Forum
On Resolving Multiple Optima in Linear Programming

Ross G. Drynan*

Although the solution to a linear programming problem is not necessarily unique, the theoretical and practical implications of multiple optima have not been thoroughly discussed by agricultural economists. Recently there have been calls for greater recognition, resolution and exploitation of multiplicity. In this note, it is argued that previous recommendations for resolving multiplicity are restrictive both procedurally and conceptually. An alternative and more general approach is outlined.

Introduction

Multiple solutions do occur in linear programming problems and in the view of some, not infrequently (McCarl 1977, Paris 1981). When such solutions occur, there are dangers of misinterpretation of standard linear programming output (Evans and Baker 1982). On the other hand, multiplicity may offer a richer body of information to the decision maker (Paris 1981). The main contribution of this note is the suggestion of a more convenient way for resolving multiplicity than previously advanced in the literature. The approach amounts to a more detailed linear programming representation of the decision problem, and as such can be used in resolving not only multiple optimal solutions but also alternative near-optimal solutions to the simpler linear programme. The presentation is informal and couched in terms useful to users of linear programming.

Multiplicity vs. Degeneracy in Optimal Solutions

Multiplicity is best approached by considering degeneracy. The potential cycling problems created by degeneracy in iterating towards an optimum are well known (for example, Hadley 1962). But in this note, the concern is with degeneracy of the optimal solution, namely when one or more of the basic activities in the optimal solution is/are at zero level. Similarly, the occurrence of multiple optima is also widely mentioned in linear programming texts, including Hadley. But the link between multiple optima and degenerate optima is not frequently discussed, perhaps because it is intuitively obvious from the separate results. Miller (1963) is an early exception. Further references are given in Miller (1985). Certainly there appears to be little in the agricultural economics journals, or in the conventional programming texts to which agricultural economists are exposed in their training, relating to multiplicity/degeneracy of optimal solutions. The most substantive contributions by agricultural economists are those of McCarl and his colleagues (for example, McCarl 1977 and McCarl *et al.* 1977) and Paris (1981, 1983, 1985).

The principal relationship is that a degenerate dual (primal) optimum is a necessary but not sufficient condition for multiple primal (dual) optima.[1] This condition is considered first.

A primal linear programming problem involves choosing x to maximize $c'x$ subject to $Ax \leq b$, $x \geq 0$, where A is an $m \times n$ coefficients matrix and c, b and x are conformable vectors. As pointed out by Hadley, degeneracy in a particular primal basic solution arises due to linear dependence between the vector of constraint levels (b) and some set of $m-l$ columns of the basis. An analogous condition (on c and the rows of A) may be identified for dual degeneracy.

Suppose in the final tableau of the primal formulation, some non-basic activity has a zero marginal opportunity cost. This is the case of a degenerate dual optimum. Such, and only such, an activity can be introduced into the basis to produce an alternative optimal primal basis. Generally this will represent a second

* University of Queensland, St Lucia.

The author is grateful to the *Review's* referees for helpful comments and for drawing his attention to papers which had escaped his notice. Special thanks are due to Noel Russell with whom the author had a number of helpful discussions.

[1] Throughout the discussion, attention is solely confined to problems which have bounded optima.

optimal primal solution, that is a different set of optimal activity levels. The adverb "generally" is appropriate here because the activity with the zero marginal opportunity cost may only be able to enter the basis at a zero level. In this case, both primal optimal bases correspond to the one solution. Dual degeneracy is necessary but not sufficient for multiple primal optima.

By casting the problem in the dual form, it is clear that a similar result to that for dual degeneracy holds for primal degeneracy: a degenerate primal solution is necessary but not sufficient for there to be multiple dual optima.

Having solved a problem, the existence of primal and/or dual degenerate solutions will be obvious. The existence of multiple optima will require further analysis of the final tableau. Multiple primal (dual) solutions are easily identified from the final primal (dual) tableau as discussed above. The identification of dual (primal) multiplicity from the final primal (dual) tableau warrants further comment.

