A Numerical Stochastic Control model of Range Management *

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Abstract

An optimal stochastic control model of grazing that incorporates both pasture and livestock dynamics is presented. The model is solved numerically using Markov chain approximation methods. Markov chain approximation methods have a number of advantages as a means of solving stochastic optimal control methods compared with the usual alternatives. In particular, optimal control of the approximating Markov chain may be determined using Linear programming methods, thus making optimal stochastic control methods accessible to a wider audience.

1 Introduction

The use of Ito stochastic control techniques in the study of renewable resources generally is quite well established but this approach has rarely been applied to the study of optimal stocking in rangelands.

Ito stochastic control theory has been applied in agricultural and resource economics by a number of authors. Pindyck (1984) (Pindyck 1984)gives a survey of applications to renewable resources and Hertzler (1991) surveys applications to Agriculture. Applications involving common property have typically been restricted to non-renewable resources and have invariably involved postulating certainty to simplify the analysis[1].

From the late 1970's onwards a series of paper examine the application of stochastic control theory to the management of renewable resources, predominantly fisheries has emerged. Much of this literature is surveyed by Mangel (1985).

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[1] See Clemhout and Wan (1985)

Gleit (1978) extends Clark's (1976) analysis of deterministic bioeconomic (fisheries) models to a stochastic setting by using Ito calculus. Gleit considers two applications,

1) Harvesting from a natural population (sole harvester).

2) Harvesting from a farm population[2].

Each of these situations is characterized by Gleit as having different cost structures analogous to our distinction between the continuous stocking strategy followed by western style ranch systems and the "intensive" herding system of mobile pastoralism. Gleit criticizes the use of discounted profit as a payoff functional as this "ignores risk"(**?**, p. 113). As an alternative Gleit suggests maximization of the expected flow of discounted future utility. He achieves this by using an isoelastic utility function with profit as the argument. Gleit concludes that the optimal harvest rate should increase with the variance of the underlying resource but that more noisy systems lead to lower utility.

Ludwig (1979), and Ludwig and Varah (1979) analyze a similar problem to that of Gleit over an infinite time horizon.

Pindyck builds on the work of Ludwig, adn Ludwig and Varah

Although a continuous-time deterministic optimal stocking model has been developed by Torell, Lyon and Godfrey (1991), the emphasis in the literature has been on discrete-time stochastic models using either dynamic programming or Hamiltonian approaches. Undoubtedly, the inclusion of stochastic aspects is a central feature of any model of range dynamics, nvertheless there is still considerable room for the analysis of the nonlinear deterministic dynamics of rangeland ecosystems that cannot be appropriately analysed from within a stochastic framework. In particular, because, the *qualitative theory of differential equations* has been primarily developed for deterministic differential equations, stability issues are often more appropriately analysed within a deterministic framework.

Swanson (1994) recently attempted to remedy some of the deficiencies of the early literature on dynamic optimal stocking. Unfortunately, Swanson's model stops short of interspecific competition by treating the base resource (Land) as a "parameter or decision variable" but not as a state variable[3]. Consequently, rangeland degradation cannot be analyzed within Swanson's model. Nevertheless, his work does point in the right direction. Perrings (1994) goes further than Swanson in incorporating the base resource land not as a constant but as a state variable. Perrings does this by viewing carrying capacity as the state variable and showing how carrying capacity evolves through time in a grazed system.

Another approach is that of Huffaker (1995) who use "fast-slow dynamics" to capture the idea of the state and transition approach to pasture dynamics. although this model

[2]This begs the question as to what a renewable farm resource might be. Most crops are not that long lived. Pastoral systems tend to involve two and not one trophic level and thus differ conceptually from Gleit's model. Gleit's model when applied to farm management therefore begs the question as to what a real-world application might look like.

[3]Swanson's use of a generic variable for all land resources is probably too simplistic, in our context it can be interpreted as pasture biomass. In the models presented in this thesis other land resources are also viewed as constants in the same way as Swanson does this.

represents an interesting approach to modelling state and transition type dynamics, it fails to incorporate livestock dynamics and thus fails to capture an essential feature of extensive pastoral systems.

