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Stochastic optimal control in bio-economic simulations

A comparison of solution techniques

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Uncertainty has long been recognised as an important aspect of renewable resource assessment and management. Stochastic optimal control provides a framework in which to incorporate uncertainty, whether arising from fluctuations in the biological or economic environment or from lack of a precise understanding of inter-relationships within a system. However, overlaying complex and interdependent biological, physical and economic relationships with uncertainty often results in an optimal control problem which is analytically complex.

In this paper, a parametric approximation to the control equation is combined with genetic search algorithms to solve the stochastic control problem. The parametric approximation to the solution of optimal control problems is compared with a collocation approach. The use of these two numerical solution techniques is explored in the context of a harvest model for a multi-species fishery.

While the two techniques yielded similar solutions, they offered different advantages and disadvantages. The use of collocation methods facilitates the understanding of the problem and the nature of the solution. However, for multi-dimensional state space problems, collocation techniques require exponentially increasing computational time. Parametric approximation techniques require prior specification of an explicit relationship between the state and control variables. As a result, the approximation may impose or miss features of the solution. However, when combined with a genetic search algorithm, the technique is very robust and computation time is significantly less than for the collocation technique. The use of collocation techniques to characterise the solution to the problem followed by the application of an appropriate approximation technique may to prove to be an expedient method for dealing with larger scale problems.

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1. Introduction

The value of a renewable resource stock is derived from both the returns available to resource users from extraction of the stock now and the inherent value of the resource as the basis for the future stock. In determining the value of access to a resource, it is therefore important to consider the integration of both the physical characteristics that describe the evolution of the resource through time, and management regimes and individual incentives that direct use of the resource.

Optimal control theory provides a framework in which the individual or collective incentives of resource users can be embedded within a model which specifies the natural evolution of the resource through a set of differential or difference equations. Uncertainty, whether from fluctuations in the biological or economic environment or from lack of a precise understanding of inter-relationships within a system, is an important aspect of resource assessment and management.

In fisheries, the magnitude of the initial stock of the resource is difficult to determine. The relationships, which determine biological or physical development, may change with seasonal conditions and may not be well known. Hence, the impact of resource use or extraction on the remaining stock is highly uncertain. Stochastic optimal control extends the control framework to incorporate uncertainty that can arise from factors such as weather and imperfect information regarding biological relationships. However, overlaying complex and interdependent biological, physical and economic relationships with uncertainty often results in an optimal control problem which is analytically complex.

While offering a robust approach in theory, few optimal control problems can be solved analytically (Miranda and Fackler 1997). To obtain a solution, either essential features of the system must be ignored to derive an algebraically tractable model, or numerical techniques must be applied. Numerical techniques allow the consideration of problems that are closer to those faced in reality, by resource managers. These techniques generally employ some form of approximation to reduce the control problem to a finite dimensional optimisation that can be solved using a search algorithm. While straightforward in principle, numerical techniques are often highly problematic (Judd 1997). Problems can include the accumulation of rounding errors when response surfaces are flat, problems associated with discontinuities, and the presence of local optima. However, even within a well-behaved system, computational requirements tend to expand exponentially as the number of state and stochastic variables increases.

In this paper, a parametric approximation to the control equation is combined with genetic search algorithms to solve the stochastic control problem. Genetic search algorithms provide a robust technique for solving non-linear programming problems (Holland 1975). While genetic algorithms (GAs) have been successfully applied for a number of years to engineering and mathematical problems, it is only relatively recently that GAs have been adopted to fill the void in computational techniques for solution of complex numerical problems in economics (for example: Alemdar and Ozyildirim 1998; Beare, Bell and Fisher 1998; Birchenhall et al 1997; Ching-Tzong and Wen-Tsuan 1997). The use of a parametric approximation to the control equation limits the growth in the dimensionality as the number of state variables increases. For a linear approximation the number of parameters to be estimated increase multiplicatively with the number of state and control variables. One limitation of the parametric approach is that the choice of functional approximation may impose features on the solution, which are not characteristic of the original problem. Parametric approaches have been used in economics to determine rational expectations solutions (Wright and Williams1991).

The parametric approximation to the solution of optimal control problems is compared with the collocation approach, discussed in detail in von Stryk (1993). Collocation approaches employ Galerkin discretistion to represent the state and control space equations. Miranda and Fackler (1997) have described economic applications of collocation techniques in detail. While computational requirements grow exponentially as the dimensions of the state space increase, collocation techniques offer a number of advantages. First, the convergence properties of a range of specific collocation schemes have been established (von Stryk 1993). Second, the values of the costate variables are readily determined from the estimation procedure (von Stryk 1993). Both the collocation and the parametric approach are readily adapted to use of randomisation (random sampling of the state transition equations) and quadrature (discrete probability space approximations to the state transition equations) techniques to deal with the stochastic control problem (discussed respectively, by Judd 1997 and Miranda and Fackler 1997).

The use of parametric approximation and the collocation approach for the solution of stochastic optimal control problems are explored in the context of a harvest model for a multi-species fishery. A multi-species fishery is a dynamically complex environment. Commercial fishing is overlaid on populations which may be competing for limited resources, exhibit predator-prey relationships and be subject to substantial fluctuations due to changes in climatic conditions. Logistic models provide a relatively simple framework in which to represent population dynamics (Clark 1990) and are easily incorporated into an optimal control formulation of the classical problem for an idealised resource manager. At the same time, the non-linear dynamics of a logistic specification provide a reasonably high level of computational difficulty for testing numerical solution techniques.

In this paper, a series of simulations were conducted to examine how uncertainty in stock growth influences the optimal management strategy and returns in a multi-species fishery.

2. A stochastic bio-economic model

2.1 Description of biological and harvesting model

Consider two fish species with population growth represented by logistic difference equations

(1)
$$\begin{aligned} S_{1,t+1} &= (\alpha_{11} + \alpha_{12} S_{2t} - \alpha_{15} S_{1t}) S_{1t} \\ S_{2,t+1} &= (\alpha_{21} + \alpha_{22} S_{1t} - \alpha_{25} S_{2t}) S_{2t} \end{aligned}$$

where S_1 and S_2 are stocks of species one and two respectively. The α_{i1} parameter is the natural growth rate of the population and α_{i5} is a self limiting factor reflecting a finite limit to the population growth of a single species. The parameter α_{i2} represents the interaction between species. For example, if both species are competing for a common food resource then both α_{i2} parameters are negative. If the first species preys on the second then α_{12} would be positive and α_{22} would be negative.