On first sight, it would appear that any non-basic primal activity with a positive marginal rate of substitution for the basic activity at zero level could be introduced into the basis (at a zero level) and maintain optimality.[2] Certainly, the primal solution remains unchanged and optimal. Only the basis has changed. However, this basis would generally have to be considered non-optimal because its associated dual solution would generally be infeasible. In particular, one or more of the dual variables will generally be at a negative level. For example, when an activity with a positive marginal opportunity cost is introduced into the basis replacing the zero level basic activity, the latter necessarily has a negative marginal opportunity cost in the new tableau. Eilon and Flavell (1974) and McCarl (1977) report such bases. However, if the introduced activity had a zero marginal opportunity cost the new basis would be considered optimal. In this case, both the primal and dual problems have multiple optimal bases, though only one optimal solution.

The correct approach is similar to that employed in parametric resource programming. A non-basic activity with a negative marginal rate of substitution for the zero-level basic activity must be introduced to the basis. The activity would enter at a zero level, and the primal solution would remain unchanged. The dual solution would alter, levels rising or falling depending on the sign (positive and negative respectively) of the coefficients in the outgoing row. The replaced basic variable would certainly have a positive marginal opportunity cost. Others may or may not. Provided there is some non-basic variable which can be entered in this way and which produces all non-negative marginal opportunity cost values, there exist multiple dual optima.

Implications of Multiplicity/ Degeneracy

Experience led McCarl (1977) to the view that degeneracy of the primal optimal solution was not uncommon in farm planning problems. To him, one of the major difficulties was how to explain alternative sets of shadow prices to farmer clients. Paris (1981), arguing from a more theoretical standpoint, reached a similar conclusion about the occurrence of dual degeneracy in the context of sector models. He argued that a linear programming sector model with less constraints than there are real world activities could only reflect the pursuit of all these activities if alternative primal optima existed. Failure to find multiple optima would suggest that the model is an imperfect description of reality.

Paris' point is well taken. However, from a practical perspective, an aggregate model must be seen as an approximation both in primal and dual variables. A model may be adequate for predicting levels of important primal activities, or changes in levels, without being particularly good at modelling dual variables or other primal variables. Given the observed diversity of farms with differing optima, it is overly restrictive to require that an optimal solution for the representative farm (or the aggregate farm defined on aggregate resources) involve all activities.

A common concern with multiplicity (Hadley 1962, p. 166, McCarl 1977, Paris 1981) is that a standard linear programming package will only identify a single (and arbitrary) optimal primal and dual solution. Alternative solutions may be more appealing or simply more informative. Strum (1969), Knolmayer (1976) and McCarl (1977) have all suggested that where primal degeneracy occurs,

[2] The relevant marginal rate of substitution is the coefficient in the non-basic activity column and the basic activity row of the final primal tableau.

"directional" dual solutions should be reported. That is, the values of a resource when increased and when decreased should both be reported. While there may be still further acceptable dual solutions, they have no economic significance. Knolmayer (1976) and Aucamp and Steinberg (1982) have examined procedures for determining all alternative solutions.

Other authors have emphasized the dangers of misinterpretation when degeneracy occurs. Evans and Baker (1982) have noted that the range information provided by linear programming packages on objective function coefficients may be misinterpreted. Such ranges are usually interpreted as the allowable variation in an objective function coefficient for which the optimal primal solution would remain unchanged. More correctly, however, a range indicates the variation for which the *basis* would remain unchanged. Only when there is a one to one correspondence between the optimal primal basis and solution, that is when the dual solution is not degenerate, do the reported ranges relate to stability of the optimal solution. When there are multiple optimal bases corresponding to the one primal optimal solution, the range over which that solution is stable is the union of the ranges over which the bases are optimal. Algorithms for deriving the correct ranges are suggested by Evans and Baker (1982), Steinberg and Aucamp (1983) and Knolmayer (1984).

Resolving Multiplicity

From one standpoint, the idea of resolving multiplicity of optimal solutions is nonsense. If the model captures the real decision problem, then the occurrence of multiple optima means that all of these solutions are equally good. There is no need or way to identify one that is better than the others. To attempt to do so is to pursue a spurious improvement.

