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# **Application of Comparative Dynamics in Stochastic Invasive Species Management in Agricultural Production**

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# **Application of Comparative Dynamics in Stochastic Invasive Species Management in Agricultural Production**

## **Abstract**

In this study, we formulate a stochastic dynamic framework for pest control over the growing season taking into account forecasts of weather conditions and pest infestation expectations. Using stochastic envelope theorem and stochastic comparative dynamics, we analytically show how the stochastic correlation between the prediction errors should affect optimal pesticide usage path. As a case study, we apply the analytical results of the paper for pesticide use in the Palouse region of Washington where pea aphid is the primary threat for lentil production. By stochastic dynamic programming, our simulation shows the optimal dimethoate usage path, which illustrates our findings in the analytical part.

Key words: Stochastic Optimal Control, Climate Change, Pest Management

JEL classification: Q10, Q54

## **Introduction**

The purpose of this paper is to examine conditional, forecast-based dynamic pest management in agricultural crop production under stochastic pest infestations and stochastic climate dynamics throughout the growing season. Forecasts of pest outbreaks in conjunction with forecasts of climatic conditions can be used to improve effectiveness of pest management decisions. In this paper we show that forecasts of climatic conditions and pest outbreaks can be used to optimize applications of pesticides, given potential correlation between prediction errors of stochastic pest infestations and climatic conditions considering effects of both of these factors on yields(Elbakidze, Lu, and Eigenbrode 2011; Cobourn

et al. 2011). Using stochastic optimal control we show that correlation between forecast errors for climate prediction and forecast errors for pest outbreaks can be a factor for improving pesticide application efficiency.

The literature on pest management typically specifies the pest management problem in terms of a damage (or damage abatement) function in conjunction with the production function (Lichtenberg and Zilberman. 1986; Fox and Weersink 1995; Saha, Shumway, and Havenner 1997; Carpentier and Weaver 1997). The advantage of such formulation is that it allows the modeler to separate the effects of direct production inputs from the effectiveness of pest control inputs via damage function specification. While the earlier studies focused on static and deterministic specifications, several later studies have extended the literature to dynamic examinations (Zivin, Hueth, and Zilberman 2000; Marsh, Huffaker, and Long. 2000; Olson and Roy 2002; Zhang and Swinton 2009). As Olson (2006) pointed out, dynamic models provide more insight than static models in that the value of pesticide application in such models includes not only the benefits of removing the pests in the current period but also the discounted sum of benefits from precluding future pests. Following this logic, we construct a dynamic model corresponding to a planning horizon lasting from planting to harvesting. We assume that the decision on planted crop acreage has been made, but the decisions about pesticide use are made throughout the growing season.

Another important aspect of pest management problem is uncertainty associated with pest infestation. The dynamics of pest populations can be expressed in terms of predicted (or expected) pest population growth and a stochastic fluctuation as a result of unexplained factors that may cause the realized population of the pest to be higher or lower than what is expected. In stochastic pest management studies a typical assumption is that the dynamics of pest infestation follows a diffusion process based on Weiner process type of formulation (Saphores 2000; Sunding and Zivin 2000; Saphores and Shogren 2005; Richards et al. 2005). Hertzler (1991) uses stochastic optimal control and Ito stochastic calculus to study dynamic agricultural decisions under risk. He suggests that diffusion process-

based stochastic dynamic models and Ito calculus can be used for economic studies of pest control in agricultural production. Olson and Roy (2002) approach the problem of managing biological invasions in terms of minimizing expected value of discounted sum of costs and damages subject to pest growth dynamics. They solve the minimization problem using stochastic dynamic programming and provide conditions for when it is optimal to eradicate the invasive species. Cobourn (2009) also uses stochastic dynamic programming to study pest management options when activities of heterogeneous producers can influence effectiveness of pesticide use. Kim (2006) study optimal allocation of resources between prevention and control for invasive species management using dynamic formulation of stochastic invasion and subsequent discovery. We extend the previous formulations by incorporating two related stochastic variables in our optimal control model: climatic conditions and pest invasions. Furthermore, we examine how potential correlation between these stochastic variables may affect optimal pesticide use.

The roles of climate conditions in agriculture (Costello, Adams, and Polasky (1998); Rubas (2008); Chen, McCarl, and Schimmelpfennig (2004); Chen and McCarl (2001)) as well as the role of climatic condition in pest management (Chen and McCarl 2001; Cobourn et al. 2011; Elbakidze, Lu, and Eigenbrode 2011) have been addressed by economists. However, the economists have given little attention to dynamic pest management when stochastic climatic conditions affect crop growth as well as stochastic pest populations simultaneously. Olson and Roy (2002) formulated their model assuming that pest growth is affected by stochastic environmental disturbances but did not account for potential randomness of pest outbreaks beyond randomness implied by environmental conditions. Elbakidze, Lu, and Eigenbrode (2011) examined the effects of climate and pests on agricultural productivity in a simultaneous fashion taking into account the effect of climate conditions on pest infestations. However, their analytical framework is set in a static setting. Another static model is given by Cobourn et al. (2011); they have also examined how climatic variables may positively affect crop yields which in turn can attract more pests via

improved habitat. In this paper we combine the effects of climatic conditions on pest infestations and on crop yields in a stochastic optimal control setting. Both, weather conditions and pest invasion dynamics are assumed to be stochastic. Furthermore, correlation between weather and pest population prediction errors is incorporated in optimal pest management decisions.

The rest of this paper is organized as follows: In the next section, we provide general framework for stochastic optimal control analysis of pest management in the context of stochastic climate and stochastic pest outbreaks. Optimality conditions are discussed and the dependence of optimal pest management on correlation between stochastic climate and stochastic pest population is demonstrated. Next, we examine a specific analytical case with specific functional forms and provide conditions for optimal pesticide use path as a function of the correlation coefficient between pest and climate forecast errors. The empirical section is based on the formulations in the preceding section and is presented for the case of lentil production in Pacific North West. We conclude with presentation of empirical results and closing comments.

## **The General Framework**

In this section we analyze the problem of pest management under stochastic pest infestation and stochastic climate dynamics using general theoretical dynamic framework, avoiding specific functional forms as much as possible. This allows us to avoid imposing implicit assumptions (see discussion in Fox and Weersink 1995 for instance), and allows for exploration of comparative dynamic properties of the optimal pesticide usage path. Our framework is based on minimization of total expected losses and costs associated with pest infestations and pest management. The analysis reflects that crop growth depends on stochastic pest infestation and stochastic climatic conditions.

We use a nonnegative climate index  $\theta(t)$  to denote climate conditions (degree day accumulation, see Marsh, Huffaker, and Long. 2000 for discussion, and/or precipitation for

instance). Climate index is assumed to follow a diffusion process (Sunding and Zivin 2000; Saphores and Shogren 2005):

$$(1) \quad d\theta = \mu^\theta(\theta, t)dt + \sigma^\theta(\theta, t)d\tilde{\theta}$$

where  $\mu^\theta$  and  $\sigma^\theta$  denote expected changes in the climate index over time and corresponding standard deviation over time respectively.  $\tilde{\theta}$  is the standard wiener process, i.e.  $\tilde{\theta} \sim N(0, t)$  with  $\text{var}(d\tilde{\theta}) = dt$ . Consequently, standard deviation of  $\tilde{\theta}$  is expanding with constant increment over time. At any time  $t$ ,  $\mu^\theta$  can be interpreted as the predictable rate of change of climate index with standard deviation  $\sigma^\theta$ . Notice that both  $\mu^\theta$  and  $\sigma^\theta$  vary with time as well as climate conditions.

Pest population is specified as:

$$(2) \quad A(t) = A(u(t), \theta(t), t, \tilde{A})$$

where  $u(t)$  denotes pesticide usage path,  $\tilde{A}$  represents another Wiener process (that is  $\tilde{A} \sim N(0, t)$ ) which can be interpreted as all other uncontrolled factors that affect pest population beyond climatic factors, pest management activities, and time.  $A$  is assumed to be twice differentiable in all of its arguments. Applying Ito's Lemma (Hertzler 1991; Kamian and Schwartz 1991) and substituting (1), we have<sup>1</sup> (see Appendix 1a):

$$(3) \quad dA = \left( A_t + A_\theta \mu^\theta + \frac{1}{2} A_{\theta\theta} (\sigma^\theta)^2 + \frac{1}{2} A_{\tilde{A}\tilde{A}} + A_{\theta\tilde{A}} \rho^{\tilde{\theta}\tilde{A}} \sigma^\theta \right) dt + A_\theta \sigma^\theta d\tilde{\theta} + A_{\tilde{A}} d\tilde{A}$$

Reflecting a possibility of interaction between stochastic climatic index and pest population (Elbakidze, Lu, and Eigenbrode 2011; Cobourn et al. 2011) we assume that errors in climate predictions can be correlated with stochastic factors that contribute to unexplained variability of pest outbreaks,  $d\tilde{\theta}d\tilde{A} = \rho^{\tilde{\theta}\tilde{A}}dt$  (Kamian and Schwartz 1991), where  $\rho^{\tilde{\theta}\tilde{A}}$  denotes the correlation<sup>2</sup> between  $d\tilde{\theta}$  and  $d\tilde{A}$ . Non zero  $\rho^{\tilde{\theta}\tilde{A}}$  implies that deviations from expected (or predicted) changes in climate index can be correlated with remaining unexplained variability in pest dynamics.

