Rules For Optimal Fertilizer Carryover: An Alternative Explanation

John O. S. Kennedy*

A method of deriving rules for optimal fertilizer application when there is carryover is presented. The carryover situations considered are: carryover a deterministic function of fertilizer *available* in the previous period; carryover a deterministic function of fertilizer *applied* in more than one previous period; and carryover a stochastic function of fertilizer *available* in the previous period. The method of derivation is more direct than an inductive one previously presented. The rules are given an economic interpretation, and the gains from using them in practice are considered.

Introduction

The rule for determining the level of fertilizer which leads to the maximum net return from a crop is well-known. Assuming crop yields are subject to diminishing marginal returns to fertilizer input, fertilizer should be applied at the level at which the value of the marginal product of fertilizer equals the price of fertilizer. Probably less well-known are rules for maximizing the present value of net returns from a sequence of crops when fertilizer applied in one season promotes growth not only in that season but also in subsequent seasons. The problem of finding optimal fertilizer levels when there are carryover effects was initially discussed by Heady and Dillon (1961, pp. 524-5), Fuller (1965) and Anderson (1967, pp. 53-4). They set out the principles involved, but no general rules were presented. Lanzer and Paris (1981) have emphasized the problem is one of optimal control, requiring dynamic analysis taking account of the current level of soil nutrients.

Rules for optimal fertilizer application when there is carryover of various types have been derived by Kennedy *et al.* (1973), Dillon (1977), Kennedy (1981) and Taylor (1983) by a process of induction based on dynamic programming. However, experience from explaining the logic of dynamic programming and comments made by Godden and Helyar (1980) suggest that the method used for deriving the rules may have appeared complex and the results may have lacked intuitive appeal. The purpose of this article is to present a more direct method of deriving the optimal rules for a range of deterministic carryover functions. The method is also extended to problems with stochastic yield and carryover functions, and stochastic prices. The rules derived are given an economic interpretation which is a simple extension of the rule for optimal fertilizer application in the absence of carryover. The rules can be summarized as requiring that fertilizer be applied at the level at which the expected present value of the current crop and of the savings in future fertilizer applications obtained from the marginal unit of fertilizer equals the current price of fertilizer.

The article ends by considering what incentives there are to use a carryover rule. In particular, there is investigation of the size of the gains from applying an optimal multiperiod rule compared with applying the suboptimal single-period rule when there actually are fertilizer carryover effects from one period to the next.

Carryover Proportional to Fertilizer Available

Consider the simplest possible problem situation. The way in which crop yield responds in period i to available fertilizer r_i is given by the response function $y_i\{r_i\}$. Available fertilizer (r_i) is the sum of applied fertilizer (a_i) and carryover (x_i). Carryover is a proportion (v_i) of available fertilizer in period $i-1$. The price of the crop, p_i^c, and of the fertilizer p_i^f, are known for all periods. Fertilizer costs are paid at the beginning of each period, and crop returns are received at the end of each period. The objective is to maximise the present value of net returns from fertilizer applications over n periods, given a discount factor α and initial fertilizer carryover, x_1. For convenience, the value of the residual fertilizer x_{n+1} at the end of the planning horizon is assumed to have zero

* Department of Economics, La Trobe University.
Constructive comments from David Godden and Gordon MacAulay are gratefully acknowledged.

value[1]. For an atomistic firm facing perfectly competitive input and output markets, the problem can be formulated as

$$\max_{a_1, \ldots, a_n} \sum_{i=1}^{n} \alpha^{i-1} (\alpha p_i^y y_i \{x_i + a_i\} - p_i^f a_i)$$

$$\text{subject to } a_i \geq 0, \quad x_{i+1} = v_i (x_i + a_i) \qquad (i = 1, \ldots, n),$$

with x_1 given.

