A Note on Maximizing Utility in Quadratic Programming Models for Farm Planning*

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Methods of obtaining a utility maximizing solution from quadratic programming models for farm planning are presented. The methods require only simple modifications to models constructed to produce an EV frontier from a portfolio selection model.

One method of whole farm planning under conditions of risk is to build a portfolio selection model of the possible farm activities, using quadratic programming (see for example, Anderson, Dillon and Hardaker (1977, Chapter 7) referred to hereafter as ADH). The analyst who uses quadratic programming may produce the entire EV frontier for a given set of possible farm activities and farm resources. The analyst may then encourage the farmer to choose the farm plan that best suits his risk-return attitudes. Alternatively, the analyst may use the farmer's utility function (reflecting the same risk-return attitudes) to directly prescribe an optimal farm plan. This would be a point on the EV frontier, tangential to an indifference curve derived from the utility function (see ADH, Figure 7.4, p.201).

While utility functions with a variety of functional forms can be found in the literature, only two of these forms will be considered in this note. They are the quadratic and negative exponential functions.

The Quadratic Utility Function

The quadratic utility function is used because it is easy to estimate and to manipulate mathematically. It has been criticized, first because it has a maximum point beyond which total utility decreases, and secondly because it implies increasing risk aversion with increasing wealth. The first criticism can be overcome to some extent, in a pragmatic way by bounding a function, and/or by splicing two or more quadratic functions together.

ADH (p.202) show a method for deriving an optimal farm plan with a quadratic utility function, using a weighted average of successive change of basis solutions on the EV frontier. Their method requires considerable calculation to be made after the EV frontier solutions have been determined from the quadratic programming algorithm[1]. The primary purpose of this note is to show that the same optimal solution can be found using a simple re-specification of the original quadratic programming model. This allows the optimal farm plan to be obtained directly from the solution of the quadratic programming problem.

The portfolio selection problem which produces an EV frontier can be solved using a quadratic programming model (Markowitz 1956) specified as

$$\text{Maximize } Z = \lambda c'x - x'Dx$$

subject to

$$Ax = b$$

and

$$x \geq 0$$

where λ is a parametric variable
c is a vector of prices and costs associated with each activity
x is a vector of activity levels
A is a matrix of input-output coefficients
b is a vector of resource constraints or requirements
D is the variance-covariance matrix and represents the stochastic relationships between activities.

* I wish to thank Gordon MacAulay, John Kennedy and one other anonymous reviewer for their constructive comments on this note.

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[1] It requires the additional calculation of

$$H_k = x_k^{*\prime} D x_{k+1}^{*}$$

where x_k^* and x_{k+1}^* are successive change of basic solutions.

This matrix is a covariance of returns between successive solutions. It is used in formula 7.18 of ADH (p.202) to give a utility maximizing solution.

The farmer's utility function might be described by the quadratic form

$$U = aR + bR^2$$

where U is utility
R is return from a given portfolio (farm plan)
a and b are constants estimated by one of the hypothetical lottery or experimental methods (see ADH (pp.69-76) and Binswanger (1980) respectively).

Expected utility is then defined as

$$E(U) = a\,E(R) + bE\,(R^2)$$
$$= a\,M + b(M^2 + \sigma^2)$$

where M is the expected return from the portfolio
σ^2 is the variance of return from the portfolio.

This result can be substituted into the objective function[2] of the quadratic programming model, which becomes

$$\text{Max } U = ac'x + b((c'x)^2) + bx'Dx$$

The problem then becomes one of introducing the variable $(c'x)^2$ into the objective function. This can be achieved by inserting an equality constraint into the programming model. This constraint has entries corresponding to the linear part of the objective function. A related transfer activity is also inserted into the model. It transfers the value of the constraint (equal to $c'x$) into the quadratic part of the objective function, after multiplying it by b. This entry in the quadratic part of the objective function thus becomes $b((c'x)^2)$.

This can be illustrated in a programming tableau, using the problem solved by ADH (pp.197-202). The first tableau is for the portfolio selection problem where the full EV frontier is derived. The Rand QP program (Cutler and Pass 1971) is particularly useful in solving this problem, as it includes in its algorithm automatic parameterization of λ.

Some minor modifications of the tableau data are required for this program. Only one triangle of the variance-covariance matrix

Tableau 1: *The Full EV Portfolio Problem from Anderson, Dillon and Hardaker adapted for the Rand QP Program*[1]

		Wheat	Oats	New Wheat	Fixed Cost	Constraint Type[2]	RHS
Quadratic part of the objective function	(Wheat	3600	3310	7814			
	(Oats		1980	4940			
	(New Wheat			5476			
Linear part of the objective function	(GM	−72	−53.4	−88.8	200	N	
Linear constraint	(Land	1	1	1		L	12
	(Rotation	1		1		L	8
	(Capital	30	20	40		L	400
	(Labour	5	5	8		L	80
	(Fixed (Cost Row				1	E	1

[1] The Rand QP Program minimizes an objective function. If maximization is required, the elements of the objective function are multiplied by −1. The program has a further characteristic of relevance to data entry. It is possible to enter the symmetric quadratic part of the objective function by using one triangle of this matrix. If this option is used, then the off-diagonal elements of the triangle should be doubled.

