ESTIMATING THE CHARACTERISTICS OF HOMOGENEOUS FUNCTIONS USING FLEXIBLE FUNCTIONAL FORMS

by

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Abstract

A flexible functional form can provide a second-order approximation to an arbitrary unknown function at a single point. Except in special cases, the parameters of flexible forms will vary from one point of approximation to another. I use this property to show that, in general, if an unknown function is homogeneous then i) Euler's Theorem gives rise to linear equality constraints involving both the data and a set of observation-varying flexible form parameters, ii) the common practice of imposing homogeneity on flexible functional forms is unnecessarily restrictive, and iii) it is possible to obtain estimates of the observation-varying parameters of approximating flexible forms using a Singular Value Decomposition (SVD) estimator. Two illustrations are provided: artificially-generated data is used to estimate the characteristics of a generalised linear production function; and Canadian data is used to estimate the characteristics of a consumer demand system.

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1. Introduction

Much of econometrics is concerned with estimating the parameters of second-order flexible functional forms (eg. translog, quadratic, generalised McFadden and generalised Leontief). Flexible forms are popular in empirical work because they have the desirable property that, at a point of approximation, their first- and second-order derivatives can be set equal to those of an arbitrary unknown function. Unfortunately, most empirical flexible form models have the characteristic that the parameters are fixed across points of approximation. This implies they can usually only provide an approximation to the unknown function at a single point. In turn, this can severely limit the ability of the model to capture the behaviour of the unknown function across a range of points, including the points represented in the data. In this paper I show how to overcome this problem by allowing the parameters of flexible forms to vary deterministically from one point to another. Estimation of these point-varying parameters is accomplished by i) specifying linear homogeneity constraints in the exact form prescribed by Euler's Theorem, and ii) imposing these constraints using the Singular Value Decomposition (SVD) estimator of Doran, O'Donnell and Rambaldi (1999). Finally, with the aid of examples, I show that the usual practice of imposing homogeneity on flexible functional forms is unnecessarily, and sometimes severely, restrictive.

To fix these ideas, consider a constant returns to scale generalised linear production function defined over the single output, y_t, and the nonstochastic $N \times 1$ input vector $\mathbf{x}_t = (x_{1t}, ..., x_{Nt})'$ (Diewert, 1973):

(1) $$y_t = h(\mathbf{x}_t) = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_{ij} x_{it}^{1/2} x_{jt}^{1/2}$$

where t identifies a particular observation ($t = 1, ..., T$) and the parameters satisfy the identifying restrictions $\alpha_{ij} = \alpha_{ji}$ ($i, j = 1, ..., N$). A quadratic function which can provide a second-order approximation to $h(.)$ at the point $\mathbf{x}_m$ is

(2) $$y_t = f_m(\mathbf{x}_t) = \beta_{0m} + \sum_{i=1}^{N} \beta_{im} x_{it} + \sum_{i=1}^{N}\sum_{j=1}^{N} \beta_{ijm} x_{it} x_{jt}$$

where the subscripts on the parameters, β_{0m}, β_{im} and $\beta_{ijm} = \beta_{jim}$ ($i, j = 1, ..., N$), make it explicit that they vary across points of approximation $m = 1, ..., M$. To see this flexibility, simply note that $f_m(.)$ has enough independent parameters to enable its first- and second-order derivatives to be set equal to those of $h(.)$ at $\mathbf{x}_m$. That is, there is a unique solution to the system of $N + 1 + N(N + 1)/2$ equations:

$$(3) \qquad f_m(\mathbf{x}_t)\big|_{\mathbf{x}_m} = h(\mathbf{x}_t)\big|_{\mathbf{x}_m}$$

$$(4) \qquad \left.\frac{\partial f_m(\mathbf{x}_t)}{\partial x_{it}}\right|_{\mathbf{x}_m} = \left.\frac{\partial h(\mathbf{x}_t)}{\partial x_{it}}\right|_{\mathbf{x}_m}$$

and

$$(5) \qquad \left.\frac{\partial^2 f_m(\mathbf{x}_t)}{\partial x_{it}\partial x_{jt}}\right|_{\mathbf{x}_m} = \left.\frac{\partial^2 h(\mathbf{x}_t)}{\partial x_{it}\partial x_{jt}}\right|_{\mathbf{x}_m}.$$

The solution for β_{ijm} $(i \neq j)$, for example, is:

$$(6) \qquad \beta_{ijm} = \alpha_{ij} x_{im}^{-1/2} x_{jm}^{-1/2}.$$

It is apparent from equations such as (6) that the parameters of flexible forms will usually[1] vary with variations in the point of approximation, $\mathbf{x}_m$. Further examples of these types of equations can be found in Diewert and Ryan and Wales (1998).

This 'point-sensitivity' of parameters has important implications for the way in which flexible forms such as (2) are used to estimate the characteristics of homogeneous functions such as (1). Euler's Theorem (and its converse – see Silberberg, 1990, p.100) states that $h(.)$ will be homogeneous of degree k (HDk) if and only if:

$$(7) \qquad \sum_{i=1}^{N} \left.\frac{\partial h(\mathbf{x}_t)}{\partial x_{it}}\right|_{\mathbf{x}_t} x_{it} = k\, h(\mathbf{x}_t)\big|_{\mathbf{x}_t} \qquad t = 1, ..., T.$$

Thus, using (3) and (4) and setting $m = t$, the parameters of the functions $f_m(.)$ must satisfy:

$$(8) \qquad \sum_{i=1}^{N} \left.\frac{\partial f_t(\mathbf{x}_t)}{\partial x_{it}}\right|_{\mathbf{x}_t} x_{it} = k\, f_t(\mathbf{x}_t)\big|_{\mathbf{x}_t} \qquad t = 1, ..., T.$$

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1 The parameters may not vary across points of approximation if $f_m(.)$ and $h(.)$ have identical functional forms, or if all third-order derivatives of both functions are everywhere zero.

Importantly, Euler's Theorem and second-order flexibility *does not* necessarily mean

$$(9) \qquad \sum_{i=1}^{N} \frac{\partial f_m(\mathbf{x}_t)}{\partial x_{it}}\bigg|_{\mathbf{x}_t} x_{it} = k\, f_m(\mathbf{x}_t)\big|_{\mathbf{x}_t} \qquad m \neq t;\ t = 1, ..., T.$$

Thus, even though $h(.)$ is HDk, there is no requirement that the approximating functions $f_m(.)$ be HDk, and it is not usually necessary to impose this property on flexible forms in empirical work. In the case of the constant returns to scale (ie. HD1) generalised linear function (1) and the quadratic function (2), for example, equations (8) and (9) take the form:

$$(10) \qquad \beta_{0t} - \sum_{i=1}^{N}\sum_{j=1}^{N} \beta_{ijt} x_{it} x_{jt} = 0$$

and

$$(11) \qquad \beta_{0m} - \sum_{i=1}^{N}\sum_{j=1}^{N} \beta_{ijm} x_{it} x_{jt} = 0.$$

Equation (10) is the necessary and sufficient condition to ensure the generalised linear function (1) is constant returns to scale. Equation (11) is an unnecessary constraint which will ensure that the quadratic function (2) is constant returns to scale. Note that (11) will be satisfied if and only if (Diewert, p.294):

$$(12) \qquad \beta_{0m} = \beta_{ijm} = 0 \qquad \text{for all } i, j \text{ and } m.$$

The usual way forward is to set $M = 1$ (ie. use a single approximation point) and impose constraints of the form given by (12). The effects of over-constraining the parameter space in this way can be severe and will be illustrated below in Section 3.

It should be apparent from this discussion that a better way forward involves setting $m = t$ for all t (ie. using observed data points as points of approximation) and directly estimating the observation-varying parameter model (2) subject to the necessary conditions for homogeneity given by (10). There are two characteristics of this system which are relevant to the choice of estimator. First, both the model (2) and the constraints (10) are linear in the parameters. This makes it possible to estimate the model using either the Kalman filter estimator of Doran and Rambaldi (1997) or the SVD estimator of Doran, O'Donnell and Rambaldi. The SVD estimator has considerable computational advantages over the Kalman filter estimator and, in addition, possesses a number of desirable properties. Second, variations in the parameters arise from variations in points of approximation and not, for example, from variations in underlying economic behaviour. Thus, the parameters are deterministic, and this rules out most other estimators, including the estimators frequently used to estimate random coefficient models.