An alternative way of capturing fast-slow dynamics is to use stochastic processes, whereby random variables represent fast processes and constants represent slow processes relative to the time scale being examined.

1.1 Modelling Rangeland Ecosystems as Systems of Stochastic Differential Equations

Rangeland ecosystems are characterised by at least two trophic levels(**?**, Noy-Meir). Further their dynamics is characterised by considerable uncertainty. One way of incorporating both these aspects into a single model is to model the ecosystem as a system of stochastic differential equations(**?**, turelli).

In the following a general time-indexed stochastic process is assumed which may or may not be Markovian gives s the following system of stochastic differential equations:

$$\dot{x} = \hat{x}(x, y, t) \tag{1}$$
$$\dot{y} = \hat{y}(x, y, t) \tag{2}$$

where x represents pasture biomass and y represents livestock.

Typically, this system of stochastic differential equations would have an additive specification:

$$\dot{x} = F(x, y, t) + g(x)\xi(t) \tag{3}$$
$$\dot{y} = G(x, y, t) + h(y)\zeta(t) \tag{4}$$

where $g(x)$ and $h(x)$ are measures of the intensity of noise and ξ and ζ are noisy processes. The functions $F(x, y, t)$ and $G(x, y, t)$ are the deterministic component of the differential equation.

It should be noted that a version of the "State and Transition model" may be recovered from the model presented here in the following manner.

Given a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ a random variable $x(t, \omega)$ with $\omega \in \Omega$ and interpreting $x(t, \omega)$ as a vector of key range condition indicators then the State and Transition model is characterized by the following additional assumptions:

1) $F(x, y, t) = \mu$

2) $G(x, y, t) = 0$

3) $\xi(t)$ is a continuous-time Markov process

This approach is characteristic of those approaches which have interpreted state and transition models as Markov processes.

Interestingly, the claims of Westoby, Walker and Noy-Meir that the state and transition model is a disequilibrium model are for the most part not fulfilled by any of the attempts to implement the state and transition approach in a concrete modelling framework (See chapter 2 for further details).

In a stochastic context, equilibrium may be interpreted in a number of different ways. The underlying stochastic process may be interpreted as a,

1 Strongly Stationary (strong equilibrium concept - see appendix for a definition).

2 Weakly Stationary (weak equilibrium concept - see appendix for a definition)

process[4].

If all forms of equilibrium were totally rejected then analysis would become impossible. Such a situation would be characterized as "chaotic" in the sense of non-linear dynamic systems theory. The existence of chaotic behaviour in real biological systems is however contentious to say the least[5].

The following model of the ecosystem, is therefore proposed:

$$dx = [f(x) - h(x,y)]\, dt + \sigma(x,y)dB$$

$$dy = [g(y,x) - u]\, dt + \sigma(x,y)dB$$

where x is forage plant biomass, y the total number of animals which may also be interpreted as the stocking rate, $f(x)$ the biomass regeneration function, $g(x,y)$ the herbivore regeneration (reproduction) function, and the control variable u the turnoff rate of livestock and dB a Wiener or Brownian motion increment, with the properties that $E(dB) = 0$ and $E(dB)^2 = dt$. This model represents an extension of Noy-Meir (1975) in that livestock numbers are also treated as a state variable. In other respects it is a continuous-time generalization of Perrings (1994) model. Note, that such a system is called an indirect feedback control system[6]. To see this note that $y$in equation 1 is influenced indirectly by the control u acts on x only indirectly via the term dy.

2 Livestock as a Derivative Asset of Pasture and the use of Weight-Gain Functions

An alternative approach is to view livestock as a derivative asset, which derives its "value" from that of of an underlying asset - pasture. This idea draws on the financial literature on

[4]The term ergodic is sometimes used instead of stationary.

[5]Renshaw (1991), pp. 4-5.

[6]Lefschetz (1965): p. 18.

derivatives which make use of Ito's lemma in valuing derivative assets. Instead of deriving a monetary value we derive a physical measure of weight gain. it is assumed that this functional relationship is known and a stochastic control approach is then applied in order to determine an optimal management "policy" for the derivative asset livestock.