The problem could be solved by forming the Lagrangian and setting all of the partial derivatives with respect to a_i, x_i and the Lagrange multipliers equal to zero. A simpler method relies on the logic used in dynamic programming. The following recursive functional equation applies:

$$f_i \{x_i\} = \max_{a_i} [\alpha p_i^y y_i \{r_i\} - p_i^f a_i + \alpha f_{i+1} \{v_i r_i\}]$$

$$= \alpha p_i^y y_i \{x_i + a_i^*\} - p_i^f a_i^* + \alpha f_{i+1} \{v_i (x_i + a_i^*)\} \qquad (i = n, \ldots, 1) \quad (1)$$

$$\text{subject to } a_i \geq 0, \quad r_i = x_i + a_i \qquad (i = n, \ldots, 1),$$

with x_1 given

$$f_{n+1} \{x_{n+1}\} = 0,$$

where $f_i\{x_i\}$ is the present value of net returns from carryover obtained by implementing the optimal sequence of fertilizer applications $a_i^*, \ldots, a_n^*$. The equation stipulates that, for optimality, fertilizer must be applied in all periods i at the rate which maximizes the sum of net return in period i (the first two terms of the RHS) and the present value of the resulting fertilizer carryover to period $i + 1$ (the last term).[2]

If the response function is concave, differentiating the term in square brackets with respect to a_i gives the first-order condition for an interior maximum[3]

$$\alpha p_i^y \partial y_i / \partial a_i - p_i^f + \alpha v_i df_{i+1} / dx_{i+1} \; 0 \qquad (i = 1, \ldots, n) \quad (2)$$

Note that, because $r_i = a_i + x_i$, then $\partial r_i/\partial x_i = \partial r_i/\partial a_i = 1$, and, since $y_i = y_i \{x_i + a_i\}$ then, by the chain rule of differentiation $\partial y_i/\partial r_i = \partial y_i/\partial a_i = \partial y_i/\partial x_i$; also that, because $x_{i+1} = v_i (x_i + a_i)$, $\partial x_{i+1}/\partial a_i = v_i$.

If df_{i+1}/dx_{i+1} were known, then (2) could be solved for optimal a_i^*. In fact it is easy to show that df_{i+1}/dx_{i+1} is a constant, equal to the price of fertilizer in period $i + 1$ for all $i \leq n-1$. To show this, differentiate (1) with respect to x_i to obtain

$$df_i / dx_i = \alpha p_i^y \partial y_i / \partial a_i + \alpha v_i df_{i+1} / dx_{i+1} \qquad (i = 1, \ldots, n),$$

where $\partial y_i / \partial a_i$ is evaluated at $x_i + a_i^*$. But from (2),

$$\alpha p_i^y \partial y_i / \partial a_i + \alpha v_i df_{i+1} / dx_{i+1} = p_i^f \qquad (i = 1, \ldots, n),$$

and therefore

$$df_i / dx_i = p_i^f \qquad (i = 1, \ldots, n). \quad (3)$$

This result is not surprising. Equation (3) merely states that the value of an additional unit of fertilizer carried over from period $i - 1$ to period i should equal the price of fertilizer in period i. This holds whatever the carryover level. After substitution in (2), the condition for optimality becomes[4]

$$\alpha p_i^y \partial y_i / \partial a_i = p_i^f - \alpha v_i p_{i+1}^f \qquad (i = 1, \ldots, n - 1). \quad (4)$$

If residual fertilizer at the end of the planning horizon has no value, then $df_{n+1}/dx_{n+1} = 0$.

---

[1] The optimal rule for $i = 1, \ldots, n - 1$ is not dependent on this assumption. However, to find the optimal rule for the final decision period, the money value of the residual fertilizer x_{n+1} must be known.

[2] The logic of the recursive functional equation is elaborated in any of the many texts on dynamic programming, such as Nemhauser (1966), Hastings (1973) and Kennedy (1986).

[3] The solution is interior if the non-negativity constraint on a_i is not binding. The method of deriving optimal carryover rules fails if it is binding. However, the constraint is unlikely to be binding unless there are marked parameter changes between periods, such as a significant rise in p^f/p^y.

[4] Fuller (1965, p. 117) anticipated the special version of (4) for constant prices and response functions, and $a = 1$.