Let  $\mu^A(u, A, \theta, t) \equiv A_t + A_\theta \mu^\theta + \frac{1}{2} A_{\theta\theta} (\sigma^\theta)^2 + \frac{1}{2} A_{\tilde{A}\tilde{A}} + A_{\theta\tilde{A}} \rho^{\tilde{\theta}\tilde{A}} \sigma^\theta$  denote the deterministic growth rate of pest population which includes second order components from Ito's Lemma but does not include stochastic elements. We can then rewrite the pest population dynamics equation as:

$$(4) \quad dA = \mu^A(u, A, \theta, t) dt + A_\theta \sigma^\theta(\theta, t) d\tilde{\theta} + A_{\tilde{A}} d\tilde{A}$$

Equation (4) shows that  $A$  is also following a diffusion process. Specifications based on similar diffusion processes can be found in prior studies (Saphores and Shogren 2005 and Mbah et al. 2010 for instance). However, rather than having a single source of randomness as in previous literature, in our formulation the change in pest population has two sources of randomness. One associated with climate ( $\tilde{\theta}$ ), and the other associated with other unaccounted environmental, ecological or other factors which can affect pest population dynamics ( $\tilde{A}$ ).  $\mu^A$  denotes the intrinsic deterministic pest growth rate, which includes deterministic effect of climate on pest growth and the second order terms from Ito's Lemma. We assume that  $\mu^A$  is decreasing in the control variable,  $u$ . Keeping in mind that  $d\tilde{\theta}d\tilde{\theta} = dt$ ,  $d\tilde{A}d\tilde{A} = dt$ , it can be easily shown that, the variance of  $dA$  is:

$$\begin{aligned} \text{Var}(dA) &= \text{Var}(A_\theta \sigma^\theta d\tilde{\theta} + A_{\tilde{A}} d\tilde{A}) \\ &= (A_\theta \sigma^\theta)^2 dt + (A_{\tilde{A}})^2 dt + 2A_\theta A_{\tilde{A}} \rho^{\tilde{\theta}\tilde{A}} \sigma^\theta dt \equiv (\sigma^A)^2 dt \end{aligned}$$

Next, we formulate losses in crop growth as a function of stochastic pest population and stochastic climate dynamics.

$$(5) \quad L(t) = L(A(u(t), \theta(t), t, \tilde{A}), \theta(t, \tilde{\theta}), t)$$

We assume that loss in crop growth,  $L$ , is twice differentiable with respect to all of its arguments and is increasing in  $A$ . Notice that the climate index ( $\theta$ ) is an exogenous variable. It affects the state variables, pest ( $A$ ) and loss ( $L$ ) dynamics, but is not a function of control variable ( $u$ ). This implies that in our optimal control problem state variables are  $A$  and  $L$ . Again, using Ito's lemma and equations (1) and (4), the dynamics of losses in crop growth



can be expressed as (see Appendix 1b):

$$\begin{aligned}
(6) \quad dL &= \left( L_t + L_\theta \mu^\theta + L_A \mu^A + \frac{1}{2} L_{\theta\theta} (\sigma^\theta)^2 + \frac{1}{2} L_{AA} (\sigma^A)^2 + L_{\theta A} \rho^{\tilde{\theta}\tilde{A}} \sigma^\theta \sigma^A \right) dt \\
&\quad + (L_\theta \sigma^\theta(\theta, t) + L_{AA} \sigma^\theta(\theta, t)) d\tilde{\theta} + L_{AA} d\tilde{A} \\
&\equiv \mu^L(u, A, L, \theta, t) dt + (L_\theta \sigma^\theta(\theta, t) + L_{AA} \sigma^\theta(\theta, t)) d\tilde{\theta} + L_{AA} d\tilde{A}
\end{aligned}$$

where  $\mu^L$  denotes the expected (or deterministic) loss rate as a function of weather and pest arguments and is assumed to be decreasing in  $u$ . Variance of  $dL$  is the variance of the sum of two stochastic components (last two terms of equation (6)) and can be expressed as follows keeping in mind that Wiener process specification implies  $d\tilde{\theta}d\tilde{\theta} = dt$ ,  $d\tilde{A}d\tilde{A} = dt$  and  $d\tilde{\theta}d\tilde{A} = \rho^{\tilde{\theta}\tilde{A}} dt$ :

$$\begin{aligned}
(\sigma^L)^2 dt &= (L_\theta \sigma^\theta + L_{AA} \sigma^\theta)^2 dt + (L_{AA} \tilde{A})^2 dt + 2\rho^{\tilde{\theta}\tilde{A}} (L_\theta \sigma^\theta + L_{AA} \sigma^\theta) L_{AA} \tilde{A} dt \\
&= \left[ (L_A \sigma^A)^2 + L_\theta \sigma^\theta (L_\theta \sigma^\theta + 2L_{AA} \sigma^\theta + 2L_{AA} \tilde{A} \rho^{\tilde{\theta}\tilde{A}}) \right] dt
\end{aligned}$$

The objective is to minimize expected losses and costs associated with pest infestation and management, which can be expressed as the following stochastic optimal control problem:

$$\begin{aligned}
(7) \quad J(t, A, L) &= \min E \left\{ e^{-rT} pL(T) + \int_0^T e^{-rt} wu(t) dt \right\} \\
&\text{subject to (4) and (6)}
\end{aligned}$$

where  $T$  is the terminal crop harvest period,  $r$  is discount rate,  $u(t)$  denotes the path of pesticide usage, and  $p$  and  $w$  denote the prices of harvested crops and costs of pesticide use respectively. At terminal time  $T$ ,  $L(T)$  can be interpreted as yield loss. Thus, the first term in the objective function can be interpreted as the discounted value of lost yield and the second term is the accumulated discounted cost of pesticide use. This formulation is essentially the continuous version of the objective function in Olson and Roy (2002).

The optimal solution for the value function is given by Hamilton-Jacobi-Bellman equation (8) (Kamian and Schwartz (1991)):

$$(8) \quad -J_t = \min_u \left\{ e^{-rt} wu + J_L \mu^L + J_A \mu^A + \frac{1}{2} J_{LL} (\sigma^L)^2 + \frac{1}{2} J_{AA} (\sigma^A)^2 + J_{AL} \rho^{AL} \sigma^L \sigma^A \right\}$$

with the boundary condition:

$$(9) \quad J(T) = e^{-rT} pL(T)$$

Equation (8) tells us that on the optimal pesticide usage path, the change in optimal total cost over time is function of discounted pesticide cost, shadow prices of yield loss and pest growth state equations, and the standard deviations and correlation of the state variables which enter the value function due stochastic nature of the problem (Kao 1996). The terminal value of total cost is determined by the boundary condition (9) and is equal to the value of lost yield. It should be noted that  $\rho^{AL}$  is the correlation coefficient between the dynamics of pest population and crop growth loss, not to be confused with  $\rho^{\tilde{\theta}\tilde{A}}$ . However, the two correlation coefficients are closely related. Notice that  $COV(dA, dL) = \sigma^A \sigma^L \rho^{AL} dt$ . Meanwhile,

$$\begin{aligned} COV(dA, dL) &= COV(A_\theta \sigma^\theta d\tilde{\theta} + A_{\tilde{A}} d\tilde{A}, (L_\theta \sigma^\theta + L_A A_\theta \sigma^\theta) d\tilde{\theta} + L_A A_{\tilde{A}} d\tilde{A}) \\ &= \left[ A_\theta (\sigma^\theta)^2 (L_\theta + L_A A_\theta) + A_{\tilde{A}} L_A A_{\tilde{A}} + \sigma^\theta \rho^{\tilde{\theta}\tilde{A}} (A_{\tilde{A}} (L_\theta + L_A A_\theta) + A_\theta L_A A_{\tilde{A}}) \right] dt \end{aligned}$$

Therefore,

$$\begin{aligned} \sigma^A \sigma^L \rho^{AL} &= A_\theta (\sigma^\theta)^2 (L_\theta + L_A A_\theta) + A_{\tilde{A}} L_A A_{\tilde{A}} + \sigma^\theta \rho^{\tilde{\theta}\tilde{A}} (A_{\tilde{A}} (L_\theta + L_A A_\theta) + A_\theta L_A A_{\tilde{A}}) \\ (10) \quad \rho^{AL} &= \frac{A_\theta (\sigma^\theta)^2 (L_\theta + L_A A_\theta) + (A_{\tilde{A}})^2 L_A}{\sigma^A \sigma^L} + \frac{A_{\tilde{A}} \sigma^\theta (L_\theta + 2L_A A_\theta)}{\sigma^A \sigma^L} \rho^{\tilde{\theta}\tilde{A}} \end{aligned}$$

which implies that  $\rho^{AL}$  is a linear function of  $\rho^{\tilde{\theta}\tilde{A}}$ . The direction of the relationship is not clear without further assumptions about functional forms of  $A$  and  $L$  and implied signs of the derivatives. For simplicity, we use  $\rho$  for  $\rho^{AL}$  in all of the following texts. Next we

define the following matrices:  $\mathbf{J}^1 = [J_A, J_L]$ ;  $\mathbf{J}^2 = \begin{bmatrix} J_{AA} & J_{AL} \\ J_{AL} & J_{LL} \end{bmatrix}$ ;  $\boldsymbol{\mu} = \begin{bmatrix} \mu^A(u, A, \theta, t) \\ \mu^L(L, A, \theta, t) \end{bmatrix}$ ;

$$\Sigma = \begin{bmatrix} (\sigma^A)^2 & \sigma^A \sigma^L \rho \\ \sigma^A \sigma^L \rho & (\sigma^L)^2 \end{bmatrix}. \text{ Then, (8) can be rewritten as:}$$

$$(11) \quad -J_t = \min_u \left\{ e^{-rt} w u + \mathbf{J}^1 \boldsymbol{\mu} + \frac{1}{2} \text{tr}(\mathbf{J}^2 \Sigma) \right\}$$

To formulate the Hamiltonian version of the problem, let  $\boldsymbol{\lambda}^1 = [\lambda^1(A, L), \lambda^2(A, L)]$  denote the vector of co-state variables, where  $\lambda^1(A, L)$  is the co-state variable for the pest dynamics equation and  $\lambda^2(A, L)$  is the co-state variable for yield loss state equation respectively. Both can be functions of  $A$  and  $L$ . Let the following matrix denote the first order derivatives of the co-state variables with respect to state variables,  $\boldsymbol{\lambda}^2 = \begin{bmatrix} \lambda_A^1 & \lambda_L^1 \\ \lambda_A^2 & \lambda_L^2 \end{bmatrix}$ . The Hamiltonian then can be expressed as expected value, which includes second order terms from Ito's Lemma (Xepapadeas (1997)):

$$(12) \quad H = E \left\{ e^{-rt} w u + \boldsymbol{\lambda}^1 \boldsymbol{\mu} + \frac{1}{2} \text{tr}(\boldsymbol{\lambda}^2 \Sigma) \right\}$$

By maximum principle,

$$(13) \quad \frac{\partial H}{\partial u} = e^{-rt} w + E \left\{ \boldsymbol{\lambda}^1 \frac{\partial \boldsymbol{\mu}}{\partial u} + \frac{1}{2} \text{tr} \left( \boldsymbol{\lambda}^2 \frac{\partial \Sigma}{\partial u} \right) \right\} = 0$$

The first term on the right hand side is the discounted value of instantaneous marginal cost of pesticide at any particular time  $t$ . The first term inside of the expectation operator can be interpreted as the marginal benefit of the control. The second term inside of the expectation operator captures the control variable's influence on the variability of its benefits.