The remainder of the paper is structured as follows. In Section 2 I briefly describe the SVD estimator in the context of a general linear model where observation-varying parameters are subject to linear constraints. In this section I also provide examples of models in applied production economics and demand analysis which fall into this class. In Section 3 I use artificially generated data to illustrate the way in which the SVD estimator and the quadratic function (2) can be used to capture economically-relevant information contained in the generalised linear production function (1). A second illustration of the methodology is provided in Section 4 where I use the Canadian expenditure data of Ryan and Wales to estimate the observation-varying parameters of a Linearised Almost Ideal Demand (LAID) system. A summary of the paper is provided in Section 6.

2. Linear Models, Linear Constraints and the SVD Estimator

Most flexible functional form models consist of, or give rise to, single equations or systems of equations which are linear in the unknown parameters. Examples from demand analysis include the LAID system of Deaton and Muellbauer (1980) and the absolute price version of the Rotterdam model used by Bewley (1983). Examples from production economics are much more numerous and include the translog, generalised Leontief, and normalised quadratic production, profit and cost functions used by, for example, Lopez (1980), Binswanger (1974), Kako (1978) and Villezca-Beccerra and Shumway (1992).

All of these models can be conveniently written in the general matrix form

$$(13) \qquad \mathbf{y}_t = \boldsymbol{X}_t\boldsymbol{\beta}_t + \mathbf{e}_t$$

where $\mathbf{y}_t$ is a known $N \times 1$ vector ($N \geq 1$), $\boldsymbol{X}_t$ is a known $N \times K$ design matrix, $\mathbf{e}_t$ is an unknown $N \times 1$ disturbance vector with zero mean vector and constant covariance matrix $\sigma^2\mathbf{I}_N$, and $\boldsymbol{\beta}_t$ is a $K \times 1$ vector of observation-varying parameters to be estimated. Several observation-varying parameter models appear in the econometrics literature, and these range from simple dummy variable models (eg. Judge *et al*, 1985, p. 519-21, 530-33) to the more sophisticated random coefficients models of Swamy (1970, 1971), Hsaio (1975) and Hildreth and Houck (1968). An important difference between the model given by (13) and these other observation-varying parameter models is that in (13) the parameters are deterministic and are permitted to vary across *all t*.

Most of the economic models listed above have the property that they are HDk in at least some of their arguments. For example, the profit function of the firm is known to be HD1 in output and input prices, the consumer's income compensated demand function is HD0 in product prices, and, of course, constant returns to scale production functions are HD1 in input quantities (see, for example, Varian 1992). Accordingly, using the rationale of the previous section, Euler's Theorem implies the parameters of flexible form models will be subject to linear constraints of the form

$$(14) \qquad \boldsymbol{R}_t \boldsymbol{\beta}_t = \mathbf{r}_t$$

where $\boldsymbol{R}_t$ is a known $J \times K$ matrix of rank $J \leq K$, and $\mathbf{r}_t$ is a known $J \times 1$ vector.

Estimating the econometric model given by (13) and (14) is problematic insofar as there are as many unknown parameter vectors as there are observations. Economists typically deal with this problem by making an invariance assumption $\beta_t = \beta$, then either i) imposing a subset of the JT constraints represented by (14), or, as we have already seen, ii) devising a set of constraints which are sufficient, but not necessary, for (14) to hold. Examples of the first approach include Clements and Izan (1987) and Selvanathan (1989) who impose observation-varying constraints at a single point. Examples of the second approach include O'Donnell and Woodland (1995) and Blake and Neid (1997) who use parametric adding-up constraints to ensure homogeneity constraints hold. More details will be provided below in Sections 3 and 4. Of course, neither of these approaches is satisfactory.

A solution to these difficulties lies in a matrix decomposition of $\boldsymbol{R}_t$ and an associated reparameterisation of (13) and (14). A suitable decomposition and reparameterisation is described by Doran, O'Donnell and

Rambaldi. These authors have shown how the Singular Value Decomposition (SVD) Theorem can be used to decompose $\boldsymbol{R}_t$ as

(15) $$\boldsymbol{R}_t = \boldsymbol{U}_t \boldsymbol{S}_t [\, \boldsymbol{I}_J, \mathbf{0}_{J,K-J}]\boldsymbol{V}_t'$$

where $\boldsymbol{U}_t$ and $\boldsymbol{V}_t$ are orthogonal matrices of dimension $J \times J$ and $K \times K$ respectively, and $\boldsymbol{S}_t$ is a $J \times J$ diagonal matrix containing the singular values of $\boldsymbol{R}_t$. For more precise definitions of these matrices see, for example, Lütkepohl (1996). Doran, O'Donnell and Rambaldi partition $\boldsymbol{V}_t$ as $\boldsymbol{V}_t = [\boldsymbol{V}_{1t}, \boldsymbol{V}_{2t}]$, where $\boldsymbol{V}_{1t}$ and $\boldsymbol{V}_{2t}$ are $K \times J$ and $K \times (K - J)$, before reparameterising and combining (13) and (14) into the form

(16) $$\mathbf{w}_t = \boldsymbol{Z}_t \gamma_t + \mathbf{e}_t$$

where γ_t is a $K \times 1$ parameter vector, $\mathbf{w}_t \equiv \mathbf{y}_t - \boldsymbol{X}_t \boldsymbol{V}_{1t} \boldsymbol{S}_t^{-1} \boldsymbol{U}_t' \mathbf{r}_t$ and $\boldsymbol{Z}_t \equiv \boldsymbol{X}_t \boldsymbol{V}_{2t} \boldsymbol{V}_{2t}'$ is of less than full rank. The important feature of the model given by (16) is that it is unconstrained (the constraints have been substituted out in the transformation from $\mathbf{y}_t$ to $\mathbf{w}_t$). Thus, no logical inconsistencies or practical difficultes arise from invoking the parametric invariance assumption $\gamma_t = \gamma$. Under this identifying assumption, a least squares estimator of $\boldsymbol{\gamma} = (\gamma_1, ..., \gamma_K)'$ is

(17) $$\mathbf{g} = (\boldsymbol{Z}'\boldsymbol{Z})^{+} \boldsymbol{Z}'\mathbf{w}$$

where $\mathbf{w} = (\mathbf{w}_1', ..., \mathbf{w}_T')'$, $\boldsymbol{Z} = (\boldsymbol{Z}_1', ..., \boldsymbol{Z}_T')'$ and $\boldsymbol{A}^{+}$ denotes the Moore-Penrose generalised inverse of the matrix $\boldsymbol{A}$. Then an estimator of β_t is

(18) $$\mathbf{b}_t = \boldsymbol{V}_{1t} \boldsymbol{S}_t^{-1} \boldsymbol{U}_t' \mathbf{r}_t + \boldsymbol{V}_{2t} \boldsymbol{V}_{2t}' \mathbf{g}.$$

with variance-covariance matrix

(19) $$\mathrm{Var}(\mathbf{b}_t) = \sigma^2 \boldsymbol{V}_{2t} \boldsymbol{V}_{2t}' \, (\boldsymbol{Z}'\boldsymbol{Z})^{+} \, \boldsymbol{V}_{2t} \boldsymbol{V}_{2t}'.$$

By construction, the estimator $\mathbf{b}_t$ will yield estimates which satisfy the constraints given by (14), will yield point-invariant estimates of any parameters which are not subject to point-varying constraints, and, unlike some other reparameterisation procedures, will yield estimates which are invariant to a reordering of the regressors in $\boldsymbol{X}_t$. A proof of these properties is contained in Doran, O'Donnell and Rambaldi. Finally, unlike the Kalman filter estimator of Doran and Rambaldi, the SVD estimator given by (17) and (18) is computationally simple: the singular value decomposition of $\boldsymbol{R}_t$ can be carried out using a single command in standard packages such as GAUSS and SHAZAM and, since $\boldsymbol{R}_t$ is of dimension $J \times K$, the practical simplicity of this task is unaffected the sample size, T; and, since $\mathbf{Z}'\mathbf{Z}$ is of dimension $K \times K$, calculating the generalised inverse in (17) and (18) is also unaffected by sample size. In the remainder of this paper I demonstrate the practical usefulness of the estimator using two different models and data sets.