Thus the rule for the final decision period is found directly from (2) to be

$$\alpha p_n^y \partial y_n / \partial a_n = p_n^f.$$

Condition (4) states that for $a_i = a_i^*$, the present value of the marginal product of fertilizer at the level $x_i + a_i^*$ must equal the opportunity cost of the marginal unit of fertilizer. The latter equals the price of fertilizer in period i, less the present value of savings of v_i units of applied fertilizer in period $i + 1$ because of carryover. The condition can be compared with that for maximizing net revenue for the single-period problem without carryover:

$$\alpha p^y dy/da = p^f.$$

The only additional information required to solve the dynamic problem with carryover is the values of x_i, v_i and p_{i+1}^f. Thus in order to find the optimal application rate a_i^*, there is no need to know $y_j\{r_j\}$, v_j or p_j^y for $j > i$, nor p_j^f for $j > i + 1$.

The condition for optimality (4) is the same regardless of how small or how large n may be. The reason for the separability of a_i^* from parameters beyond period $i + 1$ is that, whatever the level $x_i + a_i^*$ and hence carryover $v_i(x_i + a_i^*)$ to period $i + 1$, the total amount of fertilizer, r_{i+1}^*, to be made available in period $i + 1$ is the same. Hence carryover to period $i + 2$, equal to $v_{i+1} r_{i+1}^*$, is independent of a_i^*.

Alternative Carryover Functions

More complex single-period carryover functions have been suggested than one which has carryover to period i directly proportional to fertilizer available in $i - 1$. For example, Fuller (1965) has used

$$x_{i+1} = r_i (b + c \exp(hr_i)),$$

and Stauber *et al.* (1975) used

$$x_{i+1} = qr_i^s,$$

where b, c, h, q and s are constants. It is simple to show that for the general case with a concave single-period carryover function

$$x_{i+1} = g_i\{r_i\}, \tag{5}$$

the optimality condition is (4) after replacing v_i with $\partial g_i / \partial a_i$. This more general case is dealt with in the next section analyzing the stochastic problem.

Available fertilizer, r_i, is made up of applications in period i, and carryover from applications in periods prior to i. In using a single-period carryover function such as (5), it is assumed that carryover is dependent only on r_i, regardless of the periods in which the fertilizer making up r_i was applied. Bowden and Bennett (1974) and Godden and Helyar (1980) have suggested more complex multiperiod functions in which the proportion of applied fertilizer which is carried over to another period is dependent on the number of periods of carryover, j. Denoting the carryover proportion by w_j, the available fertilizer in period i is

$$r_i = a_i + \sum_{j=1}^{i-1} w_{i-j} a_j, \tag{6}$$

and fertilizer carryover to period $i+1$ is

$$x_{i+1} = \sum_{j=1}^{i} w_{i+1-j} a_j. \tag{7}$$

Bowden and Bennett (1974) suggested $w_j = 1/(1 + j)$, and Godden and Helyar (1980) $w_j = u/(u + j)$ where u is a constant. To incorporate (6) and (7) in a dynamic programming framework, fertilizer applications in all periods prior to i have to be state variables. Suppose m is large enough that, for practical purposes, w_j for $j > m$ can be taken to be zero. If the objective is to maximize the present value of net returns the recursive functional equation is

$$f_i\{a_{i-m}, \ldots, a_{i-1}\} = \max_{a_i} [\alpha p_i^y y_i\{r_i\} - p_i^f a_i + \alpha f_{i+1}\{a_{i-m+1}, \ldots, a_i\}] \quad (i = n, \ldots, 1), \tag{8}$$

subject to

$$\left. \begin{array}{l} a_i \geq 0 \\ r_i = a_i + \sum_{j=1}^{m} w_{m-j+1} a_{i+j-m-1} \end{array} \right| \quad (i = n, \ldots, 1),$$

with

x_1 given

$$f_{n+1}\{a_{n-m+1}, \ldots, a_n\} = 0.$$

Noting from (6) that $\partial r_i / \partial a_i = 1$, the first-order condition for the term in square brackets to be an interior maximum is

$$\alpha p_i^y \partial y_i / \partial a_i - p_i^f + \alpha \partial f_{i+1} / \partial a_i = 0 \quad (i = 1, \ldots, n). \tag{9}$$