Assuming existence of a unique solution path, we are interested in whether and how the correlation between weather and pest outbreak predictions errors influences optimal pest control decisions. To address this question we turn to the stochastic counterpart of the dynamic envelope theorem of Lafrance and Barney (1991).

**Proposition 1** (*Stochastic Dynamic Envelope Theorem*) *On the optimal solution path the following identity holds:*

$$\frac{\partial J^*(t)}{\partial \rho} = \int_0^T \frac{\partial H(t)}{\partial \rho} \Big|_{u=u^*} dt - E \int_0^T \frac{1}{2} \text{tr} \left( \lambda^2 \frac{\partial \Sigma^*}{\partial \rho} \right) dt$$

This identity expresses the effect of the correlation coefficient on the trajectory of the value function. On the left hand side, the derivative of the value function is taken on the optimal control path. However, on the right hand side, the first component is the derivative of Hamiltonian evaluated on the optimal control path minus the expected value of the derivative of the second moment under optimal solution. Notice that, in the deterministic case, the above equation reduces to the dynamic envelope theorem proved by Lafrance and Barney (1991), which does not contain the second term on the right hand side of the above identity.

**Proof** Differentiating both sides of equation (7) on the optimal path of  $u^*$  w.r.t.  $\rho$  and making use of the Leibniz Theorem and we have:

$$(14) \quad \frac{\partial J^*(t)}{\partial \rho} = E \left\{ \int_0^T e^{-rt} w \frac{\partial u^*(t)}{\partial \rho} dt \right\}$$

From (13), we have:

$$(15) \quad e^{-rt} w = -E \left[ \lambda^1 \frac{\partial \mu}{\partial u} + \frac{1}{2} \text{tr} \left( \lambda^2 \frac{\partial \Sigma}{\partial u} \right) \right]$$

Substituting (15) into (14), we get:

$$(16) \quad \frac{\partial J^*(t)}{\partial \rho} = -E \left\{ \int_0^T \left( \lambda^1 \frac{\partial \mu}{\partial u} + \frac{1}{2} \text{tr} \left( \lambda^2 \frac{\partial \Sigma}{\partial u} \right) \right) \frac{\partial u^*(t)}{\partial \rho} dt \right\}$$

Differentiating both sides of equation (12) w.r.t.  $\rho$  we get:

$$(17) \quad \frac{\partial H}{\partial \rho} = E \left\{ \lambda^1 \frac{\partial \mu}{\partial \rho} + \frac{1}{2} \text{tr} \left( \lambda^2 \frac{\partial \Sigma}{\partial \rho} \right) \right\}$$

Taking integral from 0 to  $T$  on both sides of (17), and using Fubini's theorem (Durrett 2010) to pull expectation operator outside of integral, we have:

$$(18) \quad \int_0^T \frac{\partial H}{\partial \rho} \big|_{u=u^*} dt = \int_0^T E \left[ \boldsymbol{\lambda}^1 \frac{\partial \boldsymbol{\mu}}{\partial \rho} + \frac{1}{2} \text{tr} \left( \boldsymbol{\lambda}^2 \frac{\partial \boldsymbol{\Sigma}}{\partial \rho} \right) \right] dt = E \int_0^T \left[ \boldsymbol{\lambda}^1 \frac{\partial \boldsymbol{\mu}}{\partial \rho} + \frac{1}{2} \text{tr} \left( \boldsymbol{\lambda}^2 \frac{\partial \boldsymbol{\Sigma}}{\partial \rho} \right) \right] dt$$

Combining (16) and (18) yields:

$$(19) \quad \begin{aligned} \frac{\partial J^*(t)}{\partial \rho} - \int_0^T \frac{\partial H(t)}{\partial \rho} \big|_{u=u^*} dt &= -E \left\{ \int_0^T \left[ \left( \boldsymbol{\lambda}^1 \frac{\partial \boldsymbol{\mu}}{\partial u} + \frac{1}{2} \text{tr} \left( \boldsymbol{\lambda}^2 \frac{\partial \boldsymbol{\Sigma}}{\partial u} \right) \right) \frac{\partial u^*(t)}{\partial \rho} + \boldsymbol{\lambda}^1 \frac{\partial \boldsymbol{\mu}}{\partial \rho} + \frac{1}{2} \text{tr} \left( \boldsymbol{\lambda}^2 \frac{\partial \boldsymbol{\Sigma}}{\partial \rho} \right) \right] dt \right\} \\ &= -E \left\{ \int_0^T \left[ \boldsymbol{\lambda}^1 \left( \frac{\partial \boldsymbol{\mu}}{\partial u} \frac{\partial u^*(t)}{\partial \rho} + \frac{\partial \boldsymbol{\mu}}{\partial \rho} \right) + \frac{1}{2} \left( \text{tr} \left( \boldsymbol{\lambda}^2 \frac{\partial \boldsymbol{\Sigma}}{\partial u} \right) \frac{\partial u^*(t)}{\partial \rho} + \text{tr} \left( \boldsymbol{\lambda}^2 \frac{\partial \boldsymbol{\Sigma}}{\partial \rho} \right) \right) \right] dt \right\} \end{aligned}$$

Let  $\mathbf{x} = [A \ L]$  and  $\mathbf{W} = [\tilde{A} \ \tilde{L}]$ . Then the state equations in differential form can be written as:  $d\mathbf{x} = \boldsymbol{\mu} dt + \boldsymbol{\sigma} d\mathbf{W}$  where  $\boldsymbol{\sigma} = [\boldsymbol{\sigma}^A \ \boldsymbol{\sigma}^L]$ . Differentiating state equations w.r.t  $\rho$  we get:

$$(20) \quad \frac{\partial d\mathbf{x}^*}{\partial \rho} = \left( \frac{\partial \boldsymbol{\mu}}{\partial \rho} + \frac{\partial \boldsymbol{\mu}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}^*}{\partial \rho} + \frac{\partial \boldsymbol{\mu}}{\partial u} \frac{\partial u^*}{\partial \rho} \right) dt + \left( \frac{\partial \boldsymbol{\sigma}}{\partial \rho} + \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}^*}{\partial \rho} \right) d\mathbf{W}$$

Taking integral from 0 to  $T$  on both sides of equation (20) and rearranging yields:

$$(21) \quad \int_0^T \left( \frac{\partial \boldsymbol{\mu}}{\partial \rho} + \frac{\partial \boldsymbol{\mu}}{\partial u} \frac{\partial u^*}{\partial \rho} \right) dt = \int_0^T \frac{\partial d\mathbf{x}^*}{\partial \rho} - \int_0^T \left( \frac{\partial \boldsymbol{\sigma}}{\partial \rho} + \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}^*}{\partial \rho} \right) d\mathbf{W} - \int_0^T \left( \frac{\partial \boldsymbol{\mu}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}^*}{\partial \rho} \right) dt$$

Notice that the term  $\int_0^T \left( \frac{\partial \boldsymbol{\mu}}{\partial \rho} + \frac{\partial \boldsymbol{\mu}}{\partial u} \frac{\partial u^*}{\partial \rho} \right) dt$  shows up in (19) and (21). Therefore, we may combine the two equations and rewrite (19) as:

$$(22) \quad \begin{aligned} \frac{\partial J^*(t)}{\partial \rho} - \int_0^T \frac{\partial H(t)}{\partial \rho} \big|_{u=u^*} dt &= E \left\{ \int_0^T \left[ \boldsymbol{\lambda}^1 \frac{\partial \boldsymbol{\mu}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}^*}{\partial \rho} - \frac{1}{2} \left( \text{tr} \left( \boldsymbol{\lambda}^2 \frac{\partial \boldsymbol{\Sigma}}{\partial u} \right) \frac{\partial u^*(t)}{\partial \rho} + \text{tr} \left( \boldsymbol{\lambda}^2 \frac{\partial \boldsymbol{\Sigma}}{\partial \rho} \right) \right) \right] dt \right\} \\ &\quad - E \int_0^T \boldsymbol{\lambda}^1 \frac{\partial d\mathbf{x}^*}{\partial \rho} - E \int_0^T \boldsymbol{\lambda}^1 \left( \frac{\partial \boldsymbol{\sigma}}{\partial \rho} + \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}^*}{\partial \rho} \right) d\mathbf{W} \end{aligned}$$

Using integration by parts,  $E \int_0^T \boldsymbol{\lambda}^1 \frac{\partial d\mathbf{x}^*}{\partial \rho} = E(\boldsymbol{\lambda}^1 \frac{\partial \mathbf{x}^*}{\partial \rho}) \big|_0^T - E \int_0^T \frac{\partial \mathbf{x}^*}{\partial \rho} d\boldsymbol{\lambda}^1$  ( See Lafrance and Barney 1991). Therefore,  $E(\boldsymbol{\lambda}^1 \frac{\partial \mathbf{x}^*}{\partial \rho}) \big|_0^T = 0$ . Then (22) becomes:

$$(23) \quad \begin{aligned} \frac{\partial J^*(t)}{\partial \rho} - \int_0^T \frac{\partial H(t)}{\partial \rho} \big|_{u=u^*} dt &= E \left\{ \int_0^T \left[ \boldsymbol{\lambda}^1 \frac{\partial \boldsymbol{\mu}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}^*}{\partial \rho} - \frac{1}{2} \left( \text{tr} \left( \boldsymbol{\lambda}^2 \frac{\partial \boldsymbol{\Sigma}}{\partial u} \right) \frac{\partial u^*(t)}{\partial \rho} + \text{tr} \left( \boldsymbol{\lambda}^2 \frac{\partial \boldsymbol{\Sigma}}{\partial \rho} \right) \right) \right] dt \right\} \\ &\quad + E \int_0^T \frac{\partial \mathbf{x}^*}{\partial \rho} d\boldsymbol{\lambda}^1 - E \int_0^T \boldsymbol{\lambda}^1 \left( \frac{\partial \boldsymbol{\sigma}}{\partial \rho} + \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}^*}{\partial \rho} \right) d\mathbf{W} \end{aligned}$$