To introduce commercial fishing effort into the model, harvest rates h_i are assumed to be proportional to fishing effort f_i, and the level of stocks (subscripts for t are dropped when there is not an explicit time dependence)

(2) $$\begin{aligned} h_{1t} &= (\alpha_{13} f_{1t} + \alpha_{14} f_{2t}) S_{1t} \\ h_{2t} &= (\alpha_{23} f_{2t} + \alpha_{24} f_{1t}) S_{2t} \end{aligned}$$

Population growth inclusive of harvests is

(3) $$\begin{aligned} S_{1,t+1} &= (\alpha_{11} + \alpha_{12} S_{2t} - \alpha_{13} f_{1t} - \alpha_{14} f_{2t} - \alpha_{15} S_{1t}) S_{1t} \\ S_{2,t+1} &= (\alpha_{21} + \alpha_{22} S_{1t} - \alpha_{23} f_{2t} - \alpha_{24} f_{1t} - \alpha_{25} S_{2t}) S_{2t} \end{aligned}$$

Uncertainty is introduced through the parameters of the model. Specifically, the value of a model parameter at time t is drawn randomly from a two parameter distribution

(4) $$\alpha_{ij}(t) \sim R(\mu_{ij}, \sigma_{ij})$$

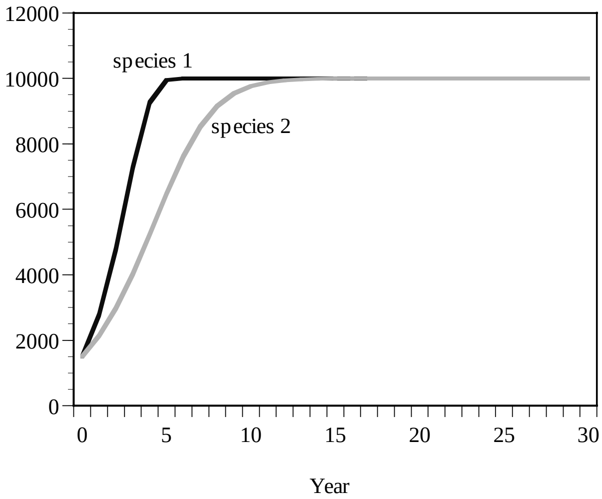
The mean level parameter values used in the model are detailed below in table 1.

Table 1: Parameters in the fish stock equations

	α_{i1}	α_{i2}	α_{i3}	α_{i4}	α_{i5}
Population 1	1.0	0.0	4e-4	1e-4	2e-5
Population 2	0.5	0.0	4e-4	1e-4	5e-6

The time paths for the natural fish populations, (that is, the populations when there is no harvesting), are shown in figure 1 for a thirty year horizon. The first population has a higher natural growth rate and reaches its long term equilibrium level after about 6 years. The second population grows much more slowly, taking about 12 years to reach a stable level.

Figure 1: Natural fish populations



2.1.1 Cost of effort

It is assumed that effort is the product of both capital and labour, represented by a simple Leontief production function

(5)
$$f_{1t} + f_{2t} \leq \beta_L L_t$$
$$f_{1t} + f_{2t} \leq \beta_K \left(S_{Kt} + v_t\right)$$

where L is labour, S_K is the capital stock, v is investment and β_i is a parameter. The change in the level of the capital stock is a function of the depreciation rate d and new investment

(6)
$$S_{K,t+1} = (1-d)S_{Kt} + v_t$$

Total costs at time t are given by the sum of current labour costs and the opportunity cost of capital

(7) $$C\left(f_{1t}+f_{2t},v_t\right)=\frac{w_L}{\beta_L}\left(f_{1t}+f_{2t}\right)+r\left(S_{Kt}+v_t\right)$$

where r is the discount rate and w_L is the wage rate.

2.2 The optimisation problem

A single manager of the fish stocks chooses the level of effort to allow expended on harvest of each species, in order to maximise the discounted stream of net revenue, given a market price p_i for the fish species and operating cost C=C_f+C_v. The manager's choice of effort level is further constrained by dynamics of the fish populations and any management framework that specifies minimum stock levels. That is,

$$\max_{\{f_1,f_2,v\}}\sum_{t=0}^{\infty}(1+r)^{-t}\left[p_1h_1\left(f_{1t},f_{2t}\right)+p_2h_2\left(f_{1t},f_{2t}\right)-C\left(f_{1t}+f_{2t},v_t\right)\right]$$

where

(8)
$$h_1\left(f_{1t},f_{2t}\right)=\left(\alpha_{13}f_{1t}+\alpha_{14}f_{2t}\right)S_{1t}$$
$$h_2\left(f_{1t},f_{2t}\right)=\left(\alpha_{23}f_{2t}+\alpha_{24}f_{1t}\right)S_{2t}$$
$$C\left(f_{1t}+f_{2t},v_t\right)=\frac{w_L}{\beta_L}\left(f_{1t}+f_{2t}\right)+r\left(S_{Kt}+v_t\right)$$

subject to

(9)
$$S_{1,t+1}=\left(\alpha_{11}+\alpha_{12}S_{2t}-\alpha_{13}f_{1t}-\alpha_{14}f_{2t}-\alpha_{15}S_{1t}\right)S_{1t}$$
$$S_{2,t+1}=\left(\alpha_{21}+\alpha_{22}S_{1t}-\alpha_{23}f_{2t}-\alpha_{24}f_{1t}-\alpha_{25}S_{2t}\right)S_{2t}$$
$$S_{K,t+1}=(1-d)S_{Kt}+v_t$$
$$f_{1t}+f_{2t}\le\beta_K\left(S_{Kt}+v_t\right)$$
$$S_i\ge 0$$
$$S_i(t=0)=S_i^{(0)}$$
$$f_{it},v_t\ge 0$$
$$\alpha_{ij}(t)\sim R\left(\mu_{ij},\sigma(\alpha_{ij})\right)$$
$$for\ i=1,2,K$$

The economic parameters used for the model are given in table 2.

Table 2: Cost of fishing parameters and market prices

Parameter	Value
w_L	1
β_L	1
β_K	0.05
d	0.05
r	0.06
p_1	5
p_2	5

3. Solution techniques

3.1 Solution by parametric approximation

While it is not possible, or at least complicated, to solve (8) algebraically to obtain an expression for each control variable in terms of the state variables, a numerical solution may be obtained by parametric approximation of the underlying equations for the decision variables, f_1, f_2 and v.

To find a solution for the optimal control problem, consider for simplicity, a first order approximation to the effort and investment relationships. That is,

$$\begin{aligned} f_1 &= \max\left(0, \omega_{11} + \omega_{12} S_1 + \omega_{13} S_2 + \omega_{14} S_k\right) \\ f_2 &= \max\left(0, \omega_{21} + \omega_{22} S_1 + \omega_{23} S_2 + \omega_{24} S_k\right) \\ v &= \max\left(0, \omega_{31} + \omega_{32} S_1 + \omega_{33} S_2 + \omega_{34} S_k\right) \end{aligned} \tag{10}$$

To impose the capital constraint (5) on fishing effort, a penalty function method was used, as suggested by Goldberg (1989).