3. Estimating the Characteristics of a Generalised Linear Production Function

In this section I use an $N = 2$ input version of the quadratic production function given by (2) (with $m = t$) to approximate a two-input constant returns to scale generalised linear production function of the form given by (1). The constant returns to scale property gives rise to the homogeneity constraint given by (10). After appending an error term, $e_t \sim N(0, \sigma^2)$, to (2), equations (2) and (10) can be written in the general matrix form given by (13) and (14). Accordingly, the observation-varying parameters of the model can be estimated using the SVD estimator given by (17) and (18). It is also possible to proceed using a restricted least squares (RLS) estimator (see Judge *et al*, p.858), by assuming the parameters are observation-invariant and imposing the sufficient conditions for homogeneity given by (12). In this section I examine i) the performance of both estimators and ii) the sensitivity of the results to the choice of approximating functional form.

I begin by generating $T + S = 200$ observations using (1) where, for illustrative purposes, I set $\alpha_{11} = \alpha_{22} = 1$, $\alpha_{12} = \alpha_{21} = 10$, $x_{1t} = 100 + t$ and $x_{2t} \sim N(200, 100)$ for t = 1, ..., $T + S$. These assumptions ensure that the production function is monotonic and concave, as required by economic theory. The first $T = 100$ observations are used for estimation purposes, and the remaining $S = 100$ observations are used for model validation.

Selected parameter estimates are presented in Table 1: column *A* reports estimates of $\boldsymbol{\gamma}_t = \boldsymbol{\gamma} = (\gamma_1, ..., \gamma_K)'$ obtained using the least squares estimator (17); column *B* reports estimates of $\boldsymbol{\beta}_t = \boldsymbol{\beta}$ obtained using the

RLS estimator; and columns C to F report estimates of β_t for $t = 1$, 50, 100 and 150, obtained using the SVD estimator (18). The numbers in parentheses are t-ratio's and indicate that all estimated slope coefficients are statistically significant at usual levels of significance. It is noteworthy that i) the RLS and SVD estimates of the first-order coefficients, β_{1t} and β_{2t}, are similar; ii) the SVD estimates of β_{1t} and β_{2t} are point-invariant owing to the fact that these parameters do not appear in the constraints (10); iii) the estimates reported in column F (ie. for $t = 150$) are out-of-sample estimates (since $T < 150$); and iv) the SVD estimates of the β_{ijt} parameters are point-varying and quite large. Of course, these estimates have no clear economic interpretation, and their magnitudes only matter to the extent that they are used to estimate y_t and other economically-interesting characteristics of the production function.

The RLS and SVD parameter estimates were used to generate within-sample estimates/predictions of y_t using equation (2). These predictions are denoted $\hat{y}^{R}(x_{1t}, x_{2t})$ and $\hat{y}^{S}(x_{1t}, x_{2t})$ ($t = 1, ..., T$). The within-sample squared correlation between y_t and $\hat{y}^{R}(x_{1t}, x_{2t})$ was 0.9982, while the squared correlation between y_t and $\hat{y}^{S}(x_{1t}, x_{2t})$ was 0.9999. In practice, goodness-of-fit statistics of this size, together with t-ratio's of the size reported in Table 1, might cause researchers to use the RLS estimates for inference, rather than calculate the theoretically-appealing but somewhat more computationally-demanding SVD estimates. In the remainder of this section I demonstrate that, even when goodness-of-fit statistics and t-ratio's are high, the consequences of persisting with the RLS estimator can be both undesirable and severe.

The parameter estimates were also used to generate out-of-sample predictions of y_t (ie. for $t = T + 1, ..., T + S$), again using equation (2). Both the within- and out-of-sample predictions were then used to obtain estimates of points on the unit isoquant, namely $\mathbf{x}_t / \hat{y}^{R}(x_{1t}, x_{2t})$ and $\mathbf{x}_t / \hat{y}^{S}(x_{1t}, x_{2t})$. The within-sample estimates of these points are presented in Figure 1 and the out-of sample estimates are presented in Figure 2. Note that the RLS estimates lie on a straight line, reflecting the fact that they have been obtained from a single approximating quadratic function (and homogeneity-constrained quadratic functions are linear, an unsatisfactory characteristic of quadratic functions which has been noted by Diewert, p.294). In contrast, each SVD estimate is obtained from a different approximating quadratic function, and collectively the SVD estimates smother the true isoquant. Interestingly, each SVD approximating function can be represented as a straight line extending from the origin to a point on or near the unit isoquant, and the problem of estimating $\boldsymbol{\gamma}$ can be viewed as one of choosing the slopes and lengths (ie. endpoints) of these straight lines.

The RLS and SVD parameter estimates have also been used to obtain estimates of the first- and second-order derivatives of (1). The RLS estimates of these derivatives are simply the estimated derivatives of the approximating quadratic function. Thus, RLS estimates of the first-order derivatives of (1) are constants, and estimates of the second-order derivatives are zero. Obtaining SVD estimates of the derivatives of (1) is not so straightforward, because the estimated first- and second-order derivatives of the SVD approximating functions cannot be regarded as estimates of the derivatives of (1), even at the appropriate points of approximation. To see this, simply note that the first- and second-order derivatives of (2) can assume an infinite number of values and still satisfy (3) and (8), and there is not enough information in the sample to ensure (4) and (5) will hold. Of course, SVD estimates of the derivatives of (1) can still be obtained as rates of change. Estimates of $\partial y_t/\partial x_{1t}$, for example, can be obtained as:

$$(20) \qquad \frac{\Delta y_t}{\Delta x_{1t}} = \frac{\hat{y}^S(x_{1t} + 0.5\Delta x_{1t}, x_{2t}) - \hat{y}^S(x_{1t} - 0.5\Delta x_{1t}, x_{2t})}{\Delta x_{1t}}$$

where, in this paper, I set $\Delta x_{jt} = 0.01x_{jt}$ (j = 1, 2). Formulae for estimating the remaining derivatives are contained in the Appendix. These formulae have been used to obtain estimates of the derivatives of (1) at each of the $T + S$ points in the data set, and these estimates are presented in Figures 3 to 7. Since the derivatives are plotted against t, the left-hand panel in each figure presents the within-sample estimates while the right-hand panel presents the out-of-sample estimates. It is apparent from these figures that the common practice of using RLS to impose the homogeneity constraints (12) is severely restrictive: the RLS estimates are constant so that, in every case, the squared correlation between the RLS estimates and the true derivatives is zero. In contrast, the SVD estimates of the derivatives of (1) are typically quite close to the true values. Table 2 reports objective measures of this closeness in the form of squared correlation coefficients and root mean square percentage errors (RMSPEs). Note from Table 2 and Figure 5 that the high RMSPE for the SVD out-of-sample estimates of $\partial^2 y_t/\partial x_{1t}^2$ reflects the fact that these estimates are incorrectly signed. In practice, incorrectly signed estimates of first- and/or second-derivatives can be easily avoided using Bayesian methodology (see, for example, Terrell, 1996; O'Donnell, Shumway and Ball, 1999; and Griffiths, O'Donnell and Tan Cruz, 2000).

To further illustrate the differences (and similarities) between RLS and SVD estimates of the characteristics of (1), I also consider the use of a translog functional form. A translog function which can provide a second-order approximation to (1) at the point $\mathbf{x}_m$ is

(21) $$y_t = f_m(\mathbf{x}_t) = \exp\{\phi_{0m} + \sum_{i=1}^{N} \phi_{im}\ln(x_{it}) + 0.5\sum_{i=1}^{N}\sum_{j=1}^{N} \phi_{ijm}\ln(x_{it})\ln(x_{jt})\}$$

where, again, the parameters, ϕ_{0m}, ϕ_{im} and $\phi_{ijm} = \phi_{jim}$ (i, j = 1, ..., N), vary across points of approximation. Conventional RLS estimates are obtained by assuming the parameters are observation-invariant (ie. $\phi_{im} = \phi_i$ and $\phi_{ijm} = \phi_{ij}$ for all i, j and m) and imposing the constraints

(22) $$\sum_{i=1}^{N} \phi_i = 1 \quad \text{and} \quad \sum_{i=1}^{N} \phi_{ij} = 0 \qquad \text{for } j = 1, ..., N.$$

In contrast, SVD estimates are obtained by setting $m = t$ for all t (ie. using sample points as points of approximation) and imposing the constraints

(23) $$\sum_{i=1}^{N} \phi_{it} + \sum_{i=1}^{N}\sum_{j=1}^{N} \phi_{ijt}\ln(x_{jt}) = 1 \qquad \text{for } t = 1, ..., T.$$

Note that observation-invariance of the parameters and the constraints given by (22) are sufficient but not necessary for the constraints given by (23) to hold.