The standard dynamics of stochastic co-state variables are expressed as follows (Yong and Zhou 1999):

$$(24) \quad d\boldsymbol{\lambda}^1 = -\frac{\partial H}{\partial \mathbf{x}} dt + \boldsymbol{\lambda}^2 \boldsymbol{\sigma} d\mathbf{W} = -\left( \boldsymbol{\lambda}^1 \frac{\partial \boldsymbol{\mu}}{\partial \mathbf{x}} + \frac{1}{2} \text{tr} \left( \boldsymbol{\lambda}^2 \frac{\partial \boldsymbol{\Sigma}}{\partial \mathbf{x}} \right) \right) dt + \boldsymbol{\lambda}^2 \boldsymbol{\sigma} d\mathbf{W}$$

Plugging (24) into (23) and rearranging we get (notice that the term  $\lambda^1 \frac{\partial \mu}{\partial x} \frac{\partial x^*}{\partial \rho}$  is cancelled out):

$$(25) \quad \frac{\partial J^*(t)}{\partial \rho} - \int_0^T \frac{\partial H(t)}{\partial \rho} dt = E \left\{ \int_0^T -\frac{1}{2} \left( tr \left( \lambda^2 \frac{\partial \Sigma}{\partial u} \right) \frac{\partial u^*(t)}{\partial \rho} + tr \left( \lambda^2 \frac{\partial \Sigma}{\partial \rho} \right) + tr \left( \lambda^2 \frac{\partial \Sigma}{\partial \mathbf{x}} \right) \frac{\partial \mathbf{x}^*}{\partial \rho} \right) dt \right\} \\ - E \int_0^T \left[ \lambda^1 \left( \frac{\partial \sigma}{\partial \rho} + \frac{\partial \sigma}{\partial \mathbf{x}} \frac{\partial \mathbf{x}^*}{\partial \rho} \right) + \frac{\partial \mathbf{x}^*}{\partial \rho} \lambda^2 \sigma \right] d\mathbf{W}$$

where  $tr \left( \lambda^2 \frac{\partial \Sigma}{\partial u} \right) \frac{\partial u^*(t)}{\partial \rho} + tr \left( \lambda^2 \frac{\partial \Sigma}{\partial \rho} \right) + tr \left( \lambda^2 \frac{\partial \Sigma}{\partial \mathbf{x}} \right) \frac{\partial \mathbf{x}^*}{\partial \rho}$  is the same as the total partial derivative of  $tr \left( \lambda^2 \frac{\partial \Sigma}{\partial \rho} \right)$  with respect to  $\rho$ , and  $E \int_0^T \left[ \lambda^1 \left( \frac{\partial \sigma}{\partial \rho} + \frac{\partial \sigma}{\partial \mathbf{x}} \frac{\partial \mathbf{x}^*}{\partial \rho} \right) + \frac{\partial \mathbf{x}^*}{\partial \rho} \lambda^2 \sigma \right] d\mathbf{W} = 0$  because mean values of the elements of  $\mathbf{W}$  are zeros. Therefore, we have:

$$\frac{\partial J^*(t)}{\partial \rho} = \int_0^T \frac{\partial H(t)}{\partial \rho} \Big|_{u=u^*} dt - E \int_0^T \frac{1}{2} tr \left( \lambda^2 \frac{\partial \Sigma^*}{\partial \rho} \right) dt$$

This concludes the proof of proposition one. The power of this proposition is that it allows us to perform comparative dynamic analysis for stochastic dynamic problems. Specifically, we can examine how the optimal solution path is influenced by the correlation between two stochastic parameters. In the following proposition we show how optimal pesticide use path depends on the correlation between weather and pest outbreak prediction errors.

**Proposition 2** (*Comparative Dynamics*). *On the optimal solution path, the comparative dynamics is given by:*

$$\frac{\partial u^*}{\partial \rho} = \frac{\left( \lambda^1 \left( \frac{\partial^2 \mu}{\partial u \partial \mathbf{x}} - \frac{\partial^2 \mu}{\partial u \partial \rho} \right) \frac{\partial^2 J^*(t)}{\partial \mathbf{x} \partial \rho} + E \left\{ \frac{1}{2} tr \left( \lambda^2 \frac{\partial^2 \Sigma}{\partial u \partial \mathbf{x}} - \frac{\partial^2 \Sigma}{\partial u \partial \rho} \right) \right\} \right) \left( \int_0^T \frac{\partial^2 H(t)}{\partial^2 \rho} \Big|_{u=u^*} dt - E \int_0^T \frac{1}{2} tr \left( \lambda^2 \frac{\partial^2 \Sigma^*}{\partial \rho^2} \right) dt - \frac{\partial^2 J^*(t)}{\partial \rho^2} \right)}{\left( \lambda^1 \frac{\partial^2 \mu}{\partial u^2} + \frac{1}{2} tr \left( \lambda^2 \frac{\partial^2 \Sigma}{\partial u \partial \mathbf{x}} \right) \right) \frac{\partial^2 J^*(t)}{\partial \mathbf{x} \partial \rho}}$$

**Proof** Differentiating both sides of equation (15) w.r.t.  $\rho$ , we get:

$$(26) \quad \lambda^1 \left( \frac{\partial^2 \mu}{\partial u \partial \mathbf{x}} \frac{\partial \mathbf{x}^*}{\partial \rho} + \frac{\partial^2 \mu}{\partial u^2} \frac{\partial u^*}{\partial \rho} + \frac{\partial^2 \mu}{\partial u \partial \rho} \right) + E \left\{ \frac{1}{2} tr \left( \lambda^2 \left( \frac{\partial^2 \Sigma}{\partial u \partial \mathbf{x}} \frac{\partial \mathbf{x}^*}{\partial \rho} + \frac{\partial^2 \Sigma}{\partial u^2} \frac{\partial u^*}{\partial \rho} + \frac{\partial^2 \Sigma}{\partial u \partial \rho} \right) \right) \right\} = 0$$

Differentiating both side of proposition 1 w.r.t  $\rho$  we get:

$$(27) \quad \frac{\partial^2 J^*(t)}{\partial \mathbf{x} \partial \rho} \frac{\partial \mathbf{x}^*}{\partial \rho} + \frac{\partial^2 J^*(t)}{\partial u \partial \rho} \frac{\partial u^*}{\partial \rho} + \frac{\partial^2 J^*(t)}{\partial \rho^2} = \int_0^T \frac{\partial^2 H(t)}{\partial^2 \rho} \Big|_{u=u^*} dt - E \int_0^T \frac{1}{2} tr \left( \lambda^2 \frac{\partial^2 \Sigma^*}{\partial \rho^2} \right) dt$$

On the optimal solution path  $\frac{\partial J^*(t)}{\partial u} = 0$ . Therefore, from (27) we can get:

$$(28) \quad \frac{\partial \mathbf{x}^*}{\partial \rho} = \frac{\int_0^T \frac{\partial^2 H(t)}{\partial^2 \rho} |_{u=u^*} dt - E \int_0^T \frac{1}{2} \text{tr} \left( \boldsymbol{\lambda}^2 \frac{\partial^2 \Sigma^*}{\partial \rho^2} \right) dt - \frac{\partial^2 J^*(t)}{\partial \rho^2}}{\frac{\partial^2 J^*(t)}{\partial \mathbf{x} \partial \rho}}$$

Substituting (28) into (26) and rearranging we confirm proposition 2.

Based on this result it is impossible to determine the direction of the influence of correlation coefficient on  $u^*$  in part because the right hand side of proposition two includes second order cross partial derivatives with indeterminate signs at this stage. Further assumptions need to be made about curvature of growth rate and variance-covariance matrix for both pest and yield loss to be able to potentially assess the sign of the effect of  $\rho$  on  $u^*$ . Furthermore, the correlation coefficient  $\rho$  represents correlation between the changes in pest population levels and crop growth losses rather than the correlation between pest infestation and climate condition prediction errors which are related via equation (10). Further assumptions, or specific functional forms, are needed to assess the relationship between the two correlation coefficients and consequently the effect of the correlation on optimal pesticide use path. We consider the following specific case.

### The Specific Case

The dynamics of climate index is assumed to follow a diffusion process. A specific functional form for equation (1) is chosen as<sup>3</sup>:

$$(29) \quad d\theta = \mu^\theta \theta dt + \sigma^\theta \theta d\tilde{\theta}$$

It is easy to check (see appendix 2) that a solution to this stochastic differential equation has the explicit form:

$$(30) \quad \theta = \theta(0) e^{[\mu^\theta - \frac{(\sigma^\theta)^2}{2}]t + \sigma^\theta \tilde{\theta}}$$

Let's assume that the natural pest growth dynamics is following a diffusion process and is influenced by climate in the following way:

$$(31) \quad A = A(0) e^{[\mu^A - \frac{(\sigma^A)^2}{2}]t + \sigma^A \tilde{A}} f^1(\theta, t)$$

where  $\mu^A$  denotes the intrinsic pest growth rate, and influence of weather takes the form  $f^1(\theta, t) = \theta^\alpha$ . Then:

$$(32) \quad A = A(0) e^{[\mu^A - \frac{(\sigma^A)^2}{2}]t + \sigma^A \tilde{A}} \theta^\alpha$$

It can be shown (see appendix 3) that the pest dynamics in differential form is:

$$(33) \quad dA = [\mu^A + \alpha \mu^\theta + \alpha \sigma^A \sigma^\theta \rho^{A\theta} + \frac{\alpha(\alpha-1)}{\theta^2}] A dt + \sigma^A A d\tilde{A} + \alpha \sigma^\theta A d\tilde{\theta}$$

Following Lichtenberg and Zilberman. (1986), crop growth losses due to pests can be expressed as:

$$(34) \quad L = f(\theta, t) D(A) g(u, t)$$

where  $f(\theta)$  is maximum yield as a function of climate index, and  $D(A)$  denotes the proportional damage function which is assumed to have the following properties:  $D(0) = 0$ ,  $\lim_{A \rightarrow \infty} D(A) = 1$ , and  $\frac{\partial D(A)}{\partial A} \geq 0$ .

Suppose  $D(A)$  is linear in  $A$ , and there exists a  $A_{\max}$  such that  $D(A_{\max}) = 1$ , or  $D(A) = \frac{A}{A_{\max}}$ .