In empirical work done by Van Heerden and Tainton (1989) a negative linear relationship between individual weight-gain and stocking rate was found[7]. A linear relationship of the type discovered by Van Heerden and Tainton conforms to the following form, if one defines average animal weight w as a linear function of livestock "numbers" y, measured as some form of pasture intake then

$w = a - by$

x pasture

y livestock units

thus $y = \frac{a-w}{b}$.

Define the feed conversion function

$$w = logx$$

Note that if one scales pasture to lie between zero and one then one must rescale the feed conversion relationship to account for the fact that the logarithm becomes negative. In this case one may use

$$w = dlog(x)$$

where $d < 0$

Substituting gives

$$y = \frac{a - dlogx}{b}$$

and the equation for pasture dynamics as

$$dx = [nx(1 - x/K) - xy]\, dt + \sigma x dB$$

Note that $x = e^{\frac{w}{d}}$.

Then use Ito's lemma to obtain an equation for livestock numbers measured in animal units.

$$dy = -\frac{1}{b}\left[n(1 - \frac{x}{K}) - y + \frac{\sigma^2}{b}\right] dt + \frac{\sigma}{b}dz$$

The last equation needs to be modified to include a turnoff rate u.

$$dy = -\frac{1}{b}\left[n(1 - \frac{x}{K}) - y + \frac{\sigma^2}{b} - u\right] dt + \frac{\sigma}{b}dB$$

From the linear weight-gain function one may derive a production function for a fixed area of land (Y) (short-run production function) in the following manner[8]

[7]See also Wheeler and Freer (1986) pp. 176-177.

[8]Humphreys (1987): p. 125.

$$Y = ay - by^2$$

This results in the revenue-cost function for a labour intensive economy. In a modern pastoral sector with no intensive herding livestock may be treate as a flight resource, so that the cost function when plotted against stock numbers or stocking rates is downward sloping. This result appears at first sight to differ from that of the fisheries literature, but some reflection shows that the result is in fact the same as the fisheries case. The same cost structure exists in fisheries when one plots costs against fish stocks rather than against effort.

Thus the cost structure of the grazing enterprise is the same as that used in the literature on renewable resources. This is important because it implies that bionomic equilibria may still exist, whereas the linear weight-gain (production) functions used by Noy-Meir preclude the existence of bionomic equilibria.

Two parameters of the revenue function are particularly important. These are the maximum sustainable yield stocking rate y_{msy} and economic grazing capacity y_{cap}. Note that economic grazing capacity may differ from the ecological grazing capacity. Economic grazing capacity is defined as the least upper bound of the set of zero revenue stocking rates. To calculate y_{msy} differentiate Y and set $\frac{dY}{dy}$ to zero

$$\frac{dY}{dy} = a - 2by = 0$$

From this one obtains $y_{msy} = \frac{a}{2b}$. To obtain y_{cap} set $Y = 0$ and solve for y. This gives $y_{cap} = \frac{a}{b}$. Note that $y_{msy} = \frac{1}{2} y_{cap}$.

Profit is define using a Jones-sandland technology as:

$$\Pi = p(a - by)u - c(y)u$$

By appropriately defining the cost function $c(y$ livestock can be treated as a flight or captive resource, thus allowing on to treat both developed ranch style pastoral systems and traditional pastoralism.

In the later numerical work it is asumed that $c(y$ is linear in y.

Then one may define following Gleit (1978) an isoelastic utility function of profit as $U(\pi) = \frac{\pi^\gamma}{\gamma} = \frac{((pw-c)u)^\gamma}{\gamma}$ for the commerical pastoralism case or $U(\pi) = \frac{\pi^\gamma}{\gamma}(y - y_{min})^\beta$ for the semi-subsistence pastoralism case.