The unknown is $\partial f_{i+1}/\partial a_i$. Differentiating (8) with respect to the state variables $a_{i-m}, \ldots, a_{i-1}$, and after some substitution and manipulation, it is found that

$$\partial f_{i+1}/\partial a_i = \sum_{j=1}^{n-i} \alpha^j w_j p^y_{i+j} \partial y_{i+j}/\partial a_{i+j} \qquad (i = 1, \ldots, n-1). \quad (10)$$

Substituting (10) in (9) leads to one way of expressing the condition for optimality as

$$\alpha p^y_i \partial y_i/\partial a_i + \sum_{j=1}^{n-i} \alpha^{j+1} w_j p^y_{i+j} \partial y_{i+j}/\partial a_{i+j} = p^f_i \qquad (i = i, \ldots, n-1). \quad (11)$$

The condition states that for $a_i = a^*_i$ the present value of the crop produced in all periods from an additional unit of fertilizer applied in period i must equal the price of fertilizer in period i. Equation (11) is a set of n m-or-lower-order difference equations. Optimal values of $\partial y_i/\partial a_i$ for $i = 1, \ldots, n$ may be found in terms of $w_1, w_2, \ldots, p^y_i, p^f_i, \ldots, p^f_n$ and α by using the matrix method of solving simultaneous equations. The solution is the same as that obtained by Kennedy (1981) through a process of induction, and is most easily expressed as[5]

$$\alpha p^y_i \partial y_i/\partial a_i = p^f_i - \sum_{j=1}^{n-i} \alpha^j \beta_j p^f_{i+j} \qquad (i = 1, \ldots, n-1) \quad (12)$$

where $\beta_1 = w_1$ and $\beta_k = w_k - \sum_{s=1}^{k-1} \beta_{k-s} w_s$ for $k > 1$.

If the terminal value of all residual fertilizer is zero, $\partial f_{n+1}/\partial a_n = 0$, and it follows directly from (9) that for the final decision period

$$\alpha p^y_n \partial y_n/\partial a_n = p^f_n .$$

In this way of expressing the optimality condition, a^*_i does not depend on future product prices or response functions. Although the right-hand side of (12) looks a lot more complicated than that of (4), it can still be rationalized as the opportunity cost of the marginal unit of applied fertilizer. It is first necessary to see that β_k is the fertilizer saved k periods into the future through applying an additional unit now. It equals something less than w_k, because by adding a unit now, less will be added over each of the next $k-1$ periods if the optimal rule is followed. That is, $\beta_1 = w_1$ is saved next year, $\beta_2 = w_2 - w_1^2$ the following year, and so on. The reduction in fertilizer available after k carryover periods is thus $\beta_1 w_{k-1} + \beta_2 w_{k-2} + \ldots$. Hence $\beta_k = w_k - \beta_1 w_{k-1} - \beta_2 w_{k-2} - \ldots$.

The right-hand side of (12) is therefore the price of fertilizer in period i, less the present value of savings in applied fertilizer over all subsequent periods in the planning horizon resulting from the application of an additional unit in period i.

If crop and fertilizer prices and response functions remain unchanged over all future years, it is simple to find the optimal $\partial y/\partial a$ for an infinite planning horizon. In this stationary case optimal $\partial y/\partial a$ will be the same for all years. It follows from (11) that

$$\alpha p^y \partial y/\partial a = p^f/(1 + \sum_{j=1}^{m} \alpha^j w_j).$$

Stochastic Analysis

The problems considered so far have been deterministic, but in practice the response and carryover functions are stochastic and prices are uncertain. Suppose the objective is to maximize the present value of expected net returns. Taylor (1983) considered the case of a stochastic response function, a single-period carryover function which is a stochastic version of (5), and crop and fertilizer prices which have a first-or-higher-order Markovian structure. He showed that the optimality condition is the same as (4), but with stochastic variables and functions set at their expected values. Taylor used the method originally suggested by Kennedy *et al.* (1973), of first finding the optimality condition for the final period in the planning horizon, then for the penultimate period, and so on. The optimality condition for all periods but the last is found by inductive logic. The result for the stochastic case can be obtained more directly by the method used above for the deterministic case.