Let's assume that the abatement function is of the form  $g(u, t) = e^{-u^\beta t}$  where  $u$  denotes pesticide use and  $0 < \beta < 1$  which assures decreasing marginal productivity of pesticide use. Assuming  $f(\theta, t) = e^{\mu^Y t} \theta^\gamma$  and  $0 \leq A < A_{\max}$  we can express losses in crop growth over time as follows, where  $\mu^Y$  is the intrinsic growth rate of the crop when the weather index is 1.

$$(35) \quad L = e^{(\mu^Y - u^\beta)t} \theta^\gamma \frac{A}{A_{\max}}$$



substituting (32) into equation (35) for  $A$  gives:

$$(36) \quad L = e^{\mu^Y t} \theta^\gamma \frac{A}{A_{\max}} = \frac{A(0)}{A_{\max}} e^{[\mu^Y + \mu^A - u^\beta - \frac{(\sigma^A)^2}{2}]t + \sigma^A \tilde{A}} \theta^{\alpha + \gamma}$$

Similar to the pest dynamics, we can show that (see appendix 4):

$$(37) \quad dL = \left[ \mu^Y + \mu^A - u^\beta + (\alpha + \gamma) \mu^\theta + (\alpha + \gamma) \sigma^A \sigma^\theta \rho^{\tilde{A}\theta} + \frac{(\alpha + \gamma)(\alpha + \gamma - 1)}{2\theta^2} \right] L dt \\ + \sigma^A L d\tilde{A} + (\alpha + \gamma) \sigma^\theta L d\tilde{\theta}$$

It should be noted that, as in the general case, the correlation coefficient  $\rho^{\tilde{A}\theta}$  is the correlation between pest infestation white noise and change of level of climate index, not be confused with the correlation between pest infestation white noise and climate index white noise ( $\rho^{\tilde{A}\tilde{\theta}}$ ). However, in the specific case, by definition  $COV(d\tilde{A}, d\theta) = \rho^{\tilde{A}\theta} \sigma^\theta dt$ , since  $Var(d\tilde{A}) = Var(d\theta) = dt$ . Also, based on equation (29),  $COV(d\tilde{A}, d\theta) = COV(d\tilde{A}, \sigma^\theta \theta d\tilde{\theta}) = \sigma^\theta \theta \rho^{\tilde{A}\tilde{\theta}} dt$ . Therefore,

$$(38) \quad \rho^{\tilde{A}\theta} = \theta \rho^{\tilde{A}\tilde{\theta}}$$

The correlation coefficients are linearly related. Since  $\theta$  is nonnegative, it must be the case that the two correlation coefficients are monotonically related in a non-negative fashion.

Let  $\mu^L = \mu^Y + \mu^A + (\alpha + \gamma) \mu^\theta + (\alpha + \gamma) \sigma^A \sigma^\theta \rho^{\tilde{A}\theta} + \frac{(\alpha + \gamma)(\alpha + \gamma - 1)}{2\theta^2}$ , then (37) can be rewritten as:

$$(39) \quad dL = (\mu^L - u^\beta) L dt + \sigma^A L d\tilde{A} + (\alpha + \gamma) \sigma^\theta L d\tilde{\theta}$$

The producer's problem is to minimize discounted expected total cost and terminal damage:

$$J = E \min_u \left\{ \int_0^T e^{-rt} w u(t) dt + e^{-rT} p L(T) \right\} \\ \text{s.t. equation (39)}$$

Notice that in the general case, both the dynamics of pest and yield loss are state equations. However, in this specific case, since there is a one-to-one correspondence between  $A$  and  $L$  through the damage function, we have only one state equation.

$$\frac{du^*}{d\rho^{A\theta}} = 0 \text{ if } \alpha + \gamma = 0.$$

where the co-state variable is  $d\lambda = -\frac{\partial H}{\partial L}dt + \sigma^A L d\tilde{A} + (\alpha + \gamma)\sigma^\theta L d\tilde{\theta}$ . Therefore,

where the co-state variable is  $d\lambda = -\frac{\partial H}{\partial L}dt + \sigma^A L d\tilde{A} + (\alpha + \gamma)\sigma^\theta L d\tilde{\theta}$ . Therefore,

Following Chang (2004), the solution to the stochastic differential equation (41) is of the following form:

Since  $\tilde{A}$  and  $\tilde{\theta}$  are Wiener processes, which have expected values of 0 at any particular time, the last two integrals yield a value of zero. Therefore, we have:

The co-state variable has to satisfy the end-point transversality condition:

Evaluating (43) at terminal time  $T$  and combining with equation (44), we have:

Therefore, we must have:

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Thus,

$$(47) \quad E\lambda = pe^{[-r+(\mu^L-u^\beta)]T}e^{[-(\mu^L-u^\beta)]t} = pe^{-rT}e^{(\mu^L-u^\beta)(T-t)}$$

Notice that on the optimal path, one should have  $E\frac{\partial H}{\partial u} = 0$ . Therefore, from (40), we get:

$$(48) \quad e^{-rt}w - E\lambda\beta u^{\beta-1}L = 0$$

Putting (47) into (48), we have:

$$(49) \quad u^{\beta-1} = \frac{e^{-rt}w}{E\lambda\beta L} = \frac{e^{-rt}w}{pe^{-rT}e^{(\mu^L-u^\beta)(T-t)}\beta L} = e^{(r-\mu^L+u^\beta)(T-t)}\frac{w}{p\beta L}$$

The optimal pesticide usage path is implicitly determined by the equation above, since  $u$  appears on both sides of the equation. Though the explicit form of solution cannot be acquired, the comparative dynamics can be derived through implicit differentiation of the equation above w.r.t  $\rho^{\tilde{A}\theta}$ :

$$\begin{aligned} (\beta-1)u^{\beta-2}\frac{du^*}{d\rho^{\tilde{A}\theta}} &= e^{(r-\mu^L+u^\beta)(T-t)}\frac{w}{p\beta L}(T-t)\left[-(\alpha+\gamma)\sigma^A\sigma^\theta + \beta u^{\beta-1}\frac{du^*}{d\rho^{\tilde{A}\theta}}\right] \\ (\beta-1)u^{\beta-2}\frac{du^*}{d\rho^{\tilde{A}\theta}} &= u^{\beta-1}(T-t)\left[\beta u^{\beta-1}\frac{du^*}{d\rho^{\tilde{A}\theta}} - (\alpha+\gamma)\sigma^A\sigma^\theta\right] \\ \left[(\beta-1)u^{-1} - \beta u^{\beta-1}(T-t)\right]\frac{du^*}{d\rho^{\tilde{A}\theta}} &= -(T-t)(\alpha+\gamma)\sigma^A\sigma^\theta \\ \frac{du^*}{d\rho^{\tilde{A}\theta}} &= -\frac{(T-t)(\alpha+\gamma)\sigma^A\sigma^\theta}{(\beta-1)u^{-1} - \beta u^{\beta-1}(T-t)} \end{aligned}$$

Notice that since  $0 < \beta < 1$ , and  $T-t \geq 0$  the effect of the correlation coefficient on optimal use of pesticide is determined by the combined sign of  $\alpha$  and  $\gamma$  as suggested in proposition 3.

Moreover, since  $\frac{d\rho^{\tilde{A}\theta}}{d\rho^{\tilde{A}\theta}} = \theta > 0$  (equation (38)), therefore,

$$\frac{du^*}{d\rho^{\tilde{A}\theta}} = \frac{du^*}{d\rho^{\tilde{A}\theta}} \frac{d\rho^{\tilde{A}\theta}}{d\rho^{\tilde{A}\theta}} = -\frac{(T-t)(\alpha+\gamma)\sigma^A\sigma^\theta}{(\beta-1)u^{-1} - \beta u^{\beta-1}(T-t)}\theta \text{ which has the same sign as } \frac{du^*}{d\rho^{\tilde{A}\theta}}.$$

The proposition is essentially saying that the marginal effect of correlation between climate and pest prediction errors on optimal pesticide usage is of the same direction as the marginal effect of climate on yield loss via the combined effect of  $\alpha$  and  $\gamma$  (equation (36)).

The greater the absolute value of combined direct effect of climate on yield loss and indirect effect of climate, through pest damages, on yield loss the greater the effect of the correlation coefficient on optimal pesticide use in absolute terms. If the combined effect of climate on yield is negative (positive) then the effect of correlation coefficient on optimal use of pesticide is negative (positive). It should also be noted that our comparative dynamics formula does not only tell the sign, but also shows that the magnitude of the marginal effect will be determined by the following factors: 1. Time remaining until harvest; 2. The marginal effect of climate on yield loss; 3. The magnitude of standard errors of pest and climate predictions; 4. The efficiency of the pesticide use; and 5. The current level of pesticide application.

### **Empirical Analyses**

In this section, we examine optimal pesticide application decisions throughout a growing season with the objective of minimizing total costs associated with pea aphid infestations in lentil production. The objective is to empirically test the hypothesis that in the context of weather forecast based conditional pest management correlation between climate prediction errors and pest infestation prediction errors can influence optimal pesticide application decisions.

We use historical lentil production, climate, and aphid infestation data from the Palouse area of northern Idaho and eastern Washington from 1983 to 2009. Idaho and Washington produce 24% and 11% of U.S. lentils (USA Dry Pea and Lentil Council, 2007). The pea aphid (*Acyrtosiphon pisum* (Harris)) reduces lentil yields through direct damage and through vectoring the Pea enation mosaic (PEMV) and Bean leaf roll (BLRV) viruses. Current practice for aphid control is to treat aphids aggressively with dimethoate when the risk of pest outbreak is considered high (Clement 2006; Clement, Husebye, and Eigenbrode 2010). Growers currently have no quantitative way to assess this risk and rely on inspection of plants for symptoms and their own perceptions of risk. On the other hand, seeking

to reduce dimethoate use some growers try to avoid treating for aphids, leaving the crops vulnerable to pest injuries.

For empirical specification we use discrete time stochastic dynamic programming (Knapp et al. 2003; Ross 1995). Due to limited availability of appropriate data and functional relationships between state and control variables we use simplified functional forms and obtain parameter values through regression analysis in cases where corresponding estimates could not be obtained from existing literature. Table 1 provides information on parameter values, corresponding units, and sources.