The validity of the approximation in (10) can be assessed by a determination of the solution using an alternative numerical optimisation approach. In this paper, the solution derived by parametric approximation is checked against that derived with the collocation technique.

The genetic algorithm is used in the parametric approximation approach to find values for the parameters ω_{ij} which give a set of effort levels corresponding to fish stocks at each period in time. These sets of effort levels in turn result in a harvest level for each species in each period, and an associated value for net revenue.

3.1.1 Genetic Algorithms

A genetic algorithm (GA) is a search technique that has been successfully applied to problems with complex dynamic structures that cannot be easily handled with traditional analytical methods. The genetic algorithm approach was first developed by Holland (1975). It has subsequently been widely employed in economics and finance research as a flexible and adaptive search algorithm (see for example: Alemdar and Ozyildirim 1998; Beare, Bell and Fisher 1998; Birchenhall 1995; Ching-Tzong and Wen-Tsuan 1997). The approach provides a globally robust search mechanism with which to optimise over a decision process involving uncertainty in the form of a lack of a priori knowledge, unclear feedback of information to decision makers and a time varying payoff function.

A GA performs a multi-directional search by maintaining a population of individual strategies, each with a potential solution vector for the problem. An objective function is employed to discriminate between fit and unfit solutions. The population undergoes a simulated evolution such that at each generation, the relatively fit solutions reproduce while the relatively unfit solutions die out of the population. During a single reproductive cycle, fit strategies are selected to form a pool of candidate strategies, some of which undergo cross over and mutation in order to generate a new population. Cross-over combines the features of two parent strategies to form two similar offspring by swapping corresponding segments of the parents. This is equivalent to an exchange of information between different potential solutions. Mutation introduces additional variability into the population by arbitrarily altering a strategy by a random change.

In determining the optimal harvest strategy for the multi-species fishery described in (8), each GA strategy or string contains a possible solution for the parameters ω_{ij}. The length of each string corresponds to the number of parameters to be estimated. How good this solution actually is, is assessed by substituting the resulting ω_{ij} values for each string into equation (8) and using the resulting set of effort levels to determine the value of the discounted stream of net revenue. Those strings that give relatively high results for net revenue are given greater weight in the formation of the next generation of strings. After a number of generations, the solution may converge, with the best individual strings representing the optimum solution.

The genetic search algorithm was implemented in Extend™ (Imagine That Inc 1997) using the approach described in Goldberg (1989). The search was conducted over 150 generations using 200 population strings. For the stochastic simulations, each string was evaluated over 50 trials, and assigned a fitness value equal to the average fitness over the trials. Following Goldberg, a cross-over rate of 0.6 and mutation rate of 0.001 was used. The genetic algorithm requires a search range to be specified. The initial values selected for the search are given in Table 1. Subsequent narrowing of the search range can refine the estimates.

Table 1: Search ranges for genetic algorithm

Variable	**f_1 Equation**	**f_2 Equation**	**v Equation**
ω_{i1}	-2000, 0	-2000, 0	-100 0
ω_{i2}	0.0, 0.5	0.0, 0.0	0.0, 3.0
ω_{i3}	0.0, 0.0	0.0, 0.4	0.0, 3.0
ω_{i4}	0.0, 0.1	0.0, 0.1	-1.0, 0.0

3.2 Solution by collocation

The collocation technique is a flexible tool for solving discrete time, continuous state dynamic economic problems. Collocation methods offer three advantages over parametric approximation. First, these methods do not require a prior guess at the functional form of the control rule. Instead, a non-parametric approximation to the control rules is provided at a number of points in the state domain, referred to as collocation nodes. Second, the problem can be solved over an infinite time horizon. Third, the method readily allows the estimation of the shadow prices of the state variables. The main problem with the approach is that as the dimensions of the state space increase, computational requirements increase much more rapidly than for parametric approximation.

The recursive Bellman equation for the problem in (8), in discrete time over an infinite horizon with continuous state variable space is given by

$$\begin{aligned} V\left[S_{1t}, S_{2t}, S_{Kt}\right] = \max_{f,v} E\Big\{ & p_1 h_1\left(f_{1t}, f_{2t}\right) + p_2 h_2\left(f_{1t}, f_{2t}\right) - C\left(f_{1t} + f_{2t}, v_t\right) + \\ & (1-r) V\left[S_{1,t+1}, S_{2,t+1}, S_{K,t+1}\right]\Big\} \end{aligned} \tag{11}$$

subject to

$$\begin{aligned} & S_{1,t+1} = \left(\alpha_{11} + \alpha_{12} S_{2t} - \alpha_{13} f_{1t} - \alpha_{14} f_{2t} - \alpha_{15} S_{1t}\right) S_{1t} \\ & S_{2,t+1} = \left(\alpha_{21} + \alpha_{22} S_{1t} - \alpha_{23} f_{2t} - \alpha_{24} f_{1t} - \alpha_{25} S_{2t}\right) S_{2t} \\ & S_{K,t+1} = (1-d) S_{Kt} + v_t \\ & f_{1t} + f_{2t} \le \beta_K \left(S_{Kt} + v_t\right) \\ & S_i \ge 0 \\ & S_i(t=0) = S_i^{(0)} \\ & f_i, v \ge 0 \\ & \alpha_{ij}(t) \sim R\left(\mu_{ij}, \sigma(\alpha_{ij})\right) \\ & \qquad\qquad for\ i = 1,2,K \end{aligned} \tag{12}$$

The basic strategy employed in collocation analysis, as discussed by Miranda and Fackler (1997), is to first approximate the unknown value function (11) with a linear combination of n polynomials or other approximating functions, where $n = n_1 * n_2 * n_K$, for n_i the number of values of state variable i selected for function evaluation. An approximation to (8) is necessary because solving Bellman's equation requires an explicit solution for an infinite number of optimisation problems, one for each possible state. With collocation, a linear combination of approximating functions is required to satisfy conditions of optimality at n points, or collocation nodes, within the domain of the state space.

3.2.1 Approximating the Bellman functional equation

A Galerkin approximation is used to replace the Bellman functional equation; that is, a system of n non-linear equations in n unknowns. The approximating function for (11) is now written

(13) $$EV(S_1, S_2, S_K, \alpha) \approx E\Phi(\mathbf{S}, \alpha)c$$

where Φ is an n by n interpolation matrix, α is the matrix of coefficients in the constraint set and c is an n by 1 vector of basis coefficients which are to be determined. A search algorithm is employed to find values for the vector of basis coefficients.

Let the vector $\mathbf{y}_i$ be the vector of nodes for the state variable S_i. Then, $\mathbf{S}$ is a matrix with rows which form the Cartesian product of $\mathbf{y}_1 \times \mathbf{y}_2 \times \mathbf{y}_K$. Each y_{ji} element represents the ith basis function evaluated at the jth collocation node.

