omega\dot{K} + \mu\dot{S} = C + \dot{K}^2/C - \beta F^2/C$ $\qquad$ [A2.16]

$\qquad = C[1 + \{F^2-C^2-\beta F^2\}/C^2] = (1-\beta)F^2/C$ $\qquad$ [A2.17]

[A2.3],[A2.14] $\Rightarrow$ $C^2 = F^2 - \dot{K}^2 = F^2 - \gamma^2 K^2 = F^2(1-\alpha\gamma/\rho)$ $\qquad$ [A2.18]

[A2.17],[A2.18] $\qquad\qquad \Rightarrow$ $Y = C(1-\beta)/(1-\alpha\gamma/\rho)$ $\qquad\qquad$ [4.14]

[4.12],[4.14] $\Rightarrow$ $Y_e/Y - 1 = (\rho-\alpha\gamma)/(\rho-\gamma)(1-\beta) - 1$

$\qquad\qquad = (-\alpha\gamma+\gamma+\beta\rho-\beta\gamma)/(\rho-\gamma)(1-\beta)$

$\qquad\qquad = \nu/(\rho-\gamma)(1-\beta)$ $\qquad\qquad\qquad$ [4.16]

Proof of sustainable income level $M(t)$ in [4.15]: $-$ to be inserted

Proof of Hartwick's rule, $\dot{K} = (F/\dot{K})F_R R$ in [4.18]: $-$ to be inserted

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