The RLS and SVD translog parameter estimates are reported in Table 3. Note that i) the RLS and SVD estimates are of similar orders of magnitude, ii) unlike the RLS estimates, the SVD estimates of the second-order coefficients are not significantly different from zero at usual levels of significance, and iii) the SVD estimates of ϕ_{0t} are point-invariant owing to the fact that these parameters do not appear in the constraints (23). Despite the statistical insignificance of some of the SVD parameter estimates, SVD estimates of y_t and the first- and second-order derivatives of (1) are still close to the true values: RLS and SVD estimates of the unit isoquant are presented in Figures 8 and 9, and goodness-of-fit statistics are presented in Table 4. Note that the SVD estimator still tends to outperform the RLS estimator. Interestingly, both estimators yield such good estimates of the characteristics of (1) that when RLS and SVD estimates of the first- and second-order derivatives of (1) are presented in graphical form, the differences between the estimated and true values are imperceptible.

As a final illustration of the way in which the SVD estimator can be used to estimate the important characteristics of (1), consider the generalised linear function

$$(24) \qquad y_t = f_m(\mathbf{x}_t) = \theta_{0m} + \sum_{i=1}^{2} \theta_{im} x_{it}^{1/2} + \sum_{i=1}^{2}\sum_{j=1}^{2} \theta_{ijm} x_{it}^{1/2} x_{jt}^{1/2}$$

where, yet again, the parameters, ϕ_{0m}, ϕ_{im} and $\phi_{ijm} = \phi_{jim}$ (i, j = 1, ..., N), are permitted to vary across points of approximation. Conventional RLS estimates are obtained by assuming $\theta_{ijm} = \theta_{ij}$ for all i, j and m, and constraining all other parameters to zero. Under these assumptions and constraints, equation (24) collapses to equation (1) and, not surprisingly, the RLS estimator can be used to obtain a perfect fit. In contrast, SVD estimates are obtained by setting $m = t$ for all t and imposing the constraints

$$(25) \qquad \theta_{0m} + \sum_{i=1}^{2} \theta_{im} x_{it}^{1/2} = 0 \qquad \text{for } t = 1, ..., T.$$

SVD estimation of (24) subject to the constraints (25) yields estimates of the θ_{ijm} parameters which are identical to the true (constant) values. Although the SVD estimates of the parameters in (25) are observation-varying and non-zero, equation (25) is satisfied and, consequently, the SVD estimator also yields a perfect fit. Thus, RLS and SVD predictions and estimates of first- and second-order derivatives coincide exactly when the approximating functional form is identical to the true functional form. The generality of this result is unproven. If proven, it may provide a basis for designing tests of functional form.

4. Estimating the Characteristics of an Unknown Demand System

In this section I use the Canadian expenditure data of Ryan and Wales to estimate the point-varying parameters of the LAID share equations (Deaton and Muellbauer)

$$(26) \qquad w_{it} = \alpha_{im} + \sum_{j=1}^{N} \phi_{ijm} \ln(p_{jt}) + \beta_{im} \ln(Y_t/P_t) \qquad i = 1, ..., N;\ t = 1, ..., T,$$

where $w_{it} = p_{it}\, q_{it}/Y_t$ is the budget share of the ith good in period t, p_{it} is the price of the ith good, q_{it} is quantity demanded, Y_t is income, α_{im}, $\phi_{ijm} = \phi_{jim}$ and β_{im} (i, j = 1, ..., N) are parameters to be estimated, and

$$\ln(P_t) = \sum_{j=1}^{N} w_{jt}\ln(p_{jt}) \tag{27}$$

is Stone's (1953) price index. After appending N random error terms, $\mathbf{e}_t = (e_{1t}, ..., e_{Nt})' \sim N(\mathbf{0}, \sigma^2\mathbf{I}_N)$, the system of N equations given by (26) can be written in the form of equation (13).

The underlying expenditure function is HD1 in prices and, by Euler's Theorem, this means the budget shares given by (26) sum to unity. Conventional RLS estimates are obtained by assuming the parameters are observation-invariant (ie. $\alpha_{im} = \alpha_i$, $\phi_{ijm} = \phi_{ij}$ and $\beta_{im} = \beta_i$ for all i, j and m) and imposing the constraints

$$\sum_{i=1}^{N} \alpha_i = 1, \quad \sum_{j=1}^{N} \phi_{ij} = 0 \quad \text{and} \quad \sum_{i=1}^{N} \beta_i = 0. \tag{28}$$

SVD estimates are obtained by setting $m = t$ for all t and imposing the constraints

$$\sum_{i=1}^{N} \alpha_{it} + \sum_{i=1}^{N}\sum_{j=1}^{N} \phi_{ijt}\ln(p_{jt}) + \sum_{i=1}^{N} \beta_{it}\ln(Y_t/P_t) = 1 \qquad \text{for } t = 1, ..., T. \tag{29}$$

The constraints given by (29) and the cross-equation identifying restrictions $\phi_{ijm} = \phi_{jim}$ can be written in the form of equation (14) with[2] $J \leq N(N-1)/2$. Note, once again, that observation-invariance of the parameters and the constraints given by (28) are sufficient but not necessary for the homogeneity constraints given by (29) to hold[3].

The Ryan and Wales data consists of $T = 47$ annual observations on Canadian per capita expenditure on $N = 3$ broad commodity groups (food, clothing and miscellaneous) covering the period 1947 to 1993 inclusive. The data can be downloaded from the *Journal of Business and Economic Statistics* website.

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[2] At some data points these constraints may be linearly dependent, implying J may vary from point to point.

[3] Deaton and Muellbauer (p.316), among others, refer to the constraints given by (28) as *adding-up constraints*, and refer to the identifying constraints $\phi_{ij} = \phi_{ji}$ as *symmetry* constraints. Together these constraints imply $\sum_{j=1}^{N} \phi_{ji} = 1$, and this constraint is referred to as an *homogeneity* constraint even though it is not, by itself, sufficient for homogeneity to hold.

RLS and SVD estimates of the LAID system parameters are reported in Table 5, and goodness-of-fit statistics are reported in Table 6. Note from Table 5 that most of the RLS and SVD parameter estimates are statistically different from zero at the 5% level of significance, and all the SVD estimates are observation-varying owing to the fact that all the parameters appear in the homogeneity constraints (29). From Table 6 it is clear that the SVD estimator dominates the RLS estimator in terms of predictive performance. Note, in particular, that the squared correlation between w_{3t} (the budget share of the miscellaneous group) and the RLS predictions is 0.79, while the squared correlation between w_{3t} and the SVD predictions is 0.87. In the case of the food and miscellaneous commodity groups, the SVD estimator also yields lower RMSPE statistics than RLS. This goodness-of-fit is evident in Figures 10 to 12 where I plot the observed and predicted budget shares. The results depicted in these figures (and the results reported in Tables 5 and 6) are consistent with the conclusion drawn in Section 3, namely that observation-invariance of the parameters and the imposition of sufficient (but not necessary) conditions for homogeneity is overly restrictive.

The SVD and RLS parameter estimates have been used to obtain estimates of the first derivatives of the commodity demand functions with respect to prices and income. A representative selection of these estimated first-derivatives is presented in Figures 13 to 16: Figure 13 presents estimates of $\partial q_{1t} / \partial p_{1t}$ at every point in the sample, Figure 14 presents estimates of $\partial q_{2t} / \partial p_{3t} = \partial q_{3t} / \partial p_{2t}$, Figure 15 presents estimates of $\partial q_{3t} / \partial p_{3t}$, and Figure 16 presents estimates of $\partial q_{1t} / \partial Y_t$. It is apparent from these figures that i) there are marked differences between the SVD and RLS estimates of some of the first-derivatives in some time periods (see, for example, the estimates of $\partial q_{2t} / \partial p_{3t}$ in periods t = 25, ..., 47, presented in Figure 14), ii) there is considerably more point-to-point variation in the SVD estimates than the RLS estimates (see, for example, Figures 15 and 16), and iii) several SVD estimates have signs which are theoretically implausible. Interestingly, I made exactly the same observations concerning the estimated first- and second-order derivatives of the generalised linear production function in Section 3, and still went on to conclude that the SVD estimates were superior to the RLS estimates. By implication, the point-to-point variations and sign patterns exhibited by the SVD estimates in Figures 13 to 16 do not imply the SVD estimates are generally further from the truth than the RLS estimates. Rather, these estimates provide reasons for, once again, obtaining SVD estimates within a framework which ensures the first- and second-derivatives are correctly signed (eg. a Bayesian framework). They also provide reasons for considering the use of alternative functional forms.