The interpolation matrix Φ depends on the functional form of the approximation. We will use a Chebychev polynomial approximation (which is detailed by Miranda and Fackler (1997) for the case of a single state variable and can be extended to consider multiple state variables).

The Chebychev nodes are used to approximate $S_{i,t}$, where

(14) $$y_{ij}(t) = \frac{1}{2}\left[a_j + b_j + (b_j - a_j)\cos\left(\frac{n_j - i + 0.5}{n_j}\pi\right)\right] \qquad i = 1, n_j$$

and the state transition equations give the approximation for $y_{i,j}(t+1)$.

For each state variable, a partial n_i*n_i interpolation matrix ϕ_{yj}, with columns can now be defined

(15) $$\begin{aligned}
\phi_{i,1,y_j} &= 1 \\
\phi_{i,2,y_j} &= y_j \\
\phi_{i,3,y_j} &= 2y_j^2 - 1 \\
\phi_{i,4,y_j} &= 4y_j^3 - 3y_j \\
&\bullet \\
&\bullet \\
&\bullet \\
\phi_{i,n_i,y_j} &= 2\phi_{i,n_i-1} - \phi_{i,n_i-2}
\end{aligned}$$

The rows of the interpolation matrix Φ are then given by

(16) $$\Phi_{i,t} = \phi_{iy_{1,t}} \otimes \phi_{iy_{2,t}} \otimes \phi_{iy_{K,t}}$$

3.2.2 Incorporating uncertainty

The uncertainty associated with the biological and harvest parameters can be incorporated in a number of ways. One approach, recommended by Miranda and Fackler (1997) based on numerical analysis theory and practice, is to adopt a Gaussian quadrature scheme (Stoer and Bulirsch 1983). When using a Gaussian quadrature scheme, each continuous random variable in the state transition function is replaced with a set of discrete approximates, the value of which the variable takes on with an assumed known probability.

Assuming that the stochastic coefficients are independent, we can consider dividing the distribution of each α_{ij} parameter into *m* percentiles. For each of the μ stochastic parameters of α, let $\mathbf{z}_j$ be an *m* by *1* vector of the expected values in each percentile. Let Z be an m^{μ} by μ matrix with rows forming the Cartesian product $(\mathbf{z}_1 \times \mathbf{z}_2 \times \ldots \times \mathbf{z}_{\mu})$. The probabilities ρ associated each row of z with Z is then a m^{μ} by *1* vector, with elements $\rho_l = m^{-\mu}$.

Clearly, with an increasing number of stochastic parameters, computational requirements escalate rapidly. A second option is to draw a limited number of random samples of the biological and harvest parameters to construct the rows of Z. Randomisation approaches are a common way to limit the dimensionality of computational problems (Rust 1997).

3.3.3 Solving the collocation problem

To satisfy conditions of optimality at the collocation nodes, we must solve on every collocation node, the non-linear programming problem, written for the i[th] node as

$$
\begin{aligned}
&\max_{x_i} \sum_{z \in Z} \rho_l \Big[p_1 h_1(x_{i,1}, x_{i,2}) + p_2 h_2(x_{i,1}, x_{i,2}) - C(x_{i,1}, x_{i,2}, x_{iK}) \\
&\qquad + (1-r)\Phi_{y_{t+1}} c \Big] \\
&\text{where} \\
&h_1(x_{i,1}, x_{i,2}) = (z_{l,1,3} x_{i,1} + z_{l,1,4} x_{i,2}) y_{1,t} \\
&h_2(x_{i,1}, x_{i,2}) = (z_{l,2,3} x_{i1} + z_{l,2,4} x_{i2}) y_{2,t} \\
&C(x_{i,1}, x_{i,2}, x_{iK}) = \frac{w_L}{\beta_K}(x_{i,1} + x_{i,2}) + r(y_{K,t} + x_{iK}) \\
&\text{subject to} \\
&y_{1,t+1} = (z_{l,1,1} + z_{l,1,2} y_{2,t} - z_{l,1,3} x_{i,1} - z_{l,1,4} x_{i,2} - z_{l,1,5} y_{1,t}) y_{1,t} \\
&y_{2,t+1} = (z_{l,2,1} + z_{l,2,2} y_{1,t} - z_{l,2,3} x_{i,2} - z_{l,2,4} x_{i,1} - z_{l,2,5} y_{2,t}) y_{2,t} \\
&y_{K,t+1} = (1-d) y_{K,t} + x_{i,K} \\
&x_{i,1} + x_{i,2} \le \beta_L (y_{K,t} + x_{i,k}) \\
&x, y \ge 0
\end{aligned}
\tag{17}
$$

where the control variables, f_1, f_2, v in (8) are approximated with *n* by *1* vectors x_1, x_2, x_k.

Starting with an initial guess at the basis vector *c*, (17) can be solved using a sequential quadratic programming algorithm. Using a Newton update rule to iteratively find a solution to the collocation problem, a new vector *c* is derived

(18) $$c_{new} = c_{old} - \left[\Phi_{y_t} - (1-r)\Phi_{y_{t+1}}\right]^{-1}\left(\Phi_{y_t} c_{old} - \tilde{V}\right)$$

Convergence is obtained when the value function approximant solves Bellman's equation to an acceptable degree of accuracy, that is

(19) $$\Phi c - \tilde{V} \cong 0$$

The interpolation equation for the controls, $\hat{x}$, at a point in the control space (y_1, y_2 y_K) is given by

(20) $$\hat{x} = \left(\phi_{y1} \otimes \phi_{y2} \otimes \phi_{y_K}\right)\Phi^{-1}_{y_{base}} x^*$$

where y_{base} is a vector of the collocation nodes and x* is a vector of the optimal control values at the collocation nodes. When interpolating a solution between the collocation nodes, the Bellman equation and the associated constraints may not be met exactly. The error can be reduced at the expense of computational time, by increasing the number of collocation nodes.

Though not used directly, the values of the costate variables at the colloction nodes is, for example, for the first state variable, given by

(21) $$\lambda_1 = \left(\frac{\partial \phi_{y_1}}{\partial y_1} \otimes \phi_{y_2} \otimes \phi_{y_K}\right)c$$

The collocation model was implemented in MATLAB5 (The Math Works 1997). Some of the routines were adapted directly from Miranda and Fackler (1997). Only four collocation nodes were used for each state variable (that is, $n = 64$), yielding a coarse approximation but still requiring significant computation time. With the limited number of collocation nodes the choice of the approximation range can significantly affect the accuracy of the solution. For the fish populations the approximation range was set at the limits of natural population growth. The approximation range on capital stock was se, through a trial and error process, to ensure that it fell within the estimation range.