Objective function:

Consistent with the specifications in previous sections and following Marsh, Huffaker, and Long. (2000), we assume that the growers minimize their expected total per acre discounted value of lost yield and costs of pest management.

$$(50) \quad \min TC = E \left\{ (1+r)^T pL(T) + \sum_{t=0}^{T-1} (1+r)^t wu(t) \right\}$$

where the notations are the same as in the previous section. Growing season for lentils starts in the middle of April but no later than first week of May and harvest starts in the beginning of August (Oplinger et al. 1991). We consequently assume that the total growing season consists of 14 weeks ( $T = 14$ ). Our discount rate for decisions over a single crop growing season is adopted from Marsh, Huffaker, and Long. (2000), where they choose  $r = 0.0005$  for a daily discount rate over a crop growing season. In our case, we let  $r = (1 + 0.0005)^7 - 1 = 0.0035$  to adjust the rate to a weekly rather than daily rate (see Ross, Westerfield, and Jaffe 2009 for discussion on discount rate adjustment). The choice variable in our model is pesticide use which is specified as a binary variable.  $u(t) = 1$  if a grower chooses to spray in week  $t$ ; and  $u(t) = 0$  otherwise.

State Equations:

The state equations are expressed as discrete time counterparts of functional forms adopted

in previous section. The climate dynamics are of the following form<sup>4</sup>:

$$(51) \quad \theta_{t+1} - \theta_t = \left( \mu^\theta + \frac{(\sigma^\theta)^2}{2} \right) \theta_t + \sigma^\theta \theta_t \Delta \tilde{\theta}$$

where the notations are the same as in equations (1) and (29).

Similarly, we assume that pest population dynamics follow:

$$\bar{A}_{t+1} - \bar{A}_t = \left( \mu^A + \frac{(\sigma^A)^2}{2} \right) \bar{A}_t + \sigma^A \bar{A}_t \Delta \tilde{A}$$

and following from equation (31) and (34), we assume that pesticide use and climate adjusted pest population level is:  $A_t = \bar{A}_t (\theta_t)^\alpha (1 - \beta u_t)$ . Then,

$$(52) \quad A_t = \left[ \left( \mu^A + \frac{(\sigma^A)^2}{2} + 1 \right) \bar{A}_{t-1} + \sigma^A \bar{A}_{t-1} \Delta \tilde{A} \right] (\theta_{t-1})^\alpha (1 - \beta u_{t-1})$$

Elbakidze, Lu, and Eigenbrode (2011) report that, without pesticide treatment, a mild (historically observed average in the region) aphid invasion can cause a 40% yield loss. However dimethoate treatment reduces yield loss to 7%. We make an extrapolating assumption that the pesticide application reduces yield loss due to aphid by 85% ( $\beta = 0.85$ ).

The intrinsic crop growth dynamics is of the form:

$$(53) \quad \bar{Y}_{t+1} - \bar{Y}_t = \mu^Y \bar{Y}_t$$

where  $\bar{Y}$  denotes the maximum potential crop growth without aphid attack with climate index fixed at 1. And the damage function is again,  $A_t/A_{\max}$ . Hence, the crop loss function is given by:

$$(54) \quad L_t = (\theta_t)^{\gamma \bar{Y}_t} A_t / A_{\max}$$

Where the other notations are the same as in equation (37). As in the specific case, the only state equation is the change of loss at any given time, which is determined by equation (54).

#### Data and model parameters:

The following Seemingly Unrelated Regression analysis<sup>5</sup> is used to obtain the parameters

used in the state equations.

$$(55) \quad \begin{cases} \ln A = \beta_{10} + \beta_{11}T + \beta_{12} \ln \theta + e_1 \\ \ln \theta = \beta_{20} + \beta_{21}T + e_2 \end{cases}$$

The data for the regression come from two sources. The daily aphid data come from aphid suction trap records maintained by the Department of Entomology at the University of Idaho. Daily temperature data comes from the National Climate Data Center. Following Marsh, Huffaker, and Long. (2000), we use degree-days as a climatic explanatory variable for number of aphids. The degree-days measures the cumulative extent to which average daily temperature exceeds a threshold temperature over time. The threshold temperature varies for different crops and pests. According to University of California, Davis IPM database, the threshold temperature is 5.5 Celsius degree for pea aphid. The parameters  $\beta_{11}$ ,  $\beta_{21}$ ,  $\beta_{12}$  in (55) correspond to the parameters  $\mu^A$ ,  $\mu^\theta$ , and  $\alpha$  in the theoretic model respectively. The standard errors of each equation in (55) represent  $\sigma^A$ ,  $\sigma^\theta$  respectively, and the correlation coefficient between the two error terms is  $\rho^{\tilde{A}\tilde{\theta}}$ .

Furthermore, we run the following OLS regression to obtain a representation of a relationship between climate (degree-days) and yield:

$$(56) \quad \ln Y = \beta_{30} + \beta_{31}A + \beta_{32} \ln \theta + e_3$$

where parameter  $\beta_{32}$  corresponds the  $\gamma$  in our theoretic model. Following Elbakidze, Lu, and Eigenbrode (2011) we use historical state level data of yields from 1983 to 2009 for Washington State. This period is chosen because of data availability on aphid records. For climate index, we use the accumulated growing degree-days on August 1st of each year (the temperature threshold for lentil is 5 Celsius degree, see Gan et al. 2005).

The regression results are shown in table 2. Also, the standard deviation for aphid prediction is 0.167, the standard deviation for temperature deviation is 0.079 and the correlation coefficient is 0.72 (not reported in the table). Since the regression is based on daily obser-

vations, weekly parameters for the simulation are calculated as  $\mu^A = 1.041^7 - 1 = 0.32$ ,  $\mu^\theta = 1.0226^7 - 1 = 0.17$ ,  $\sigma^A = \sqrt{7} \cdot 0.167 = 0.44$ , and  $\sigma^\theta = \sqrt{7} \cdot 0.079 = 0.21$ .

From the historical records of lentil yields, the maximum realized yield was 1600 lb/acre, in 2001 when according to entomological records aphid invasion was mild. Using Elbakidze, Lu, and Eigenbrode (2011) estimate of 7% reduction in lentil yields due to mild aphid invasions with the application of pesticides we calculate maximum potential yield as  $1600 \times 1.07 = 1712$ . When there is no aphid invasion, the terminal period loss is zero and the yield is given by the formula  $Y(T) = e^{\mu^Y T} \theta^\gamma(T)$ . We put in the values of terminal period potential yield (1712) and corresponding degree day value (2330) to find that the intrinsic growth rate of lentils is 0.137.

#### Stochastic Simulation:

With the assigned values for the parameters, we run the optimization 1000 times and solve using recursive dynamic programming. In our formulation, aphid and climate index are stochastic processes which follow diffusion processes specified as geometric Brownian motions. Each run generates optimal pesticide application for a growing season. The mean value and 95% confidence intervals of the optimal solution paths are given in figure 1. It is easy to see that this result is in favor of applying pesticides sooner rather than later. This coincides with Olson and Roy (2002)'s vision of early pesticide applications suppressing pests in the current period as well as suppressing offspring of pests killed in early periods. Recall that we have shown in our specific case that if  $\alpha + \gamma < 0$  and  $\rho^{\tilde{A}\tilde{\theta}} > 0$ , which is the case in our empirical example, then the optimal pesticide use is negatively affected by  $\rho^{\tilde{A}\tilde{\theta}}$ . For the sake of demonstration we compare pesticide use paths under  $\rho^{\tilde{A}\tilde{\theta}} = 0$  and under  $\rho^{\tilde{A}\tilde{\theta}} = 0.72$ . In addition to visual demonstration we test if there are statistically significant differences between optimized pesticide uses when  $\rho^{\tilde{A}\tilde{\theta}} = 0$  and when  $\rho^{\tilde{A}\tilde{\theta}} = 0.72$  for any given week. One sided t-tests for the mean values of pesticide usage for each week are reported in table 3. It is clear that between weeks 4 and 7 pesticide use when  $\rho^{\tilde{A}\tilde{\theta}} = 0.72$  is statistically significantly lower than in the case of  $\rho^{\tilde{A}\tilde{\theta}} = 0$  under 15% critical value. In the



other weeks, the differences are not significant. This comes from the fact that the growers will almost surely spray in the first few weeks and not spray in the last few weeks. Consequently, the differences in the early and late growing seasons are not significant. Overall, the empirical result shows that the optimal pesticide use when  $\rho^{\tilde{A}\tilde{\theta}} = 0.72$  in some weeks should be lower than the case of  $\rho^{\tilde{A}\tilde{\theta}} = 0$ , which coincides with our expectations from previous section.

*The Two Treatment Constraint:*

Dimethoate is considered to be highly toxic. The legal restriction, regulated by the EPA (EPA 2008), on dimethoate application is that it can be applied at most twice within a growing season. To reflect this, we reproduce the results with this constraint. The mean value and 95% confidence intervals of the optimal solution path are given in figure 2.

Figure 2 suggests that pesticide application tends to be more advantageous in the later part of the growing season if number of applications is restricted. Several factors may contribute to the shape of the figure. Since the spraying is limited to two sprays over the growing season, if a grower chooses to spray very early, he will give up the opportunity of spraying later. Depending on the predictions of aphid infestations over the growing season and the expected accuracy of such predictions the grower may choose to spray in the later periods of growing season. In addition to growth in the population of aphids which invaded the area in previous periods our model also allows for additional invasions to occur in the later stages of the growing season. Spraying in the earlier stages will kill aphids in the current period and prevent corresponding population growth that would be produced by the killed aphids. However, earlier sprays will permit infestations in the later periods to grow unchecked. Therefore, given constrained spraying frequency, the grower is likely to apply pesticide in the later stages of production when a spray kills offsprings of all previous invaders as well as new invaders. Furthermore, aphids are likely to multiply and thrive in later stages of production with greater green biomass of the crop corresponding to more favorable habitat Cobourn et al. (2011). Exponential aphid growth specification

in our model implies that incremental growth of aphid population is relatively low in the first few weeks. Therefore, it may be more important to make sure that aphid growth in the later stages is stopped if necessary. Again, to compare pesticide use when  $\rho^{\tilde{A}\tilde{\theta}} = 0$  vs. when  $\rho^{\tilde{A}\tilde{\theta}} = 0.72$ , we conduct one sided t-test for the mean values of pesticide use for each week (Table 3), The results show that there is almost no statistically significant difference between the two cases. Existing pesticide use restriction makes correlation between weather and pest outbreak prediction errors insignificant for optimal pesticide use during the growing season.