5. Summary and Conclusion

Much of econometrics is concerned with estimating functions which are homogeneous of degree *k* (HD*k*) in at least some their arguments. The profit function of the firm, for example, is HD1 in output and input prices, constant returns to scale production functions are HD1 in input quantities, and the consumer's income compensated demand functions are HD0 in product prices. The exact mathematical form of these functions is typically unknown, and in empirical work it is common practice to employ second-order flexible forms. These flexible forms are popular because, at a point of approximation, their first- and second-order derivatives can be set equal to those of the unknown function. Regrettably, they are usually estimated under the assumption that the parameters are observation-invariant, with the implication that they can only approximate the unknown function at a single point.

The assumption that the parameters are observation-invariant has important implications for the way in which homogeneity constraints are imposed. Specifically, it leads econometricians to impose parametric adding-up constraints which are sufficient, but not necessary, for homogeneity to hold. By implication, this places unnecessary restrictions on the structure of underlying preferences and technologies, and limits the ability of the model to capture relevant characteristics of underlying economic behaviour. It is not surprising, therefore, that these homogeneity constraints are often rejected in empirical work (for a discussion of this 'homogeneity puzzle', see, for example, Buse, 1998).

In this paper I have shown how these problems can be overcome by allowing the parameters of flexible form models to vary deterministically from one point of approximation (or observation) to another. By allowing the parameters to vary across observations, it is possible to impose homogeneity constraints in the exact form prescribed by Euler's Theorem. Euler's Theorem typically gives rise to equality constraints which involve both the data and the unknown observation-varying parameters. When the model and the constraints are both linear in the unknown parameters, it is possible to obtain constrained estimates of the parameters using the Singular Value Decomposition (SVD) estimator of Doran, O'Donnell and Rambaldi. Unlike other estimators which can be used to estimate observation-varying parameters, the SVD estimator yields observation-invariant estimates of any parameters which are not subject to observation-varying constraints, and yields parameter estimates which are invariant to a reordering of the regressors.

I illustrated the approach using two examples. First, I used artificially-generated data to estimate the characteristics of a constant returns to scale generalised linear production function. The results confirmed that the conventional restricted least squares (RLS) estimator is generally less 'flexible' than the SVD estimator. Interestingly, the RLS and SVD estimators yielded identical predictions only when the approximating functional form was also generalised linear. This property of the estimators may provide a basis for testing for functional form. Second, I used the Canadian expenditure data of Ryan and Wales to estimate the characteristics of an unknown expenditure function. Again, the SVD estimator appeared to outperform the RLS estimator in terms of predictive performance. Both examples reinforced the commonly-held view that the parameters of economic models should be estimated in a (Bayesian) framework which allows inequality (ie. curvature) constraints to be imposed.

Finally, despite (or perhaps because of) the plausibilty of the empirical results, there appear to be several opportunities for further research. Aside fom the imposition of curvature constraints, the most interesting and potentially useful avenues of further investigation appear to be i) relaxing the observation-invariance assumption on $\boldsymbol{\gamma}_t$, ii) specifying a more general covariance structure for the random error vector $\mathbf{e}_t$, and iii) using discrepancies between RLS and SVD estimates as a basis for testing for functional form. With or without these developments, the estimation approach outlined in this paper appears to provide numerous and wide-rangeing opportunities for empirical research.

Appendix A

Estimating Derivatives as Rates of Change

(A.1) $$\frac{\partial y_t}{\partial x_{1t}} \approx \frac{\hat{y}^{s}(x_{1t} + 0.5\Delta x_{1t}, x_{2t}) - \hat{y}^{s}(x_{1t} - 0.5\Delta x_{1t}, x_{2t})}{\Delta x_{1t}}$$

(A.2) $$\frac{\partial y_t}{\partial x_{2t}} \approx \frac{\hat{y}^{s}(x_{1t}, x_{2t} + 0.5\Delta x_{2t}) - \hat{y}^{s}(x_{1t}, x_{2t} - 0.5\Delta x_{2t})}{\Delta x_{2t}}$$

(A.3) $$\frac{\partial^2 y_t}{\partial x_{1t}^2} \approx \frac{\hat{y}^{s}(x_{1t} + 0.5\Delta x_{1t}, x_{2t}) \; + \; 2\,\hat{y}^{s}(x_{1t}, x_{2t}) \; + \; \hat{y}^{s}(x_{1t} - 0.5\Delta x_{1t}, x_{2t})}{(0.5\Delta x_{1t})^2}$$

(A.4) $$\frac{\partial^2 y_t}{\partial x_{2t}^2} \approx \frac{\hat{y}^{s}(x_{1t}, x_{2t} + 0.5\Delta x_{2t}) \; + \; 2\,\hat{y}^{s}(x_{1t}, x_{2t}) \; + \; \hat{y}^{s}(x_{1t}, x_{2t} - 0.5\Delta x_{2t})}{(0.5\Delta x_{2t})^2}$$

(A.5) $$\frac{\partial^2 y_t}{\partial x_{1t}\,\partial x_{2t}} \approx (1/\Delta x_{1t}\Delta x_{2t}) \times \Big[\, \hat{y}^{s}(x_{1t} + 0.5\Delta x_{1t}, x_{2t} + 0.5\Delta x_{2t}) - \hat{y}^{s}(x_{1t} - 0.5\Delta x_{1t}, x_{2t} + 0.5\Delta x_{2t})$$

$$- \hat{y}^{s}(x_{1t} + 0.5\Delta x_{1t}, x_{2t} - 0.5\Delta x_{2t}) + \hat{y}^{s}(x_{1t} - 0.5\Delta x_{1t}, x_{2t} - 0.5\Delta x_{2t}) \Big]$$

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Table 1
Quadratic Approximation(s) to a Generalised Linear Production Function: Parameter Estimates [a]

				SVD			
Parameter	Estimate (*A*)	Parameter	RLS (*B*)	t = 1 (*C*)	t = 50 (*D*)	t = 100 (*E*)	t = 150 (*F*)
γ_1	-48.367 (-0.95)	β_{0t}	-	-61.310 (-1.83)	-28.499 (-0.82)	-40.251 (-1.10)	-73.664 (-2.11)
γ_2	12.476 (136.70)	β_{1t}	12.607 (229.148)	12.476 (136.70)	12.476 (136.70)	12.476 (136.70)	12.476 (136.70)
γ_3	9.990 (31.02)	β_{2t}	9.531 (225.689)	9.990 (31.02)	9.990 (31.02)	9.990 (31.02)	9.990 (31.02)
γ_4	-4.582E+7 (-9.93)	β_{11t}	-	-4.569E+7 (-10.22)	-4.627E+7 (-10.76)	-4.615E+7 (-11.27)	-4.424E+7 (-12.18)
γ_5	3.670E+7 (26.85)	β_{12t}	-	3.722E+7 (20.79)	3.550E+7 (17.37)	3.601E+7 (15.75)	3.907E+7 (14.64)
γ_6	-2.640E+7 (-20.78)	β_{22t}	-	-2.588E+7 (-35.21)	-2.721E+7 (-36.64)	-2.676E+7 (-36.40)	-2.552E+7 (-33.90)

[a] Numbers in parentheses are *t*-ratios.

Table 2
Quadratic Approximation(s) to a Generalised Linear Production Function: Goodness-of-Fit Statistics

	Within-sample (t = 1, ..., T)		Out-of-Sample ($t = T + 1, ..., T + S$)		All observations ($t = 1, ..., T + S$)	
	RLS	SVD	RLS	SVD	RLS	SVD
Squared Correlation Between Actual and Predicted						
y_t	0.9982	0.9999	0.9788	0.9928	0.9943	0.9978
$\partial y_t / \partial x_{1t}$	0.0000	0.9421	0.0000	0.3281	0.0000	0.6901
$\partial y_t / \partial x_{2t}$	0.0000	0.9792	0.0000	0.7273	0.0000	0.9416
$\partial^2 y_t / \partial x_{1t}^2$	0.0000	0.9828	0.0000	0.6652	0.0000	0.9501
$\partial^2 y_t / \partial x_{1t}\, \partial x_{2t}$	0.0000	0.1008	0.0000	0.7319	0.0000	0.6880
$\partial^2 y_t / \partial x_{2t}^2$	0.0000	0.9122	0.0000	0.2088	0.0000	0.6420
Root Mean Square Percentage Error (RMSPE) Between Actual and Predicted[a]						
y_t	0.44	0.08	3.31	1.81	2.36	1.28
$\partial y_t / \partial x_{1t}$	9.12	2.54	27.81	19.74	20.70	14.07
$\partial y_t / \partial x_{2t}$	8.98	1.84	22.03	8.40	16.82	6.08
$\partial^2 y_t / \partial x_{1t}^2$	100.00	55.93	100.00	122.90	100.00	95.48
$\partial^2 y_t / \partial x_{1t}\, \partial x_{2t}$	100.00	27.03	100.00	75.10	100.00	56.44
$\partial^2 y_t / \partial x_{2t}^2$	100.00	77.51	100.00	16.27	100.00	56.00

[a] In the case of $\hat{y}_t$ (the within-sample prediction of y_t), for example, RMSPE is calculated as the square root of $(1/T) \sum_{t=1}^{T} [100 \times (y_t - \hat{y}_t)/y_t]^2$.