3. Model solutions

Two comparative simulations were conducted using the two solution methods. First, the deterministic control problem was solved. Second, a stochastic control problem was solved with uncertainty regarding the natural growth rate parameter in the biological state transition equations, α_{i1}. Quadrature at the quartiles of two independent normal distributions, with a relative standard deviation of 25 per cent, was used to represent the stochastic parameters. For comparison, solutions from both methods were calculated for a thirty year time horizon, from identical initial conditions ($S_1 = S_2 = 1500$, $S_K = 1000$).

3.1 The deterministic solution

Both methods yielded very similar approximations for the deterministic case. The graphs of the state and control space variables are shown in figures 2 and 3. Greater effort is targeted to the faster growing species. The equilibrium population levels remain slightly below the faster growing species. Both solutions exhibit an initial delay in fishing effort to allow the fish populations to build up. The building up of the capital base as effort commences is better aligned in the collocation solution. However, net present value of accumulated revenue was nearly the same for both solutions at $142,000.

Solution times were about eight-fold faster using the parametric approximation. The solution time for the parametric approximation was about 40 seconds as compared to over 300 seconds for the collocation solution. The use of a sequential quadratic-programming algorithm at each node added considerably to execution time.

3.1.1 Characteristics of the solutions

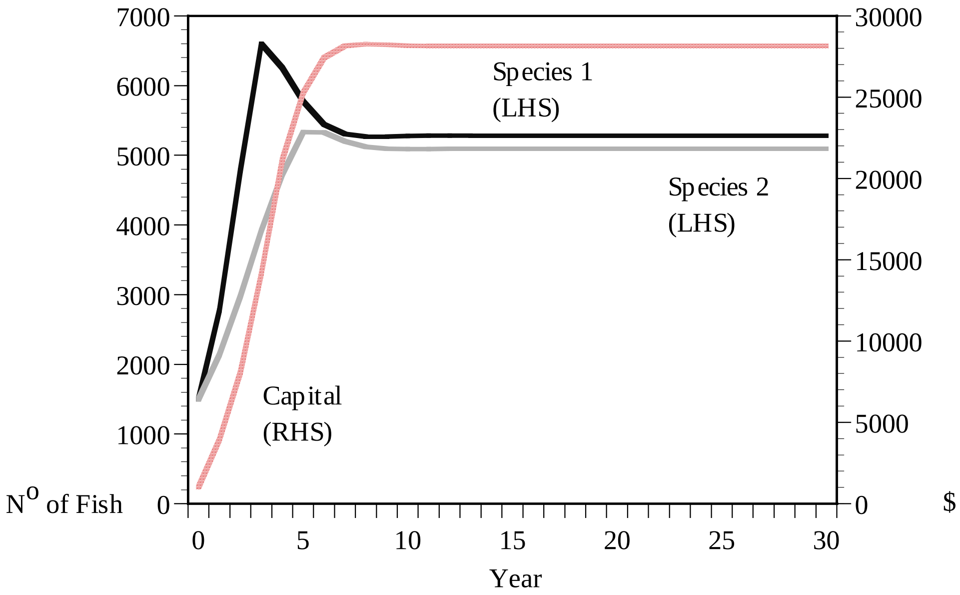
The comparatively good performance of the linear parametric approximation can be explained in part by the essentially quadratic nature of the problem. The value function, at a fixed level of capital investment, derived from the collocation estimation is shown in figure 4. Shadow prices, at a fixed level of capital investment, for the first fish stock are shown in figure 5. Both the value and costate surfaces are smooth and would be well approximated by a quadratic function, which might be expected to lead to a reasonably accurate linear control rule. Optimal effort is shown as a function of fish stocks, at a fixed level of capital investment, in figure 6. Within the capital constraint boundary and the non-negativity constraint on effort, the effort relationship is roughly linear. However, optimal investment as a function of the stocks (shown for a fixed initial level of capital in figure 7) has considerable curvature. This would explain the difference in the investment trajectories for the parametric and collocation solutions. Hence, the parametric approximation might be improved by including second order terms in the approximating equation for investment.

3.2 The stochastic solution

The graphs of the state and control space variables for the stochastic simulations are shown in figures 8 and 9. The parametric and the collocation solutions are again similar. The main difference between the deterministic and stochastic simulations is due to the capital constraint on fishing effort. In each solution the level of capital investment is higher. Maintaining a higher level of capital stock increases the costs of fishing effort on average. However, higher levels of capital are required to fully capture the benefits associated with random increases in the fish populations. Equilibrium effort and population levels are approximately the same. Again, this could be expected, as the population equations are linear in terms of the growth parameter.

Solution times for the stochastic models were of the order of four times faster using the parametric as opposed to the collocation method. The solution time was approximately 700 seconds for the parametric approximation and over 3200 seconds for the collocation solution.

Figure 2: State and control variables – deterministic parametric approximation solution



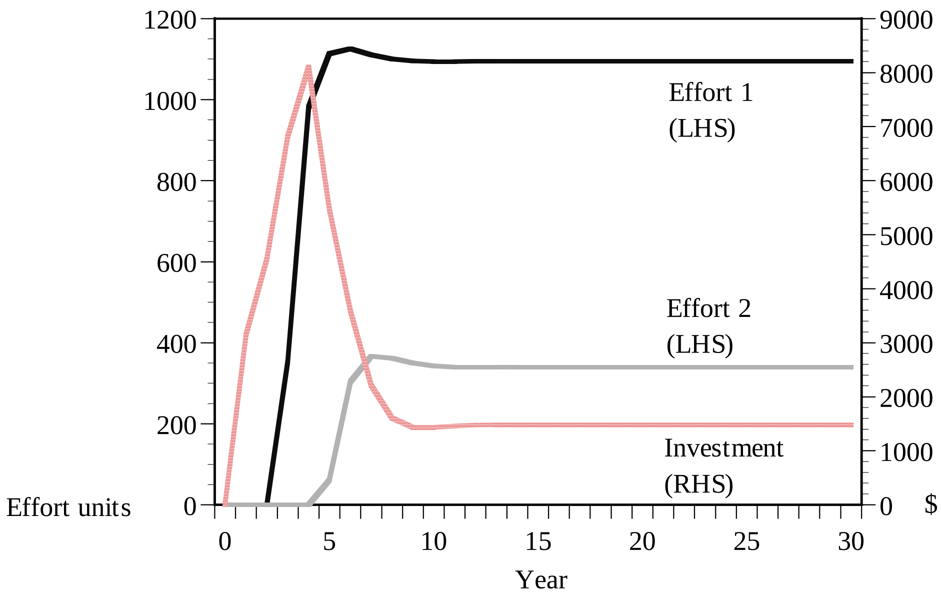
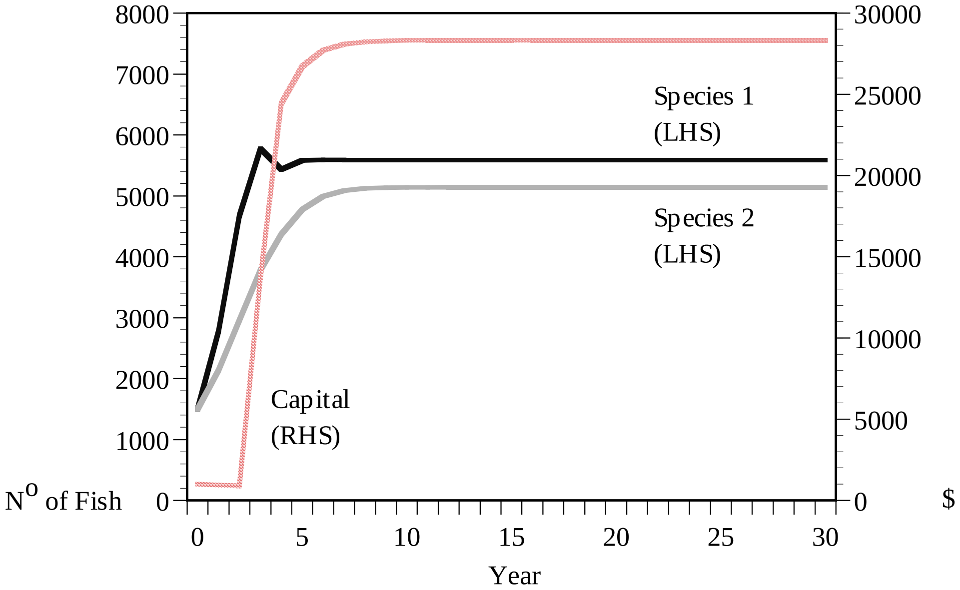