The pastoralists objective function is given by

$$\max_u E\left[\int_o^\infty U(\pi)E^{-rt}dt|F_0\right]$$

subject to

$$dx = [nx(1 - x/K) - cxy]\, dt + \sigma x dB$$

$$dy = -\frac{1}{b}\left[n(1-\frac{x}{K}) - y + \frac{\sigma^2}{b} - u\right]dt + \frac{\sigma}{b}dB$$

From this one obtains the Hamilton-Jacobi-Bellman equation

$$0 = J_t + \max_u \Bigg\{U(\pi)e^{-rt} + [nx(1-x/K) - xy]\,J_x + \left[-\frac{1}{b}\left[n(1-\frac{x}{K}) - y + \frac{1}{2}\frac{\sigma^2}{b} - u\right]\right]J_y +$$

$$\frac{1}{2}\sigma^2 x^2 J_{xx} + \frac{1}{2}\sigma^2 J_{yy}\Bigg\}$$

Maximisation gives

$$\hat{u} = (\frac{e^{rt}J_y}{(pw-c)^\gamma})^{\frac{\gamma}{\gamma-1}}$$

for the case of a utility maximising but profit oriented pastoralist.

Substituting this into the Hamilton-jacobi-bellman equation gives the PDE.

$$0 = J_t + U(\pi)e^{-rt} + [nx(1-x/K) - xy]\,J_x + \left[-\frac{1}{b}\left[n(1-\frac{x}{K}) - y + \frac{1}{2}\frac{\sigma^2}{b} - \hat{u}\right]\right]J_y +$$

$$\frac{1}{2}\sigma^2 x^2 J_{xx} + \frac{1}{2}\frac{\sigma^2}{b^2}J_{yy}$$

In order to solve this equation carryout a change of variables from y to x.

To change variables back to an equation solely dependent on x and t onemay make use of the following identities

$$\frac{\partial J}{\partial y} = \frac{\partial J}{\partial x}\frac{\partial x}{\partial y}$$

$$\frac{\partial^2 J}{\partial y^2} = \frac{\partial^2 J}{\partial x^2}\left(\frac{\partial x}{\partial y}\right)^2$$

Note that the $\frac{\partial x}{\partial y}$ term is obtained from the feed conversion and weight gain relations.

Substituting these in and using the livestock weight gain and feed conversion relations gives.

$$0 = J_t + U(\pi^*)e^{-rt} + \left[nx(1-x/K) - x\frac{a - logx}{b}\right]J_x +$$

$$\frac{1}{b}\left[n(1-\frac{x}{K}) - \frac{a - logx}{b} + \frac{1}{2}\frac{\sigma^2}{b} - \hat{u}\right]J_x bx +$$

$$\frac{1}{2}\sigma^2 x^2 J_{xx} + \frac{1}{2}\frac{\sigma^2}{b^2}J_{xx}b^2x^2$$

This equation then needs to be numerically solved in order to obtain a solution. This done in chapter 8. note that in utililising Markov chain approximation via LP it is not necessary to substitute $\hat{u}$ back into the Hamilton-Jacobi-bellman equation as the control values are set at fixed levels (see below).

3 Markov Chain Approximation

One of the probelms with using methods developed for the solution of partial differential equations to solve stochastic control problems is that convergence results for methods such as finite difference may no longer be valid in the stochastic setting.

An alternative to finite difference methods is to approximate the problem as one involving solution of a possibly controlled Markov chain on a finite state space(Kushner and Dupuis 1992, pp. 67-68) . Thus, one discretizes the model in a similar way to finite differences methods but then derives transition probabilities for a Markov chain.

Thus, the Markov chain is constructed based on transition probabilities from one point on a grid to another. With the grid representing the discretised state space.

There are two main approaches to the derivation of the approximating Markov chain: One may iterate in either the value dimesnion or in the policy (strategy) dimension.

Computation of derivatives follows the standard finite difference approach except that backward derivatives are used instead of forward derivatives if the drift is negative (upwind approximation).

In the following I will only present the general finite difference method for obtaining transition probabilities for an approximating Markov chain. This is not to be confused with the finite difference method for the numerical solution of a partial differential equation.