A more defensible position is that any linear programming model is intended only to aid decision analysis and to be only an interim approximation of reality to be adapted as required during the process of decision analysis. The occurrence of multiple optimal solutions to a particular formulation of the problem does not then imply multiple optimal solutions to the real decision problem. Instead, multiplicity emphasises explicitly that there are a number of solutions which are likely to be good answers to the real decision problem. Likewise, failure to find multiplicity does not imply that there is only one answer to the real decision problem. Clearly, it is restrictive to attempt only to resolve multiplicity of solutions and not also to focus on "near-optimal" solutions (Powell and Hardaker 1969).

Multiplicity can only be resolved if more (or at least different) information is injected into the analysis. One simple means of exploiting the freedom offered by multiple optima is to consider secondary objectives which were ignored in the initial formulation. With reference to aggregative agricultural sector models, Paris (1981) has argued that an appropriate secondary objective is to find a solution "near" to the existing real world solution. Of several plans with the same level of attainment of the primary objective, that nearest to existing practice is the least disruptive to implement, and therefore preferred. Powell and Hardaker (1969) have noted the same suggestion for farm planning problems.

Although the relevance of locating a solution near to existing practice can be quesitoned, particularly in normative models, attention here is focussed on Paris' proposal for finding that solution. He suggested solving the overall optimization in two stages. First, the optimal linear programming solution maximizing the primary objective should be sought. Second, if there are multiple solutions, the extreme points of the solution space should be determined, and quadratic programming used to search for the unique linear combination of these extreme points that minimizes the sum of squares of deviations of optimal activity levels from the real world activity levels.

This approach has several disadvantages. First, all optimal basic solutions must be located before the second stage can be implemented. Second, it employs an arbitrary metric for measuring "nearness". Third, it has the deceptively attractive feature of only having to formulate the second stage optimization when multiple solutions occur, a feature that prevents its application to resolving near-optimality. Finally, the separation of stages means that sensitivity or parametric analysis of the linear programming problem is not easily implemented.

McCarl and Nelson (1983) expressed reservations about the Paris approach, and suggested an alternative two-stage linear programming method based on the absolute deviation metric. In the first stage, the optimal value (C^*) of the objective function ($c'x$) is determined as usual. The solution nearest to actual practice is then sought subject to the usual constraints and the additional constraint that $c'x$ be no less than (C^*-e), where e is a tolerance factor. This factor, in effect, allows McCarl and Nelson to resolve specified near-optimal as well as multiple optimal solutions.

McCarl and Nelson's idea is now generalized in a way which involves solving only a single linear programme. The initial focus on one objective to the total exclusion of the "nearness" objective suggests a pre-emptive or lexicographic structure of preferences in which the latter objective has relevance in choosing between solutions only if the alternative solutions have equal (and optimal) levels of the first objective. Thus a lexicographic goal programme suggests itself (see, for example, Ignizio 1982).

Consider the following lexicographic formulation:

$$\min_{x,\, d^-,\, d^+} \quad [(-c'x), (w_1^- d_1^- + w_1^+ d_1^+ + w_2^- d_2^- + \ldots w_k^+ d_k^+)]$$

subject to

$$A\,x \leq b$$
$$J\,x + I_k d^- - I_k d^+ = x_A$$
$$x,\, d^-,\, d^+ \geq 0$$

where A is an ($m \times n$) matrix and c, b and x are conformable vectors,

x_A is a ($k \times 1$) vector of the actual production levels of k products,

J is ($k \times n$) matrix defined so as to sum the appropriate elements of x corresponding to each product,

I_k is a ($k \times k$) identity matrix,

d^- and d^+ are ($k \times 1$) vectors of under- and over-production, with elements d_i^- and d_i^+,

and w_i^-, w_i^+ are non-negative weights.

This problem can be solved by a multi-phase simplex procedure (for example, Ignizio 1982). Alternatively, the normal simplex algorithm can be used by proxying the lexicographic objective function with a weighted linear function having a sufficiently large weight on the primary objective. In both cases, sensitivity analysis and parametric analysis of the resource and objective coefficients are easily implemented (Ignizio 1982).