Figure 3: State and control variables – deterministic collocation solution



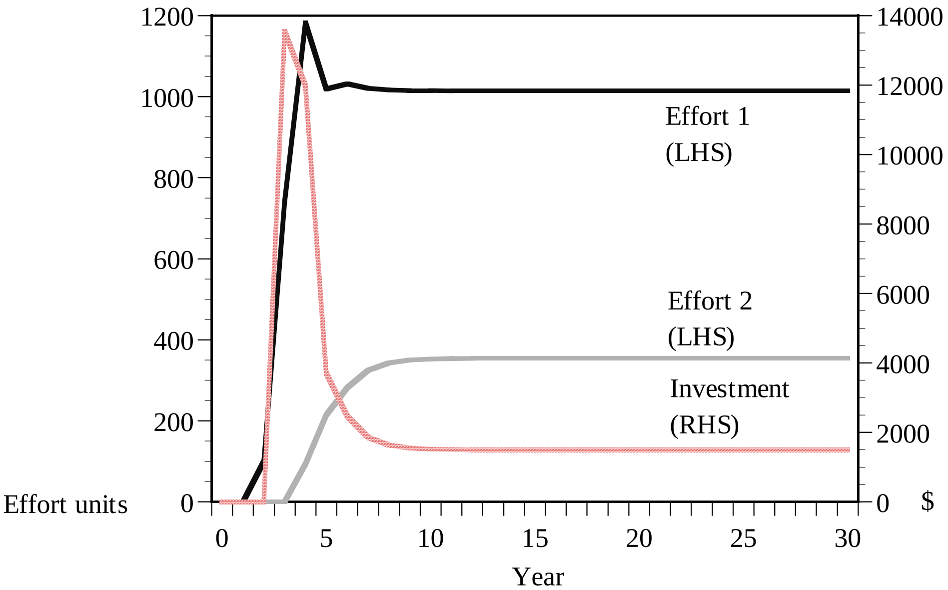


Figure 4: Value function for a fixed level of capital stock

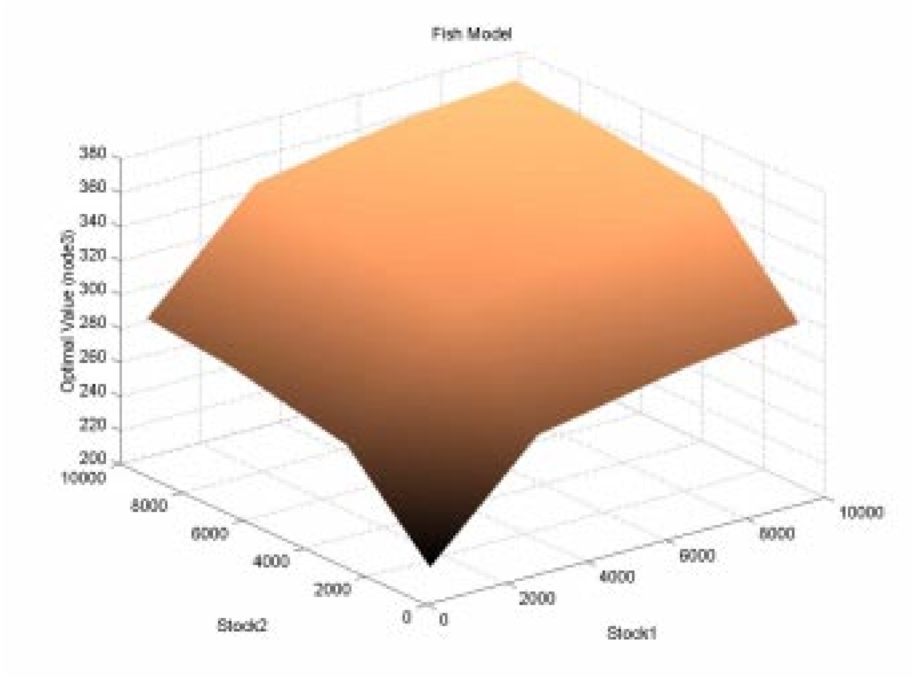


Figure 5: Shadow price for fish stock 1

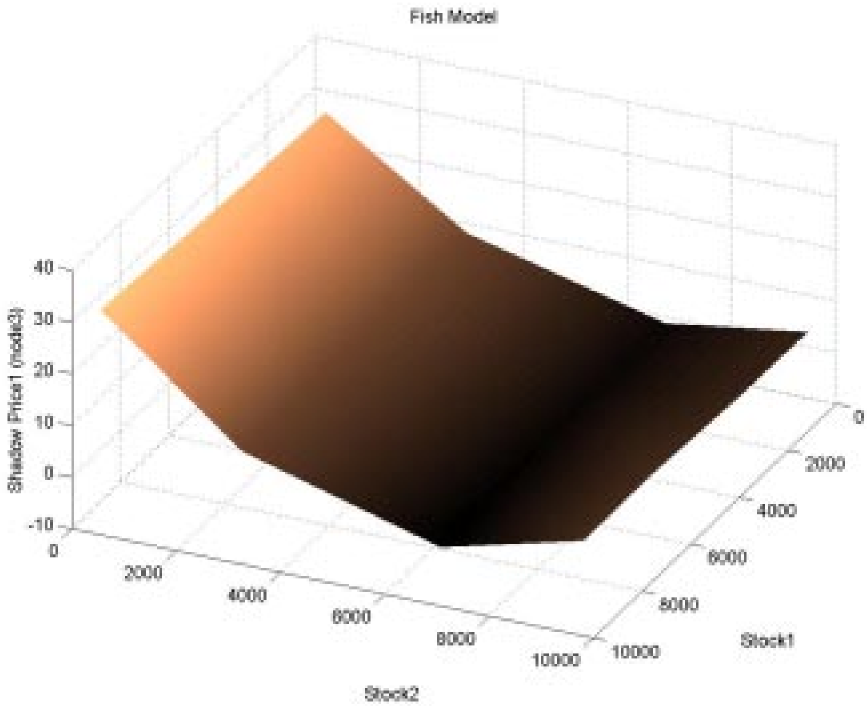