The recursive functional equation is:

$$f_i\{x_i, p^y_i, p^f_i\} = \max_{a_i} [E(\alpha p^y_i y_i\{r_i, \epsilon^y_i\} - p^f_i a_i + \alpha f_{i+1}\{g_i\{r_i, \epsilon^x_i\}, p^y_{i+1}, p^f_{i+1}\})] \qquad (i = n, \ldots, 1) \quad (13)$$

[5] In the original exposition by Kennedy (1981, p., 205), the upper limit in the corresponding equation (10), should be "$n-i$" instead of m. The relevant planning horizon remains n periods, and is not truncated to m periods as suggested in proposition (i) on p. 206.

subject to $a_i \geq 0$ $\quad (i = n, \ldots, 1)$

$r_i = x_i + a_i$

with x_1 given

$$f_{n+1}\{x_{n+1}, p^y_{n+1}, p^f_{n+1}\} = 0.$$

where ϵ^y_i and ϵ^x_i are random variables affecting yield and carryover respectively.

Because next period's crop and fertilizer prices depend on the current period's prices, current prices (assumed known) are included in the optimal return function. It is useful to refer to $E(y_i\{r_i, \epsilon^y_i\})$ as $\bar{y}_i\{r_i\}$, and to $E(g_i\{r_i, \epsilon^x_i\})$ as $\bar{g}_i\{r_i\}$. Note that $E(\partial y_i/\partial r_i) = d\bar{y}_i/dr_i$ and that $\partial r_i/\partial a_i = \partial r_i/\partial x_i = 1$. Again making the usual assumptions about concavity and that $a^*_i > 0$ for all i, the first order condition for a maximum is

$$\alpha p^y_i \partial \bar{y}_i/\partial a_i - p^f_i + \alpha E((\partial f_{i+1}/\partial g_i)(\partial g_i/\partial r_i)) = 0 \qquad (i = 1, \ldots, n). \quad (14)$$

Partially differentiating (13) with respect to x_i gives

$$\partial f_i/\partial x_i = \alpha p^y_i \partial \bar{y}_i/\partial a_i + \alpha E((\partial f_{i+1}/\partial g_i)(\partial g_i/\partial r_i)) \qquad (i = 1, \ldots, n). \quad (15)$$

From (14) and (15) it follows that

$$\partial f_i/\partial x_i = \partial f_i/\partial g_{i-1} \doteq p^f_i \qquad (i = 1, \ldots, n). \quad (16)$$

Substituting (16) in (14) gives

$$\alpha p^y_i \partial \bar{y}_i/\partial a_i - p^f_i + \alpha E(p^f_{i+1}(\partial g_i/\partial r_i)) = 0 \qquad (i = 1, \ldots, n-1).$$

Assuming that fertilizer price and carryover are independent, the condition for optimality simplifies to

$$\alpha p^y_i \partial \bar{y}_i/\partial a_i = p^f_i - \alpha(E(p^f_{i+1} \mid p^f_i))\, \partial \bar{g}_i/\partial r_i \qquad (i = 1, \ldots, n-1).$$

That is, at the optimum, the discounted value of the expected marginal product of applied fertilizer equals the expected opportunity cost of the marginal unit of applied fertilizer. If the terminal value of carried-over fertilizer is zero, the optimal rate for the final decision period is $\alpha p^y_n \partial \bar{y}_n/\partial a_n = p^f_n$. In the case of a multiperiod carryover function, a similar approach can be employed to derive a "certainty equivalent" version of (12).