## Conclusion/Discussion

In this paper, we examine stochastic dynamic pest management in agricultural crop production under two stochastic factors that influence agricultural productivity: climate and pest populations. Predictions, or expected values, of climatic variables and pest populations can be used to improve pest management practices. We extend this idea by explicitly showing that the pest management practices can potentially be further improved by taking into account potential correlation between prediction errors for climatic variables and pest populations.

We first set up a general discounted cost minimization problem with stochastic climate and pest population variables. We provide necessary condition for optimal pesticide use path and discuss properties of the solution. Choosing functional forms that allow for mathematical tractability we find a closed form solution for pesticide use as a function of the correlation coefficient between pest and climate forecast errors. Moreover, we provide conditions for when pesticide use is monotonically increasing, and when it is decreasing in the correlation coefficient.

Although our theoretical analysis demonstrates potential role of correlation between prediction errors in optimal pest management decisions over the growing season, our empirical case study shows that for strictly regulated toxic pesticides like dimethoate the correlation

coefficient does not play a significant role. If dimethoate use was not regulated then our results show that pesticide use application rates can be suboptimal in some periods of growing season if correlation between weather and pest infestation prediction errors is not considered. However, for the case of pea aphid control in lentil production using dimethoate the correlation between weather and aphid infestation prediction errors does not affect pesticide use throughout the growing season because of tight restrictions imposed on dimethoate application.

Our paper further extends the stochastic optimal control and pest management literature in two notable ways. First, though our general framework is within the pest management context, the tools developed in this paper, i.e. the stochastic envelope theorem and stochastic comparative dynamics can be found applicable in many other fields that require a dynamic and stochastic environment. Second, our paper extends pest management literature by considering the dynamic climate influence and the stochastic relationship between weather prediction and pest invasion prediction, which can be viewed as an extension of Marsh, Huffaker, and Long. (2000) and Olson and Roy (2002). For future studies, we suggest to relax the condition that, in our analytical model,  $\rho^{\tilde{A}\tilde{\theta}}$  is constant over time. If the mathematical sophistication allows, one can examine how dynamic changes in the correlation could impact the results. The limitations of our model lie in the simplifying assumptions on the functional forms and assumptions driven by limited data availability pertaining to the effects of pests on crop growth, effectiveness of pesticides, the role of climate, and historical aphid records.

## Notes

<sup>1</sup>We use subscripts to denote derivatives throughout the current and the following section.

<sup>2</sup>Notice that the covariance and correlation are equal because standard errors for  $d\tilde{\theta}$  and for  $d\tilde{A}$  are  $dt$ .

<sup>3</sup>In the specific case we simplify the formulation for analytical tractability and assume that  $\mu^\theta$ ,  $\mu^A$ ,  $\sigma^\theta$ ,  $\sigma^A$  are constant over time. In other words, the rate of change of weather conditions and pest populations and standard errors of corresponding prediction errors throughout the growing season are assumed to be constant.

<sup>4</sup>In this section, the subscripts are used to denote time index (weeks).

<sup>5</sup>Note that  $\beta_{11}$  in equation (55) stands for the growth rate of aphid ( $\mu^A$ ), since the parameter has the meaning of how much percentage of aphid change could happen if time increase by one unit.

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## Appendix

### Appendix 1a

According to Ito's Lemma, if  $X_1, \dots, X_n$  are stochastic diffusion processes and  $Y = f(X_1, \dots, X_n)$  where  $f$  is twice differentiable, then (See Kao 1996):

$$dY = \sum_{i=1}^n f_{X_i} dX_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n f_{X_i X_j} dX_i dX_j$$

To show (3), apply Ito's Lemma to equation (2) to get:

$$dA = A_t dt + A_\theta d\theta + A_{\tilde{A}} d\tilde{A} + A_{\theta \tilde{A}} d\theta d\tilde{A} + A_{\theta t} d\theta dt + A_{\tilde{A} t} d\tilde{A} dt + \frac{1}{2} (A_{\tilde{A} \tilde{A}} d\tilde{A} d\tilde{A} + A_{\theta \theta} d\theta d\theta)$$

According to white noise properties (See Kamien and Schwartz, 1991 for instance), we have:  $d\theta d\tilde{A} = \sigma^\theta \rho^{A\theta} dt$ ;  $d\theta d\theta = (\sigma^\theta)^2 dt$ ;  $d\tilde{A} d\tilde{A} = dt$ ;  $d\theta dt = 0$ , and  $d\tilde{A} dt = 0$ .

Then,

$$dA = A_t dt + A_\theta d\theta + A_{\tilde{A}} d\tilde{A} + A_{\theta \tilde{A}} \rho \sigma^\theta dt + \frac{1}{2} A_{\tilde{A} \tilde{A}} dt + \frac{1}{2} A_{\theta \theta} (\sigma^\theta)^2 dt$$

substituting equation (1) for  $d\theta$ , we get equation (3):

$$dA = \left( A_t + A_\theta \mu^\theta + \frac{1}{2} A_{\theta \theta} (\sigma^\theta)^2 + \frac{1}{2} A_{\tilde{A} \tilde{A}} + A_{\theta \tilde{A}} \rho^{\tilde{\theta} \tilde{A}} \sigma^\theta \right) dt + A_\theta \sigma^\theta (\theta, t) d\tilde{\theta} + A_{\tilde{A}} d\tilde{A}$$

### Appendix 1b

Similarly, since  $L(t) = L(A(u(t), \theta(t), t, \tilde{A}), \theta(t, \tilde{\theta}), t)$ , by Ito's Lemma, we have:

$$dL = L_t dt + L_\theta d\theta + L_A dA + L_{\theta A} d\theta dA + L_{\theta t} d\theta dt + L_{A t} dA dt + \frac{1}{2} (L_{AA} dA dA + L_{\theta \theta} d\theta d\theta)$$

Using equations (1), (4) and  $dA dA = (\sigma^A)^2 dt$  and  $d\theta d\theta = (\sigma^\theta)^2 dt$ , we have

$$dL = \left( L_t + L_\theta \mu^\theta + L_A \mu^A + \frac{1}{2} L_{\theta \theta} (\sigma^\theta)^2 + \frac{1}{2} L_{AA} (\sigma^A)^2 + L_{\theta A} \rho^{\tilde{\theta} \tilde{A}} \sigma^\theta \sigma^A \right) dt + (L_\theta \sigma^\theta (\theta, t) + L_{A \theta} \sigma^\theta (\theta, t)) d\tilde{\theta} + L_A A_{\tilde{A}} d\tilde{A}$$

## Appendix 2

To show that  $\theta = \theta(0) e^{[\mu^\theta - (\frac{\sigma^\theta}{2})^2]t + \sigma^\theta \tilde{\theta}}$  is indeed a solution to (29). Take derivatives of

theta with respect to its arguments to get:  $\theta_t = \left[ \mu^\theta - \frac{(\sigma^\theta)^2}{2} \right] \theta(0) e^{[\mu^\theta - \frac{(\sigma^\theta)^2}{2}]t + \sigma^\theta \tilde{\theta}} = \left[ \mu^\theta - \frac{(\sigma^\theta)^2}{2} \right] \theta$

$$\begin{aligned}\theta_{\tilde{\theta}} &= \sigma^\theta \theta(0) e^{[\mu^\theta - \frac{(\sigma^\theta)^2}{2}]t + \sigma^\theta \tilde{\theta}} = \sigma^\theta \theta \\ \theta_{\tilde{\theta}\tilde{\theta}} &= (\sigma^\theta)^2 \theta(0) e^{[\mu^\theta - \frac{(\sigma^\theta)^2}{2}]t + \sigma^\theta \tilde{\theta}} = (\sigma^\theta)^2 \theta\end{aligned}$$

Then by Ito's lemma, we have:

$$d\theta = \left( \theta_t + \frac{\theta_{\tilde{\theta}\tilde{\theta}}}{2} \right) dt + \theta_{\tilde{\theta}} d\tilde{\theta} = \mu^\theta \theta dt + \sigma^\theta \theta d\tilde{\theta}$$

wich is the differential expression in (29).

### Appendix 3

Similar to appendix 2, for  $A = A(0) e^{[\mu^A - \frac{(\sigma^A)^2}{2} - u^\beta]t + \sigma^A \tilde{A}} \theta^\alpha$ , we calculate:  $A_t =$

$$[\mu^A - \frac{(\sigma^A)^2}{2} - u^\beta] A(0) e^{[\mu^A - \frac{(\sigma^A)^2}{2} - u^\beta]t + \sigma^A \tilde{A}} \theta^\alpha = [\mu^A - \frac{(\sigma^A)^2}{2} - u^\beta] A$$

$$A_{\tilde{A}} = \sigma^A A(0) e^{[\mu^A - \frac{(\sigma^A)^2}{2} - u^\beta]t + \sigma^A \tilde{A}} \theta^\alpha = \sigma^A A$$

$$A_{\tilde{A}} = \sigma^A A(0) e^{[\mu^A - \frac{(\sigma^A)^2}{2} - u^\beta]t + \sigma^A \tilde{A}} \theta^\alpha = \sigma^A A$$

$$A_{\tilde{A}\tilde{A}} = (\sigma^A)^2 A(0) e^{[\mu^A - \frac{(\sigma^A)^2}{2} - u^\beta]t + \sigma^A \tilde{A}} \theta^\alpha = (\sigma^A)^2 A$$

$$A_\theta = \alpha A(0) e^{[\mu^A - \frac{(\sigma^A)^2}{2} - u^\beta]t + \sigma^A \tilde{A}} \theta^{\alpha-1} = \frac{\alpha}{\theta} A$$

$$A_{\theta\theta} = \alpha(\alpha-1) A(0) e^{[\mu^A - \frac{(\sigma^A)^2}{2} - u^\beta]t + \sigma^A \tilde{A}} \theta^{\alpha-2} = \frac{\alpha(\alpha-1)}{\theta^2} A$$

$$A_{\tilde{A}\theta} = \alpha \sigma^A A(0) e^{[\mu^A - \frac{(\sigma^A)^2}{2} - u^\beta]t + \sigma^A \tilde{A}} \theta^{\alpha-1} = \frac{\alpha \sigma^A}{\theta} A$$