The only real disadvantage of this approach *vis-a-vis* a two-stage approach is the inefficiency of an increased matrix size. It offers several advantages. Most importantly, formulations recognizing the existence of near-optimal as well as multiple optimal solutions can be considered by using those forms of goal programming (weighted and minimax formulations) permitting tradeoffs between the primary objective, the "nearness" goals and other goals of the decision maker. This is desirable, not just because the decision maker may have a complex objective structure, but also because of the futility in seeking an optimum, an extreme point, to an approximate model of the real problem.

Concluding Remarks

The real question is not one of resolving multiplicity, but of how best to use linear programming in aiding decision making. It may be best to solve an initial linear programme and, if there are multiple optima, select the best in a second stage. Alternatively, it may be best to subjectively evaluate the many near-optimal and optimal solutions. Finally, it may be best to solve a comprehensive linear programme, in which case the tradeoffs between goals need to be represented by a set of prespecified weights. How such weights can best be selected *a priori* is not clear, though multi-objective programming analysts face this task all the time. What is clear is that they should not be selected by blind acceptance of the pre-emptive weights of a lexicographic resolution of multiplicity, either in a two stage or single stage procedure.

It is perhaps only proper to end on a cautionary note: although "resolution" is the fashionable term, all one accomplishes by any of the proposals is the identification from a set of multiple or near-optimal solutions that solution which is uniquely optimal in terms of a more specifically defined objective. One is inevitably still left with the situation that there are many good solutions and relatively minor changes in problem specification could make other solutions optimal.

References

Aucamp, D. C. and Steinberg, D. I. (1982), "The computations of shadow prices in linear programming", *J.Opl.Res.Soc.* 33, 557–65.

Eilon, A. and Flavell, R. (1974), "Note on 'Many-sided shadow prices' ", *Omega* 2, 821–3.

Evans, J. R. and Baker, N. R. (1982), "Degeneracy and the (mis)interpretation of sensitivity analysis in linear programming", *Decision Sciences* 13, 348–54.

Hadley, G. (1962), *Linear Programming*, Addison-Wesley, Reading, Mass.

Ignizio, J. P. (1982), *Linear Programming in Single and Multiple-Objective Systems*, Prentice-Hall, Englewood Cliffs, N.J.

Knolmayer, G. (1976), "How many-sided are shadow prices at degenerate primal optima?" *Omega* 4, 493–4.

Knolmayer, G. (1984), "The effects of degeneracy on cost-coefficient ranges and an algorithm to resolve interpretation problems", *Decision Sciences* 15, 14–21.

McCarl, B. A. (1977), "Degeneracy, duality, and shadow prices in linear programming", *Can.J.Agric.Econ.* 25, 70–3.

McCarl, B. A., Candler, W. V., Doster, D. H. and Robbins, P. R. (1977), "Experiences with farmer oriented linear programming for crop planning", *Can.J.Agric.Econ.* 25, 17–30.

McCarl, B. A. and Nelson, C. H. (1983), "Multiple optimal solutions in linear programming models: comment", *Amer.J.Agric.Econ.* 65, 181-3.

Miller, R. E. (1963), "Alternative optima, degeneracy, and imputed values in linear programs", *J.Reg. Sci.* 5, 21-39.

Miller, R. E. (1985), "Multiple optimal solutions in linear programming models: a further comment", *Amer.J.Agric.Econ.* 67, 153.

Paris, Q. (1981), "Multiple optimal solutions in linear programming models", *Amer.J.Agric.Econ.* 63, 724-7.

Paris, Q. (1983), "Multiple optimal solutions in linear programming models: reply", *Amer.J.Agric.Econ.* 65.

Paris, Q. (1985), "Multiple optimal solutions in linear programming models: a further reply", *Amer.J. Agric.Econ.* 67, 154-5.

Powell, R. A. and Hardaker, J. B. (1969), "Sub-optimal programming methods for practical farm planning", *Rev.Mktg.Agric.Econ.* 37, 1–9.

Steinberg, D. I. and Aucamp, D. C. (1983), "On ranging cost coefficients in dual degenerate linear programming problems", *Decision Sciences* 14, 440-1.

Strum, J. E. (1969), "Note on 'Two-sided shadow prices' ", *J.Acc.Res.* 7, 160-2.