An additional factor likely to affect fertilizer carryover to period i, besides fertilizer available in period $i-1$, is the crop yield in period $i-1$, and possibly earlier periods. For the deterministic case, where carryover function (5) applies, this conceptually does not introduce further complexities provided the function is based on data from experiments with the crop in question produced in every period. If yield is a function of a random variable such as rainfall, crop yield in period i may be low because of low rainfall, leading to low fertilizer uptake, and leaving more fertilizer than usual to be carried over to period $i+1$. Again this is not a problem if all of the random variables which affect crop yield are included appropriately in the carryover function. There is then no need for crop yields in previous years to be additional state variables in the recursive equations.

The fertilizer decision problem may not be just how much fertilizer to apply each year but, also, whether to apply any. Application costs may be high enough to make it more economic to apply fertilizer infrequently at high rates. Stauber *et al.* (1975) have shown that this problem is equivalent to the inventory problem of when and how much to re-order, and therefore has the same type of solution. They used stochastic dynamic programming numerically to find optimal nitrogen fertilization policies for seeded grasses in semi-arid regions in the U.S.

Implications for Practice

What hinders the greater use of the rule for optimal fertilizer application in the case of carryover? There would be an obvious impediment if the net returns from using the optimal rule for a carryover case (**R1**) were little higher than from using the optimal rule for the no-carryover case (**R0**) in a situation in which there actually was carryover. Doll (1972) presented evidence to show that, in the context of no carryover, net returns are insensitive to large variations in fertilizer application around the optimal level, and Jardine (1975) argued that this result is not dependent on the parameters of some response functions such as the quadratic and square root. This suggests that for such functions, in the absence of carryover, the optimal application of fertilizer is highly sensitive to the ratio of fertilizer to product prices. From (4) and (12), the optimal rule for the carryover case may be viewed as the optimal rule for the no-carryover case with the price of fertilizer discounted to an extent dependent on the degree of carryover. Thus it might be supposed that for such functions the carryover rule might imply quite different optimal fertilizer levels and net returns.

Consider the calculation of benefits in a very simple case: the carryover function is the same as that used in (1) with v_i constant for all i, the discount rate is zero (*i.e.* $\alpha = 1$) and the steady-state annual net return is to be found after (i) **R0** and (ii) **R1** have been implemented over a large number of periods. For the steady-state situation, subscripts can be dropped so that $x = v(a+x)$ and therefore

$$r = x+a = a/(1-v).$$

Note that, whereas $\partial r_i/\partial a_i = 1$, $dr/da = 1/(1-v)$ which takes account of the change in steady-state carryover, x, resulting from a change in applied fertilizer, a. Application rates under **R0** and **R1** are therefore probably best given in terms of dy/dr instead of $\partial y/\partial a$. The steady-state application rate under **R0** is $a^0 = r^0$ which satisfies

$$p^y dy/dr = p^f. \qquad (17)$$

From (4) the steady-state application rate under **R1** is $a^1 = r^1 - x = r^1 - r^1 v = r^1(1-v)$ which satisfies

$$p^y dy/dr = p^f(1-v). \qquad (18)$$

Let the net return from following rule $\mathbf{R_j}$ be

$$b^j = p^y y\{a^j/(1-v)\} - p^f a^j$$

where $j = 0$ or 1. The percentage gain from using the carryover rule is

$$G = ((b^1/b^0)-1)\times 100.$$

It follows from (17) and (18) that if the response function is concave as assumed, then $r^1 > r^0 = a^0$; however, whether, $a^1 > a^0$ depends in general on model parameters. This indeterminacy occurs because $a^1 = r^1(1-v)$ with $v > 0$, so that $a^1 < r^1$. In other words, the requirement of the optimal carryover rule that more fertilizer should be made available does not necessarily mean that more fertilizer should be applied. The optimal rate of application is the optimal rate to be made available, less calculated carryover. Thus the difference between a^1 and a^0 is not so great as might initially be supposed.

As a rough guide to the gain from using **R1** over **R0** when there is carryover, values of G are presented in Table 2 for the four crop response functions and price parameters described in Table 1.