Table 3
Translog Approximation(s) to a Generalised Linear Production Function: Parameter Estimates [a]

				SVD			
Parameter	Estimate (*A*)	Parameter	RLS (*B*)	t = 1 (*C*)	t = 50 (*D*)	t = 100 (*E*)	t = 150 (*F*)
γ_1	3.278	ϕ_{0t}	3.091	3.278	3.278	3.278	3.278
	(6.902)		(7.337E+6)	(6.902)	(6.902)	(6.902)	(6.902)
γ_2	0.503	ϕ_{1t}	0.500	0.464	0.466	0.467	0.467
	(68.961)		(1.652E+5)	(5.120)	(5.345)	(5.539)	(5.604)
γ_3	0.499	ϕ_{2t}	0.500	0.460	0.462	0.463	0.463
	(143.843)		(1.652E+5)	(4.576)	(4.758)	(4.913)	(4.966)
γ_4	0.312	ϕ_{11t}	0.023	0.134	0.126	0.121	0.115
	(0.426)		(2557.617)	(0.476)	(0.483)	(0.487)	(0.493)
γ_5	0.269	ϕ_{12t}	-0.023	-0.113	-0.114	-0.114	-0.114
	(0.363)		(-2557.617)	(-0.495)	(-0.494)	(-0.493)	(-0.493)
γ_6	0.314	ϕ_{22t}	0.023	0.109	0.117	0.121	0.127
	(0.426)		(2557.617)	(0.499)	(0.491)	(0.487)	(0.481)

[a] Numbers in parentheses are *t*-ratios.

Table 4
Translog Approximation(s) to a Generalised Linear Production Function: Goodness-of-Fit Statistics

	Within-sample (t = 1, ..., T)		Out-of-Sample ($t = T + 1, ..., T + S$)		All observations ($t = 1, ..., T + S$)	
	RLS	SVD	RLS	SVD	RLS	SVD
Squared Correlation Between Actual and Predicted						
y_t	1.0000	1.0000	0.9999	1.0000	1.0000	1.0000
$\partial y_t / \partial x_{1t}$	1.0000	1.0000	1.0000	1.0000	0.9999	1.0000
$\partial y_t / \partial x_{2t}$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\partial^2 y_t / \partial x_{1t}^2$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\partial^2 y_t / \partial x_{1t} \partial x_{2t}$	0.9998	1.0000	0.9998	1.0000	0.9998	1.0000
$\partial^2 y_t / \partial x_{2t}^2$	0.9999	1.0000	1.0000	1.0000	0.9999	1.0000
Root Mean Square Percentage Error (RMSPE) Between Actual and Predicted[a]						
y_t	0.2150	0.0027	0.1025	0.0019	0.1684	0.0023
$\partial y_t / \partial x_{1t}$	0.2178	0.0030	0.1176	0.0062	0.1750	0.0049
$\partial y_t / \partial x_{2t}$	0.2124	0.0085	0.0877	0.0114	0.1625	0.0100
$\partial^2 y_t / \partial x_{1t}^2$	0.3171	0.0614	0.1215	0.1369	0.2401	0.1061
$\partial^2 y_t / \partial x_{1t} \partial x_{2t}$	0.3171	0.1138	0.1215	0.0517	0.2401	0.0884
$\partial^2 y_t / \partial x_{2t}^2$	0.3171	0.2401	0.1215	0.2937	0.2401	0.2683

[a] See footnote to Table 2.

Table 5
LAID System Approximation(s) to an Unknown Demand System: Parameter Estimates [a]

		SVD					
Parameter	RLS (*A*)	t = 1 (*B*)	t = 10 (*C*)	t = 20 (*D*)	t = 30 (*E*)	t = 40 (*F*)	t = 47 (*G*)
α_{1t}	0.502	2.100	2.087	2.087	2.076	2.023	1.989
	(52.799)	(2.791)	(2.784)	(2.768)	(2.718)	(2.761)	(2.808)
ϕ_{11t}	0.081	-0.341	-0.357	-0.350	-0.344	-0.313	-0.289
	(3.935)	(-1.346)	(-1.381)	(-1.386)	(-1.384)	(-1.185)	(-1.126)
$\phi_{12t} = \phi_{21t}$	-0.094	-0.576	-0.589	-0.583	-0.577	-0.551	-0.522
	(-3.817)	(-2.625)	(-2.624)	(-2.652)	(-2.651)	(-2.433)	(-2.412)
$\phi_{13t} = \phi_{31t}$	0.014	-0.536	-0.553	-0.545	-0.538	-0.506	-0.469
	(0.871)	(-2.181)	(-2.203)	(-2.229)	(-2.247)	(-2.001)	(-1.940)
β_{1t}	-0.023	-1.069	-1.066	-1.076	-1.102	-1.203	-1.272
	(-3.478)	(-1.931)	(-1.935)	(-1.950)	(-2.030)	(-2.119)	(-2.095)
α_{2t}	0.183	1.808	1.795	1.796	1.784	1.731	1.698
	(11.141)	(2.504)	(2.495)	(2.481)	(2.438)	(2.475)	(2.513)
ϕ_{22t}	0.069	-0.279	-0.288	-0.283	-0.278	-0.257	-0.223
	(1.665)	(-1.308)	(-1.330)	(-1.332)	(-1.324)	(-1.166)	(-1.058)
$\phi_{23t} = \phi_{32t}$	0.025	-0.586	-0.600	-0.593	-0.587	-0.560	-0.517
	(1.031)	(-2.294)	(-2.303)	(-2.321)	(-2.319)	(-2.127)	(-2.078)
β_{2t}	0.009	-1.054	-1.051	-1.061	-1.087	-1.187	-1.257
	(0.790)	(-2.011)	(-2.017)	(-2.032)	(-2.117)	(-2.194)	(-2.164)
α_{3t}	0.315	2.177	2.164	2.165	2.153	2.100	2.067
	(31.617)	(2.840)	(2.833)	(2.817)	(2.769)	(2.809)	(2.856)
ϕ_{33t}	-0.039	-0.612	-0.631	-0.621	-0.614	-0.581	-0.531
	(-1.930)	(-2.556)	(-2.573)	(-2.608)	(-2.627)	(-2.351)	(-2.290)
β_{3t}	0.014	-1.237	-1.234	-1.244	-1.270	-1.371	-1.440
	(1.959)	(-2.188)	(-2.194)	(-2.206)	(-2.286)	(-2.364)	(-2.326)

[a] Numbers in parentheses are *t*-ratios. Subscripts refer to commodity groups (1 = food, 2 = clothing and 3 = miscellaneous).

Table 6
LAID System Approximation(s) to an Unknown Demand System: Goodness-of-Fit Statistics

	RLS	SVD
Squared Correlation Between Actual and Predicted		
w_{1t} (food)	0.95	0.96
w_{2t} (clothing)	0.92	0.92
w_{3t} (miscellaneous)	0.79	0.87
RMSPE Between Actual and Predicted [a]		
w_{1t} (food)	1.62	1.43
w_{2t} (clothing)	3.30	3.66
w_{3t} (miscellaneous)	3.51	2.73

[a] See footnote to Table 2.