Figure 6: Optimal effort for a fixed level of capital stock

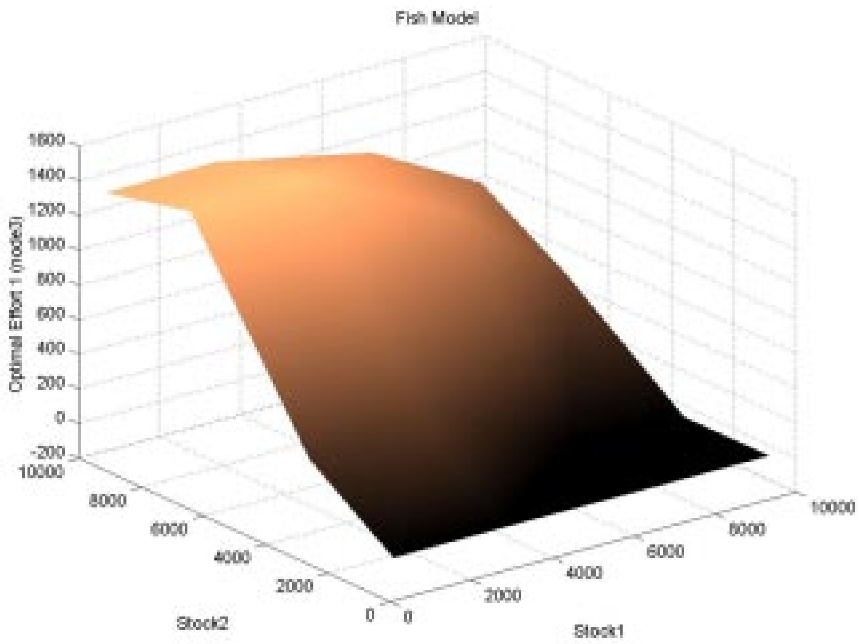


Figure 7: Optimal investment for a fixed initial level of capital stock

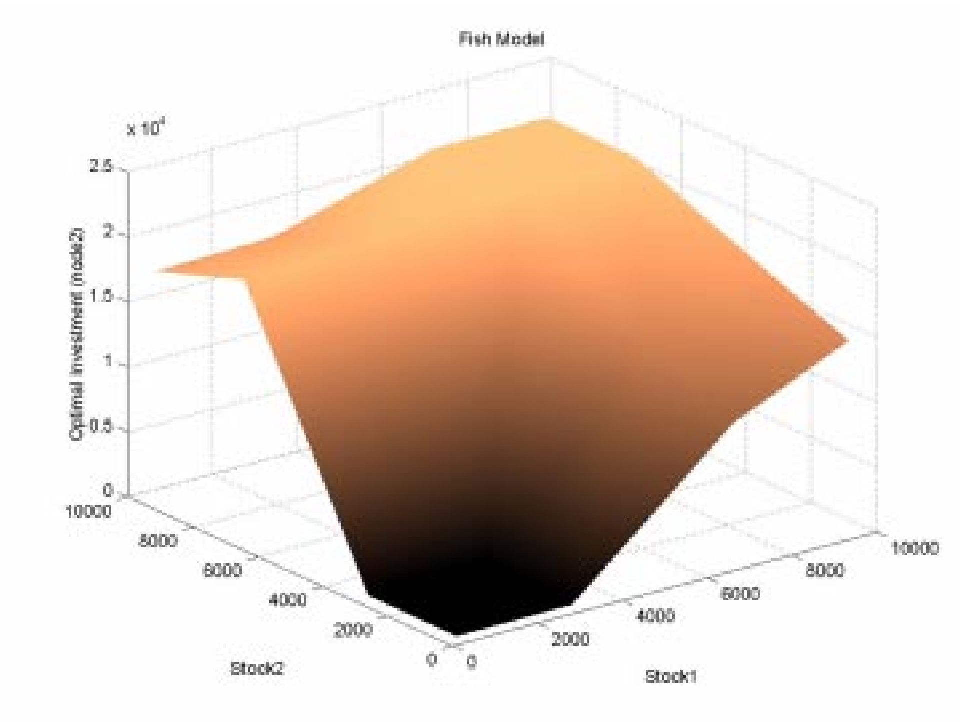


Figure 8: Expected values of state and control variables
– stochastic parametric approximation solution