Given a stochastic control problem,

$$J(x, t_0) = \max_u E\left[\int_{t_0}^{T} U(x,u)e^{-r(T-t_0)}dt|F_{t_0}\right]$$

$$dx = b(x,u)dt + a(x,t)dB$$

Then the partial differential equation

$$0 = J_t + \left\{U(x,u)e^{-r(T-t_0)} + [b(x,u)]\, J_x + \frac{1}{2}ba(x,t)^2 J_{xx}\right\}$$

may be approximated by a Markov chain in the following manner.

$$W(x,u,n) = \sum_y U(x,u)\Delta t + e^{-r\Delta}p^h(x,y|u)W(y)$$

Where $p^h(x,y|u)$ represents the transition probability between state x and state y. Transition probabilities may be computed in the following manner

$$p^h(x,y|u) = \frac{a_{ii}(x)/2 + hb_i^{\pm}(x,u)}{Q^h(x,u)}$$

The time step is:

$$\Delta t^h(x,u) = \frac{h^2}{Q^h(x,u)}$$

The scaling term is given by

$$Q^h(x,u) = \sum_i \left[a_{ii}(x) + h|b_i(x,u)|\right]$$

Two problems may arise in implementing Markov chain approximation techniques. if the diffusion term is dependent on the control variable it may be necessary to eliminate the control variable from the scaling term by an appropriate transformation. In this case define

$$\bar{p}^h(x,x|u) = 1 - \sum_{y:y\neq x} \bar{p}^h(x,y|u)$$

and

$$\bar{Q}^h = \max_{u\in C} Q^h(x,u)$$

The terms with a bar are then used to replace those without a bar.

Secondly, a similar problem to stiffness in partial differential equations arises if there is a high degree of heterogeneity in the values taken on by the variance-covariance matrix.

The latter problem arises if the following condition is not fulfilled:

$$a_{ii}(x) - \sum_{j:j\neq i} |a_{ij}(x)| \geq 0$$

It should be noted that this will always hold univariate systems but in the multivariate setting problems may arise. Following Noy-meirs definition of grazing problems as those problems with two or more trophic levels. It is clear that numerical solution of such problems may be a problem. the difference in scale between pasture and stock numbers makes this almost inevitable.

Possible remedies include some form of rescaling and allowing transitions to states of the grid other than nearest neighbours.

This involves "splitting the operator", ie.e. the value function and decomposing the transition probability matrices into separate transition probability matrices for the diffusion and drift.

The transition probabilities are given by

$$p_a^h(x,y|u) = \frac{a_i i(x)}{2Q_a^h(x,u)}$$

and

$$\Delta t_a^h(x,u) = \frac{h^2}{Q_a^h(x,u)}$$

and

$$p_b^h(x, y|u) = \frac{b_i^{\pm}(x, u)}{Q_b^h(x, u)}$$

and

$$\Delta t_b^h(x, u0 = \frac{h}{Q_b^h(x, u)}$$

and the scaling terms are

$$Q_a^h(x, u) = \sum_i a_{ii}(x)$$

$$Q_b^h(x, u) = \sum_i |b_i(x, u)|$$

and combining terms:

$$p^h(x, y|u) = \left[\frac{a_{ii}(x)/2}{Q_a^h(x, u} \frac{h}{)Q_b^h(x, u)} + \frac{b_i^{\pm}(x, u)}{Q_b^h(x, u)} \frac{h^2}{Q_a^h(x, u)}\right] \times \text{ normalisation}$$

$$= \left[\frac{a_{ii}(x)}{2} + b_i^{\pm}(x, u)h\right] \times \text{ normalization}$$

where the normalization is $\frac{h}{Q_a^h(x,u)Q_b^h(x,u)}$
and

$$\Delta^h(x, u) = \frac{h^2}{Q_a^h(x, u) + hQ_b^h(x, u)}$$

This method can then be implemented in three possible ways:

i) Value iteration

ii) Policy iteration

iii) a mixture of value and policy iteration.

iv) Linear programming.

According to (1992) there is no apparent reason for preferring one approach over the other. It is simply a matter of expediency.

In the following subsection both the primal and dual linear programming approach to implementing the Markov chain approximation method will be reviewed.