Values of the carryover fraction v obviously depend on site-specific characteristics. Fuller (1965) estimated a value of about 0.32 for the cropping situation to which response function (3) applies. Kennedy *et al.* (1973) considered that v was in the range 0.2 to 0.4 for their study of sorghum production in the Northern Territory. The value of v implied by the multiperiod carryover function used by Godden and Helyar (1980) for the cropping situation to which response function (4) applies is about 0.7. Consequently the range of v in Table 2 is from 0.2 to 0.8.

Table 1: Selected yield-fertilizer response functions and prices

1.	Quadratic[a]	$y = -7.51 + 0.584r - 0.0016r^2$
2.	Square root[a]	$y = -5.68 - 0.316r + 6.3512r^{0.5}$
3.	Exponential[b]	$y = 107.7 - 73.4(0.69^r)$
4.	Mitscherlich[c]	$y = 2129(1-\exp(-0.04r + 8.8))$

[a] Based on equations 14.2 and 14.3 in Heady and Dillon (1961) with zero phosphate levels. y is bushels of corn per acre, r is pounds of available nitrogen per acre, p^y = US\$1.30 per bushel and p^f = US\$0.12 per pound.

[b] Fuller (1965, equation 13). y is bushels of corn per acre, r is 40 pound units of available nitrogen per acre, p^y = US\$1.00 per bushel and p^f = US\$6.00 per 40 pounds.

[c] Godden and Helyar (1980, Table 1). y is kilograms of sorghum per hectare, r is kilograms of phosphorus per hectare, p^y = A\$0.80 per kilogram and p^f = A\$0.83 per kilogram.

The results in Table 2 show that for the functions other than the quadratic the gains from using **R1** and **R0** are modest for values of v up 0.4. On the other hand, gains are spectacular in the case of the quadratic response function. Of course, these results depend on the selected price ratios p^f/p^y. If **R1** and **R0** are applied using a discount rate of 10 per cent, the percentage gains in net revenue are slightly reduced. If net revenue were calculated after deducting fixed costs, the percentage gains in net revenue would be accentuated.

Table 2: Percentage gain in net returns through following the optimal carryover rule R1 over the single-period rule R0

Response function	Carryover fraction (v)			
	0.2	0.4	0.6	0.8
1. Quadratic	5.4	57.6	†	†
2. Square root	0.6	4.0	20.1	576.5
3. Exponential	0.2	1.3	4.8	13.4
4. Mitscherlich	0.3	1.7	5.6	13.7

† Infeasible result. a^* under R0 gives $r = a^*/(1-v)$ so large that y is negative.

The results are sufficient to show that there is a risk of foregoing significant benefits by using **R0** instead of **R1**, particularly for $v > 0.4$. Lanzer and Paris (1981) provide one piece of empirical evidence supporting this conclusion. They showed that arranging for optimal carryover for wheat–soybean cropping in Southern Brazil would result in significant increases in net returns, primarily through reduced fertilizer costs.

For all of the four response situations examined in Table 2 the level of fertilizer applied was lower under **R1** than under **R0.** The percentage reduction compared with the amount which would be applied under **R0** varies from 20 to 36 per cent across the four response situations for $v = 0.4$.

Another impediment to implementing the carryover rule is the additional information required. This consists of forecasts of fertilizer prices, fertilizer carryover functions and residual fertilizer carried over from previous years. If the residual fertilizer can be described by one state variable then it may be economic to estimate residual fertilizer on the basis of soil tests instead of previous applications of fertilizer. In the stochastic formulation of the problem the random disturbance term ϵ_i^x in the carryover function would include measurement error. This type of feedback-control process has been discussed by Lanzer and Paris (1981).

Conclusion

The problem of determining the optimal sequence of fertilizer applications across seasons when there is carryover is a multistage problem which can be solved in various ways using dynamic programming. The method used in this article for deriving optimal rules makes use of the information obtained by differentiating the relevant recursive equation with respect to the state variable, fertilizer carried over or fertilizer previously applied. It is more direct than the inductive method used previously. It enables the rules to be derived more simply when carryover is proportional to fertilizer available.