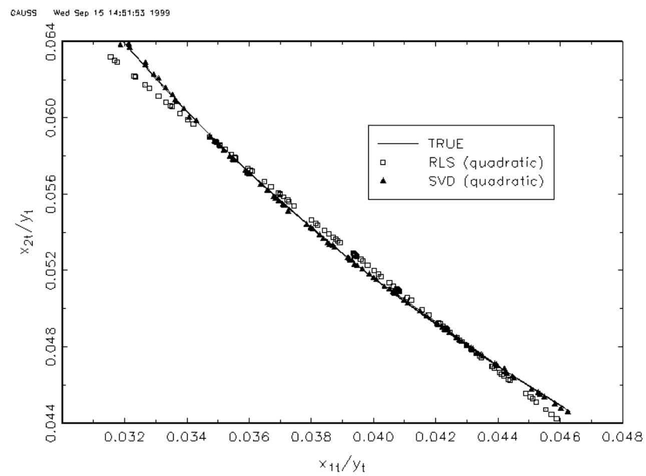

Figure 1. Within-Sample Estimates of Points on the Unit Isoquant

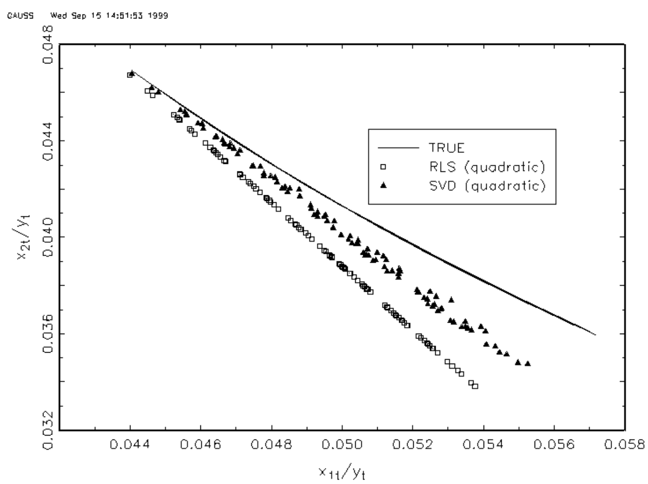

Figure 2. Out-of-Sample Estimates of Points on the Unit Isoquant

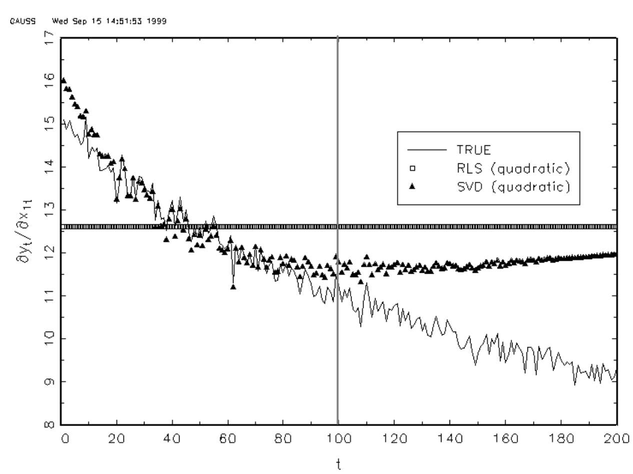

Figure 3. Estimates of $\partial y_t/\partial x_{1t}$

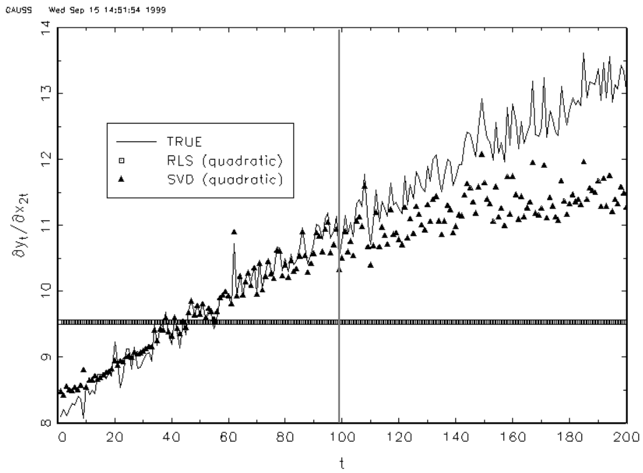

Figure 4. Estimates of $\partial y_t/\partial x_{2t}$

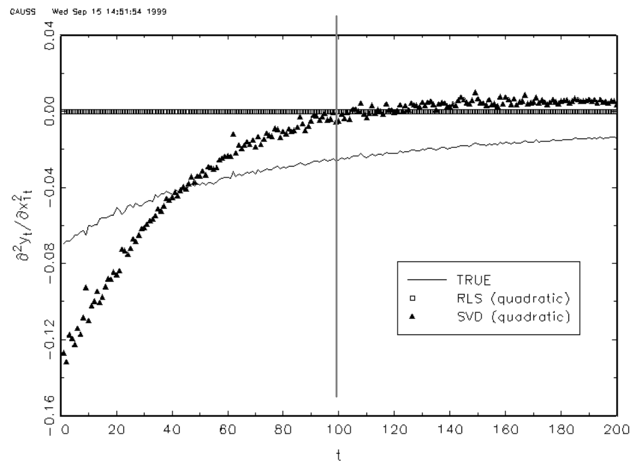

Figure 5. Estimates of $\partial^2 y_t/\partial x_{1t}^2$

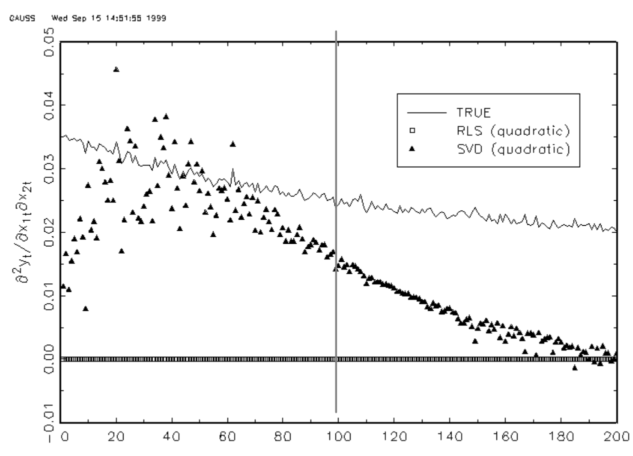

Figure 6. Estimates of $\partial^2 y_t/\partial x_{1t}\partial x_{2t}$

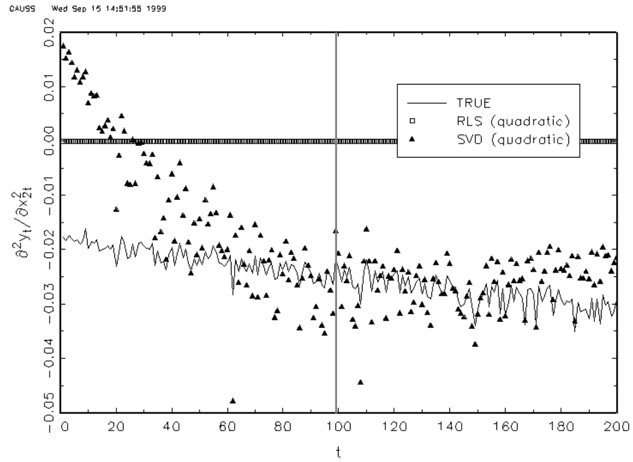

Figure 7. Estimates of $\partial^2 y_t/\partial x_{2t}^2$

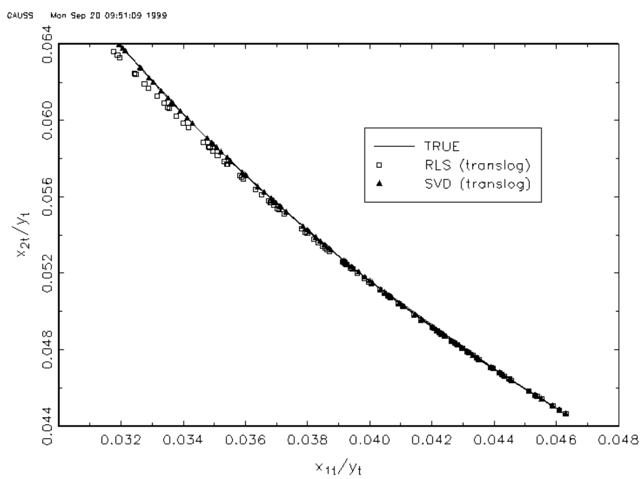

Figure 8. Within-Sample Estimates of Points on the Unit Isoquant