3.1 Solution via Linear Programming

Markov chain transition methods can be implemented in the form of a linear program. This allows one to solve stochastic control problems in a relatively simple manner. Solution of stochastic differential games via this method is also posssible by means of linear multiobjective programming methods. The method has the disadvantage that the computational complexity of the method increases exponentially with number of dimensions.

The primal problem is given by

$$\max W_\rho = \sum_{i,k} U(i,u_k) M_{ik}$$

subject to

$$M_i = \rho_i + \sum_{j,k} r(j,i|u_k) M_{jk}$$

$$M_{ik} \geq 0$$

where

$M_i = \sum_k M_{ik}$ are the mean occupancy times in thei-th state

ρ_i the initial probabilities of the Markov chain for each state.

$r(j,i|u_k)$ are the transition probabilities between the j-th and i-th states.

$U(i,u_k)$ the utilities given each state of the system.

In the discounted case $r(j,i|u_k) = e^{-r\Delta t} p^h(x,y|u)$. In addition in the case of multiple state variables one neeeds to split the operator and consequently the transition probabilities.

The dual LP problem may be formulated in the following manner:

$$\min \sum_i \rho_i Y_i$$

subject to

$$Y_i \leq \sum_j r(i,j|u_k) Y_j + U(i,u_k) \text{ for all } i,k$$

The complementary slackness condfitions of the dual minimisation problem give

$$Y_i = \min_k \left[\sum_j r(i,j|u_k) Y_j + U(i,u_k) \right]$$

However the right hand side of this is just the discretised HJB equation, so that the Y_i are numerical approximations to $J(x,t)$.

Whilst this approach works well for problems with a small number of dimensions (one or two) for higher dimensional problems it suffers from the curse of dimensionality. That is the size of the grid being evaluated increases exponentitally with the number of dimensions. By utilizing weight gain functions and Ito's lemma it is possible to reduce the dimensionality of the original control problem so that it is amenable to numerical solution using Markov chain approximation.

4 Results

The model presented above was implemented using Markov chain approximation using MS-Excel 97. A small grid of approximately ten gridpoints was used with a step size of 0.1. Eight different turnoff rates were used giving a total of 80 occupation times needed to be evaluated.

the LP approach to Markov chain approximation solves the model by generating occupation times, i.e. the number of times states and control values occur simultaneously. In reading the model output the M_{ik} should be interepreted thin this way. they can be easily converted to probabilities by dividing by M_i (see above). Some results are shown in the appendix.

Preliminary results indicate a number of interesting features. Higher turnoff rates appear to occur in the presence of more abundant pasture. Although high levels of turnoff were observed for all pasture states. Low turnoff levels also at the highest level of pasture availability. Whilst it was expected that increased prices and reduced costs increased utility it was found quite unexpectedly that increased volatility in weather actually increased utility.

It would seem that uncertainty in fact provides opportunities for exploitation. Interesstingly price increases do not appear to effect stocking behaviour as such but do appear to affect welfare. Increased uncertainty does appear to have an impact on stocking behaviour and seems to lead to reduced frequency of turnoff at high pasture levels and increased turnoff at lower pasture levels. This appears to make sense and seems to indicate that it pays pastoralists to practice conservative destocking strategies in highly uncertain systems whereas at low to moderate levels of uncertainty good pasture is likelty to produce more and heavier stock that is turned off more frequently.

These are just some of the preliminary findings that the model seems to indicate a more thorough sensitivity analysis is however necessary to determine what other factors may be important. Some preliminary model output is to be found in the attached appendicesm with a listing of the parameter values used for each model run.

5 Conclusion

In this paper I have presented a stochastic optimal control model of livestock as a derivative asset of pasture. the approach utilised exploits livestock weight-gain and feed conversion relationships to employ Itos lemma to reduce the dimensionality of the problem to a level amenable to analysis using Markocv chain approximation methods. Markov chain approximation methods are easily implementable for problems involving only a single state variable. These emthods can be implemented as linear progams in which the original continuous-time stochastic control problem is approximated by a discrete Markov chain. The widespread availability of linear programming packages including in spreadsheets makes numerical stochastic control for low dimensional problems using Markov chain approximation an accessible tool for agricultural and resource economists.

A Model Output

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