The rules which emerge for the optimal fertilizer application over time when there is carryover have a straightforward economic rationale. The value of the marginal product of fertilizer applied in the current period consists of the present value of additional product not only in the current period but also in all future periods. If fertilizer is optimally applied in all future periods, it also equals the present value of additional product in the current period plus the savings in fertilizer applications in all future periods. With these interpretations of the value of the marginal product of fertilizer, the optimal rule is the same as the one usually stated: Apply fertilizer to the level at which the present value of the marginal product of fertilizer equals the price of fertilizer. If carryover and yields are stochastic, the same rule applies, but couched in terms of expected values.

The evidence for significant increases in net returns from using the optimal multiperiod rule over the single-period rule is mixed. For some types of response function the gains are quite low. However, for all of the response functions the optimal level of fertilizer application was lower if there was carryover. Anderson (1967) pointed out that various real-world considerations such as limited funds, discounting, uncertainty and carryover effects lead to lower optimal fertilizer levels than the simple "profit maximizing" level, and perhaps explain why farmers are commonly observed to apply "lower than optimal" levels of fertilizer. As he points out, it is difficult to apportion the reduction in fertilizer application between these considerations, but the limited evidence presented here suggests if there are carryover effects then net returns can be increased at least marginally by significantly reducing fertilizer levels below the simple "profit maximizing" level.

References

ANDERSON, J. R. (1967), "Economic interpretation of fertilizer response data", *Review of Marketing and Agricultural Economics* 35 (1), 43–57.

BOWDEN, J. W. and BENNETT, D. (1974), "The Decide model for predicting superphosphate requirements", Proceedings of a symposium, *Phosphate in Agriculture,* Australian Institute of Agricultural Science (Victorian Branch).

DILLON, J. L. (1977), *The Analysis of Response in Crop and Livestock Production,* 2nd edition, Pergamon Press, Oxford.

DOLL, J. P. (1972), "A comparison of annual versus average optima for fertilizer experiments", *American Journal of Agricultural Economics* 54 (2), 226–33.

FULLER, W. A. (1965), "Stochastic fertilizer production functions for continuous corn", *Journal of Farm Economics* 47, 105–119.

GODDEN, D. P. and HELYAR, K. R. (1980), "An alternative method for deriving optimal fertilizer rates", *Review of Marketing and Agricultural Economics* 48 (2), 83–97.

HASTINGS, N. A. J. (1973), *Dynamic Programming with Management Applications,* Butterworths, London.

HEADY, E. O. and DILLON, J. L. (1961), *Agricultural Production Functions,* Iowa State University Press, Ames.

JARDINE, R. (1975), "Two cheers for optimality", *Journal of the Australian Institute of Agricultural Science* 41 (1), 30–34.

KENNEDY, J. O. S., WHAN, I. F., JACKSON, R. and DILLON, J. L. (1973), "Optimal fertilizer carryover and crop recycling policies for a tropical grain crop", *Australian Journal of Agricultural Economics* 17 (2), 104–113.

KENNEDY, J. O. S. (1981), "An alternative method for deriving optimal fertilizer rates: comment and extension", *Review of Marketing and Agricultural Economics* 49 (3), 203–209.

KENNEDY, J. O. S. (1986), *Dynamic Programming: Applications to Agriculture and Natural Resources,* Elsevier Applied Science, London.

LANZER, E. A. and PARIS, Q. (1981), "A new analytical framework for the fertilization problem", *American Journal of Agricultural Economics* 63 (1), 93–103.

NEMHAUSER, G. L. (1966), *Introduction to Dynamic Programming,* Wiley, New York.

STAUBER, M. S., BURT, O. R. and LINSE, F. (1975), "An economic evaluation of nitrogen fertilization of grass where carryover is significant", *American Journal of Agricultural Economics* 57 (3), 463–471.

TAYLOR, C. R. (1983), "Certainty equivalence for determination of optimal fertilizer application rates with carry-over", *Western Journal of Agricultural Economics* 8 (1), 64–67.