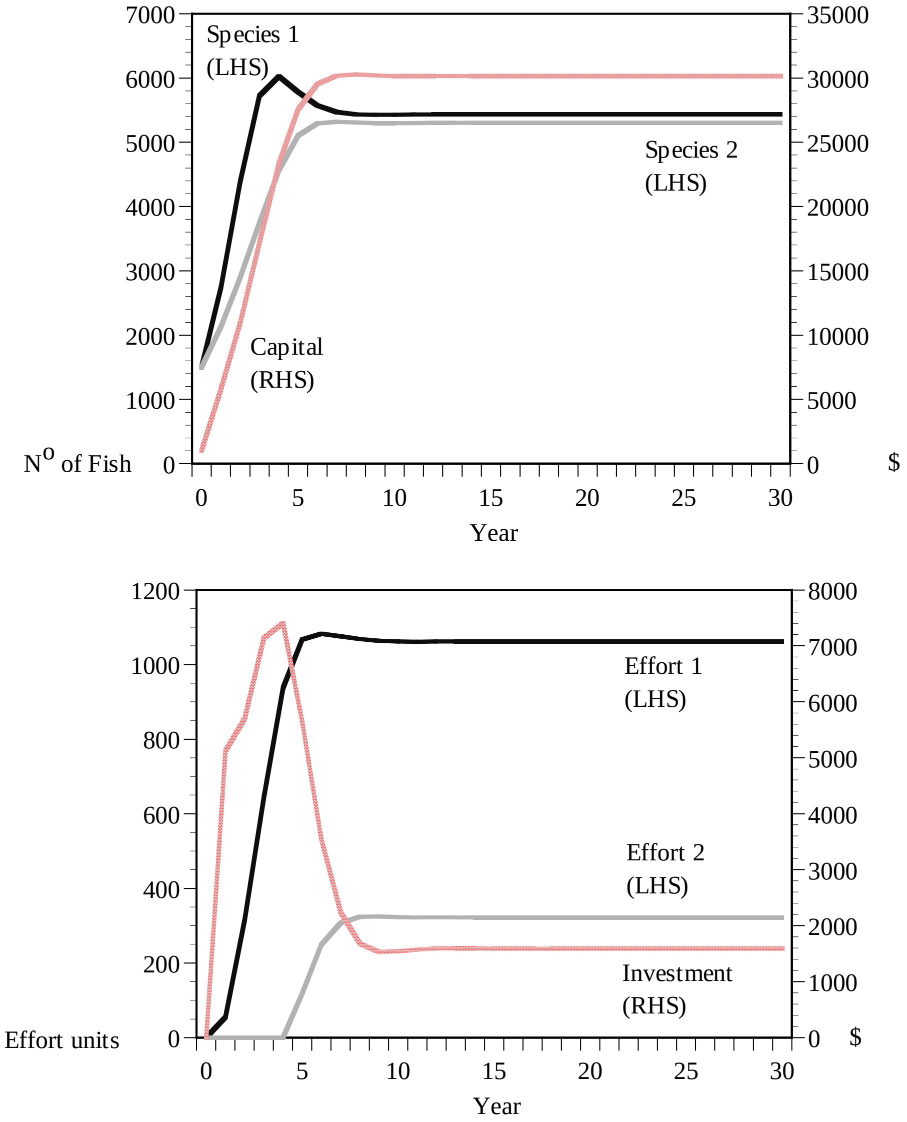
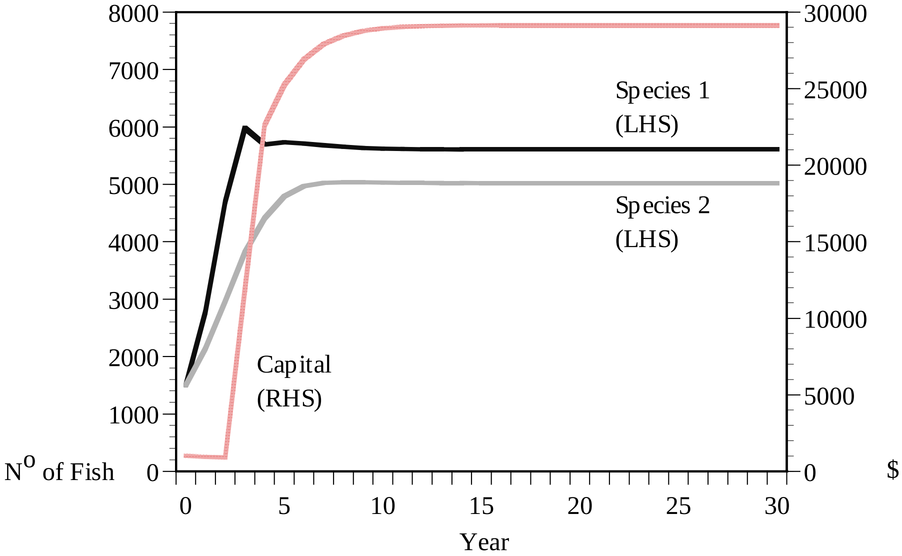
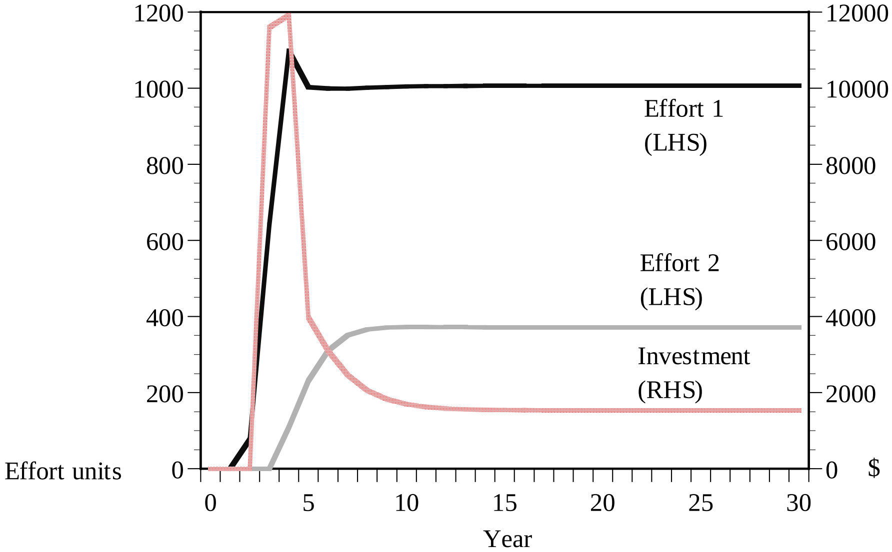


Figure 9: Expected values of state and control variables – stochastic collocation solution





4. Conclusions

Uncertainty has long been recognised an important aspect of renewable resource management. In modelling resource management problems, deterministic specifications may miss the optimal response to stochastic variation where there are binding constraints and non-linear relationships. In the fisheries example used here, the deterministic solution understates the optimal level of capital when populations are subject to random variation, due to a constraint on fishing effort imposed by the existing capital stock and the irreversible nature of investment.

Extending a basic deterministic model to allow the overlaying of the essential features of an interdependent biological, physical and economic system with uncertainty often leads to an intractable stochastic control problem. However, solution of such problems has been facilitated by recent developments in computational methods for numerical control.

Two alternative solution methods were explored in this paper in the context of a bio-economic model for a multi-species fishery. In the first method, a parametric approximation to the control equation was combined with a genetic search algorithm. In the second, a collocation method was used to solve Bellman’s equation. While both techniques yielded similar solutions, they offered different advantages and disadvantages.

Collocation methods generate non-parametric approximations to control equation. This facilitates the understanding of the problem and the nature of the solution. However, for multi-dimensional state space problems, collocation techniques require exponentially increasing computational time.

Parametric approximation techniques require prior specification of an explicit functional relationship between the state and control variables. As a result, the approximation may impose or miss features of the solution. However, when combined with a genetic search algorithm, the technique is very robust. The number of parameters to be estimated increases only multiplicatively with the number of state and control variables, reducing the rate of increase in computation time.

When faced with a non-linear problem with multiple state and control variables, the two solution techniques might be usefully combined. The use of collocation techniques to characterise the degree of non-linearity in the solution to the problem, followed by an appropriate approximation for the control equations, may prove to be an expedient method for dealing with larger scale problems that are closer to those faced in reality, by resource managers.

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