[2] The constraint type symbols are standard mathematical programming system symbols. They are
N non constrained row
L less than or equal to constraint
E equality constraint

[2] In reviewing this note both John Kennedy and Gordon MacAulay pointed out that this objective function could be expressed alternatively as

$$\text{Max } U = ac'x + bx'(cc' + D)x$$

The term cc' is a matrix of the squares and cross products of the gross margins. It would be added to the variance covariance matrix D, and the quadratic programming problem solved in the usual way. Theirs is a more elegant solution, but from a computational viewpoint there seems little to choose between this and the method outlined above.

needs to be entered, but if this is done, the covariance elements must be doubled. The objective function is

$$\text{Minimize } Z = \lambda c'x + x'Dx$$

so that the signs on the linear part of the objective function must be reversed for a maximization problem. Where a fixed cost is included in the farm planning model (as is the case for the ADH problem) it can be entered as an activity, constrained to equal one. Running the model corresponding to Tableau 1 produces the same EV solutions as those given by ADH (p.201).

The second tableau is for utility maximization. The additional row (called *GMR* in this case) is inserted into the model. It is an equality constraint with a value of zero in the right hand side. This forces the value of the activity (called *GMSQ* in this case) into the quadratic part of the objective function. The quadratic part of the objective function in Tableau 2 incorporates the quadratic part of the objective function in Tableau 1, multiplied all through by $-b$.

In the ADH problem the utility function has the form

$$U = R - 0.0005R^2$$

$$\text{i.e. } a = 1, \quad b = -0.0005$$

Thus, in this case, all elements in the quadratic part of the objective function are multiplied by 0.0005 (i.e. by $-b$). Solving the utility maximization problem using this specification gives the same solution as that calculated by ADH (p.202).

If the utility function has the form $a \neq 1$ then it may be subjected to a suitable linear transformation to make $a = 1$. Alternatively all elements in the linear part of the objective function can be multiplied by a.

The method of solving the utility maximization problem presented here is useful in that it requires only simple changes to the original portfolio selection problem to give a quadratic utility maximizing solution. These changes can be made manually, or by a computer program which converts the data for portfolio selection into that required for utility maximization, given any quadratic utility function. This computer program, written in Fortran 5, is available from the author.

The Negative Exponential Utility Function

This function was used by Freund (1956) in his pioneering paper "The Introduction of Risk into a Programming Model". The utility function has the form

$$U = 1 - e^{-aR}$$

where a is a risk aversion coefficient.

Tableau 2: The Utility Maximization Problem from Anderson, Dillon and Hardaker adapted for the Rand QP Program

		Wheat	Oats	New Wheat	GMSQ	Fixed Cost	Constraint Type	RHS
Each element in the quadratic part of the objective function is multiplied by $-b$	(Wheat	1.8	1.655	3.907	0.0			
	(Oats		0.99	2.47	0.0			
	(New Wheat			2.738	0.0			
	(GMSQ				0.0005			
Each element of the linear part of the objective function is multiplied by a	(GM	−72	−53.4	−88.8		200	N	
Row used to transfer $(c^1x)^2$ to objective function	(GMR	−72	−53.4	−88.8	1	200	E	0
Linear constraint	(Land	1	1	1			L	12
	(Rotation	1		1			L	8
	(Capital	30	20	40			L	400
	(Labour	5	5	8			L	80
	(Fixed Cost					1	E	1
	(Row							

This function has two theoretical advantages over the quadratic function. It is monotonically increasing to an asymptote, and it exhibits constant risk aversion. It also has a slight practical advantage in that it is easy to modify the portfolio selection problem to solve for a given risk aversion coefficient.

Freund shows that if net revenue is normally distributed, the solution for maximum expected utility is the same as the solution for the problem

$$\text{Max } Y = c'x - a/2\ x'Dx$$

This maximization problem can be solved using the Rand program by reversing the signs on the objective function

$$\text{Min } Z = -(c'x) + a/2\ x'Dx$$

This problem can be solved in a number of ways. The first (illustrated in Tableau 3 for the Freund problem) is to multiply D by $a/2$. This can be done manually or by use of a modified version of the Fortran program referred to above. Freund used a value of $a = 1/1250$. Thus each element of D is multiplied by 0.0004.

A second method makes use of the parameter λ in the objective function of the Rand program. The program allows this parameter to be set at any value. In this case it is set at 2500, which is the reciprocal of 0.0004.

Concluding Remarks

This note has shown methods for converting portfolio selection models used for farm planning purposes into models that maximize a particular decision maker's quadratic or negative exponential utility function. The methods outlined obviously can be generalized from farm planning problems to any portfolio selection and corresponding utility maximization problem.

Tableau 3: The Quadratic Part of the Objective Function for the Freund Utility Maximization Problem adapted for the Rand Program

	Potatoes	Corn	Beef	Cabbage
Potatoes	2.921876	0.723112	−0.550984	−1.48964
Corn		0.248064	−0.376912	0.088344
Beef			0.449856	0.60552
Cabbage				1.475812

The rest of the tableau, that is the linear part of the objective function and the linear constraint set, are the same as given by Freund (1956, table 1).

References

ANDERSON, J. R., DILLON, J. L., and HARDAKER, J. B. (1977), *Agricultural Decision Analysis*, Iowa State University Press, Ames.

BINSWANGER, H. P. (1980), "Attitudes towards risk: experimental measurement in rural India", *American Journal of Agricultural Economics* 63(3), 395-407.

CUTLER, L., and PASS, D. S. (1971), *A Computer Program for Quadratic Mathematical Models to be Used for Aircraft Design and Other Applications Involving Linear Constraints*, Rand Corporation Report R-516-PR.

FREUND, R. J. (1956), "The introduction of risk into a programming model", *Econometrica* 24, 253-63.

MARKOWITZ, H. M. (1956), "The optimization of a quadratic function subject to linear constraints", *Naval Research Logistics Quarterly* 3, 111-33.