Again, by Ito's lemma,

$$\begin{aligned}dA &= \left( A_t + \frac{A_{\tilde{A}\tilde{A}}}{2} + \frac{A_{\theta\theta}}{2} \right) dt + A_\theta d\theta + A_{\tilde{A}} d\tilde{A} + A_{\tilde{A}\theta} d\theta d\tilde{A} \\ &= [\mu^A + \alpha \mu^\theta + \alpha \sigma^A \sigma^\theta \rho^{A\theta} + \frac{\alpha(\alpha-1)}{\theta^2}] A dt + \sigma^A A d\tilde{A} + \alpha \sigma^\theta A d\tilde{\theta}\end{aligned}$$

### Appendix 4

$$L = e^{\mu^Y t} \theta^\gamma \frac{A}{A_{\max}} = \frac{A(0)}{A_{\max}} e^{[\mu^Y + \mu^A - \frac{(\sigma^A)^2}{2} - u^\beta]t + \sigma^A \tilde{A}} \theta^{\alpha+\gamma}$$

$$L_t = [\mu^Y + \mu^A - \frac{(\sigma^A)^2}{2} - u^\beta] e^{\mu^Y t} \theta^\gamma \frac{A}{A_{\max}} = [\mu^Y + \mu^A - \frac{(\sigma^A)^2}{2} - u^\beta] L$$

$$L_{\tilde{A}} = \sigma^A \frac{A(0)}{A_{\max}} e^{[\mu^Y + \mu^A - \frac{(\sigma^A)^2}{2} - u^\beta]t + \sigma^A \tilde{A}} \theta^{\alpha+\gamma} = \sigma^A L$$

$$L_{\tilde{A}\tilde{A}} = (\sigma^A)^2 \frac{A(0)}{A_{\max}} e^{[\mu^Y + \mu^A - \frac{(\sigma^A)^2}{2} - u^\beta]t + \sigma^A \tilde{A}} \theta^{\alpha+\gamma} = (\sigma^A)^2 L$$

$$L_\theta = (\alpha + \gamma) \frac{A(0)}{A_{\max}} e^{[\mu^Y + \mu^A - \frac{(\sigma^A)^2}{2} - u^\beta]t + \sigma^A \tilde{A}} \theta^{\alpha+\gamma-1} = \frac{\alpha+\gamma}{\theta} L$$

$$L_{\theta\theta} = (\alpha + \gamma)(\alpha + \gamma - 1) \frac{A(0)}{A_{\max}} e^{[\mu^Y + \mu^A - \frac{(\sigma^A)^2}{2} - u^\beta]_t + \sigma^A \tilde{A}} \theta^{\alpha + \gamma - 2} = \frac{(\alpha + \gamma)(\alpha + \gamma - 1)}{\theta^2} L$$

$$L_{\tilde{A}\theta} = (\alpha + \gamma) \sigma^A \frac{A(0)}{A_{\max}} e^{[\mu^Y + \mu^A - \frac{(\sigma^A)^2}{2} - u^\beta]_t + \sigma^A \tilde{A}} \theta^{\alpha + \gamma - 1} = \frac{\alpha + \gamma}{\theta} \sigma^A L$$

Thus, based on Ito's Lemma and substituting equation (29) for  $d\theta$ , keeping in mind that

$d\theta dt = 0$  and  $d\tilde{A}dt = 0$ , we get:

$$\begin{aligned} dL &= \left( L_t + \frac{L_{\tilde{A}\tilde{A}}}{2} + \frac{L_{\theta\theta}}{2} \right) dt + L_\theta d\theta + L_{\tilde{A}} d\tilde{A} + L_{\tilde{A}\theta} d\theta d\tilde{A} + L_{\tilde{A}t} dt d\tilde{A} + L_{t\theta} d\theta dt \\ &= \left( L_t + \frac{L_{\tilde{A}\tilde{A}}}{2} + \frac{L_{\theta\theta}}{2} \right) dt + L_\theta (\mu^\theta \theta dt + \sigma^\theta \theta d\tilde{\theta}) + L_{\tilde{A}} d\tilde{A} + L_{\tilde{A}\theta} \sigma^\theta \theta \rho^{\tilde{A}\theta} dt \\ &= \left( L_t + \frac{L_{\tilde{A}\tilde{A}}}{2} + \frac{L_{\theta\theta}}{2} + L_\theta \mu^\theta \theta + L_{\tilde{A}\theta} \sigma^\theta \theta \rho^{\tilde{A}\theta} \right) dt + L_{\tilde{A}} d\tilde{A} + L_\theta \sigma^\theta \theta d\tilde{\theta} \\ &= \left( \mu^Y + \mu^A - u^\beta + \frac{(\alpha + \gamma)(\alpha + \gamma - 1)}{2\theta^2} + (\alpha + \gamma) \mu^\theta + (\alpha + \gamma) \sigma^A \sigma^\theta \rho^{\tilde{A}\theta} \right) L dt + \sigma^A L d\tilde{A} + (\alpha + \gamma) L \sigma^\theta d\tilde{\theta} \end{aligned}$$

## Figures

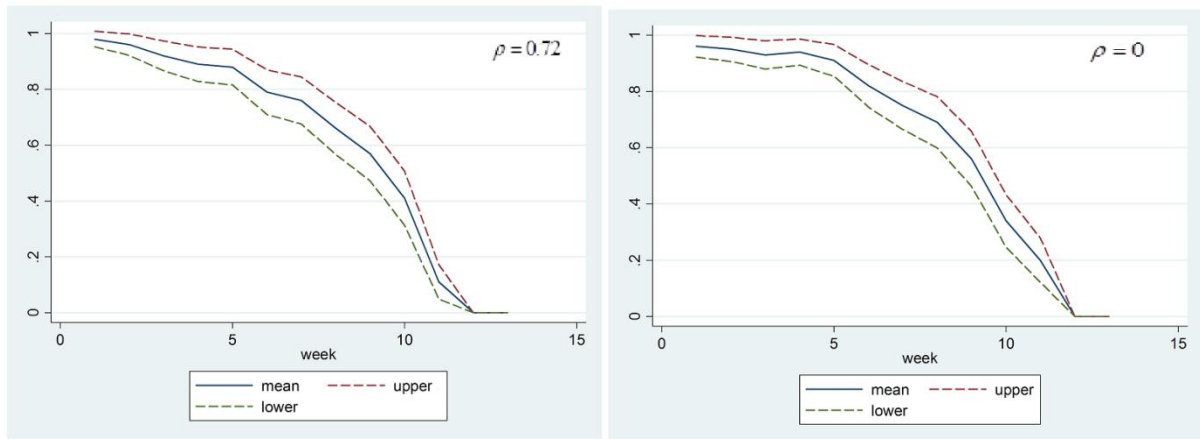
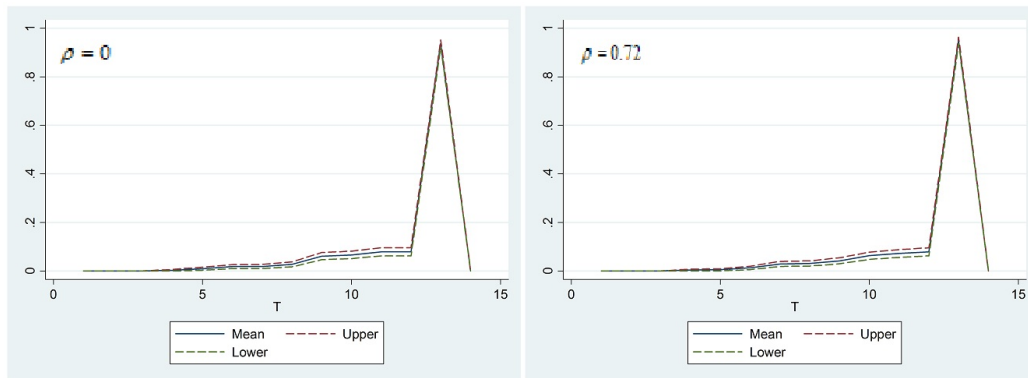


Figure 1. The optimal pesticide usage path



**Figure 2. The optimal usage path with constraint**

## Tables

**Table 1. Summary of parameters used in the empirical analyses**

Parameter	Value (unit)	Source	Description
$\sigma^A$	0.44	Regression Analysis	Standard deviation of aphid diffusion process
$\sigma^\theta$	0.21	Regression Analysis	Standard deviation of climate diffusion process
$\mu^A$	0.32	Regression Analysis	Growth rate of aphid
$\mu^Y$	0.137	Regression Analysis	Growth rate of crop
$\rho$	0.72	Regression Analysis	Correlation coefficient between weather and aphid prediction errors ( $\rho^{\tilde{A}\tilde{\theta}}$ )
$T$	14 weeks	Oplinger et al., (1990)	A full growing season
$\alpha$	-0.037	Regression Analysis	Marginal effect of climate on aphid growth
$\beta$	0.85	Elbakidze et al. (2011)	Pesticide efficiency
$\gamma$	-0.074	Regression Analysis	Marginal effect of climate on crop growth
$r$	0.0035	Marsh et al. (2000)	Discount rate
$P$	0.295 \$/lb	Painter (2011)	Per pound lentil price
$W$	4.84 \$/Acre	Painter (2011)	Per acre pesticide cost
$A(0)$	1	Assumed	Initial number of aphids
$A_{\max}$	$1.8 \times 10^7$	Oplinger et al., (1990)	Carrying capacity of aphid per acre
$\mu^\theta$	0.17	Regression Analysis	Growth rate of climate index

**Table 2. Regression Results**

VARIABLES	SUR formulation (55)		Equation (56)
	$\ln \theta$	$\ln A$	$\ln Y$
T	0.0227*** -0.00188	0.0407*** -0.00526	
$\ln \theta$		-0.0374 -0.166	-0.0742 -0.235
A			-0.0568* -0.0316
Constant	4.813*** -0.191	-0.176 -0.881	7.751*** -1.764
Observations	137	137	23
R-squared	0.514	0.463	0.16

Standard errors in parentheses

\*\* p&lt;0.01, \* p&lt;0.05, \* p&lt;0.1

**Table 3. T-test for pesticide usage comparison**

	$H_0 : u_{\rho=0} \leq u_{\rho=0.72}$													
week	1	2	3	4	5	6	7	8	9	10	11	12	13	14
p-value with constraint	0.65	0.8	0.3	0.09	0.1	0.15	0.08	0.25	0.88	0.96	0.75	0.99	0.99	0.99
p-value without constraint	0.99	0.99	0.86	0.67	0.23	0.05	0.09	0.12	0.11	0.26	0.33	0.88	0.97	0.99