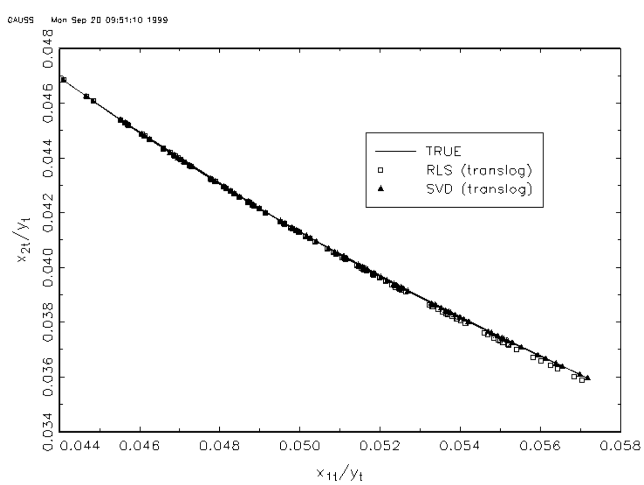

Figure 9. Out-of-Sample Estimates of Points on the Unit Isoquant

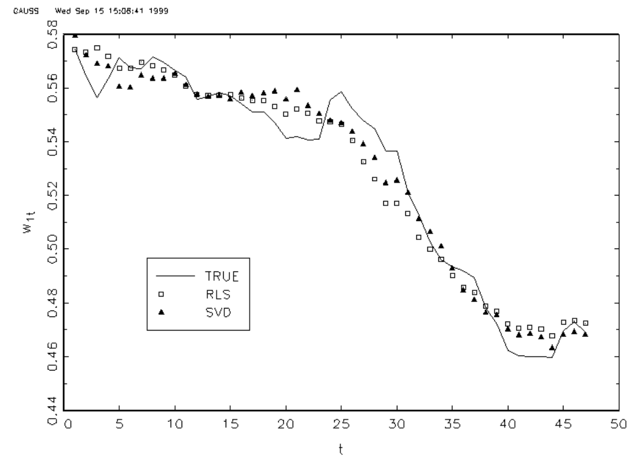

Figure 10. Estimates of the Food Budget Share (w_{1t})

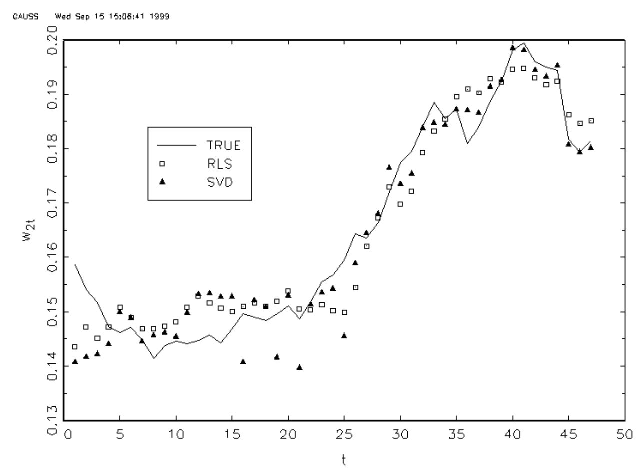

Figure 11. Estimates of the Clothing Budget Share (w_{2t})

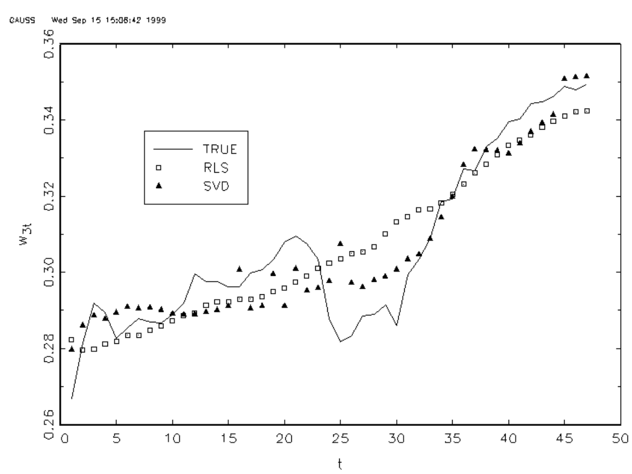

Figure 12. Estimates of the Miscellaneous Budget Share (w_{3t})

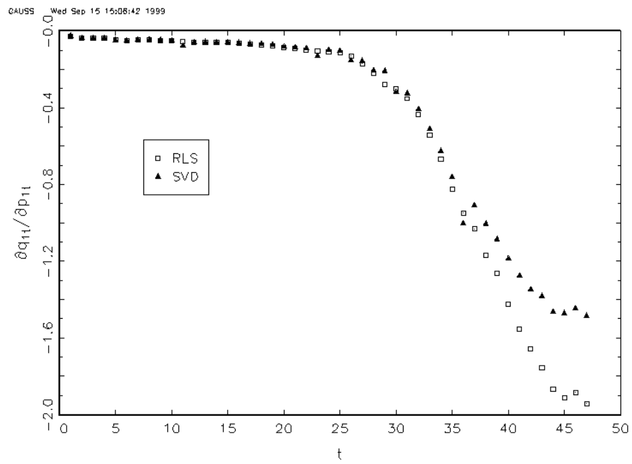

Figure 13. Estimates of $\partial q_{1t}/\partial p_{1t}$

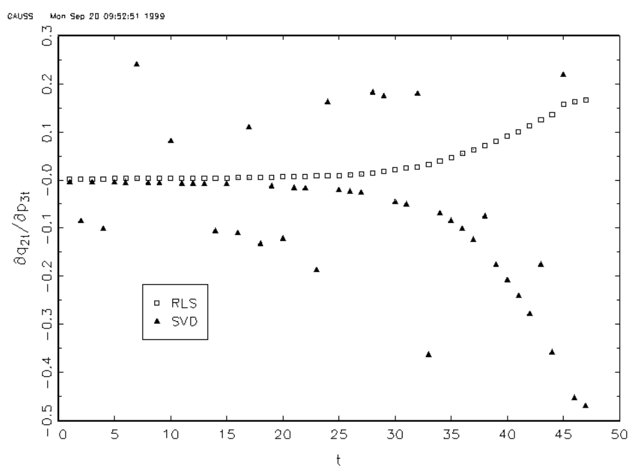

Figure 14. Estimates of $\partial q_{2t}/\partial p_{3t}$

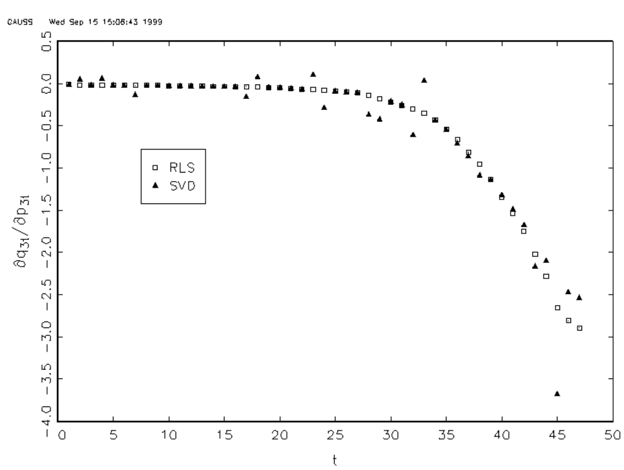

Figure 15. Estimates of $\partial q_{3t}/\partial p_{3t}$

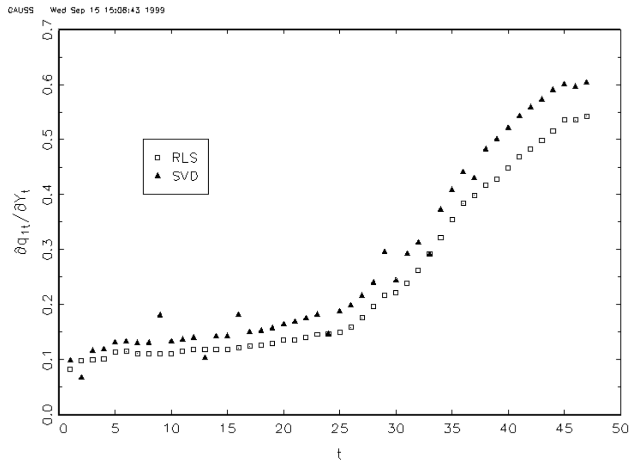

Figure 16. Estimates of $\partial q_{1t}/\partial Y_t$