# Optimal Fertilizer Carryover and von Liebigs Law of the Minimum*

Rodney Beard
r.beard@mailbox.uq.edu.au
Department of Economics
University of Queensland

February 2, 2000

**Abstract**

von Liebigs law of the minimum is frequently proposed as a model of crop production which captures agronomic reality better than substitutional production functions. Whilst many studies of the law of the minimum and optimal fertilizer carryover have been undertaken separately, the of the minimum for optimal fertilizer carryover has it would appear until now not been analysed, because of the non-differentiable nature of the agricultural producers objective function. In this paper I develop an optimal control model of fertilizer carryover subject to the law of the minimum using the differential inclusion approach to optimal control.

## 1 Introduction

The problem of optimal fertilizer carryover in agriculture has had a long tradition in agricultural economics, beginning with Heady and Dillon (1961). Optimal control methods were first applied to the fertilizer carryover problem by Lanzer and Paris (1981). Godden and Helyar (1980) have suggested that the optimal control approach lacks intuition. Kennedy (1986) therefore proposed dynamic programming as a possible alternative approach. For standard production functions whether one chooses to use optimal control or dynamic programming is probably simply a matter of taste although, it may be that the choice has implications for predictions regarding soil nutrient depletion rates, as evidenced by Reinganum and Stokey (1985) concern over tyhe choice between closed versus open loop controls in the study of nonrenewable resource problems.

*Paper presented at the 44th annual conference of the Australian Agricultural and Resource Economics Society, Sydney January 22-25, 2000

In an alternative literature on the response of crops to fertilizer application it has been suggested that crop responses are best characterised by limitational production functions rather substituional production functions. von Liebigs law off the minimum suggests that crop yields are limited by those nutrients that are most limiting. Paris (1992) has found empirical evidence to support this notion using a nonlinear response of Mitscherlich form.

It seems desirable that at some stage an attempt should be made to combine these two strands of the literature on crop responses to fertilizer and to attempt to ascertain what the implications of von Liebigs law of the minimum are for the fertilizer carryover problem.

The combination of the law of the minimum with fertilizer carryover raises a numbert of technical issues that largely make the discussion of dynamic programming versus optimal control redundant. In this paper I will argue that the law of the minimum leads to the non-differentiability of the farmers objective function and that consequently methods drawn from non-smooth analysis need to be brought to bear on the fertiliser carryover problem. In particular the differential inclusion approach to optimal control provides a fruitful approach to the analysis of the problem of optimal fertilizer carryover in the presence of the law of the minimum.

## 2 Differential Inclusions and Optimal Control

Differential inclusions are generalisations of differential equations to the case where the function mapping the value of a state variable, e.g. soil nutrients, to the time derivative of the state variable is set-valued.

Thus instead of writing:

$$\dot{x} = f(x(t), t)$$

one writes

$$\dot{x} \in F(x(t), t)$$

Note that control problems of the form

$$\dot{x} = f(x(t), t, a(t))$$

where $a(t) \in U(t)$ is a control variable in some set $U(t)$ can be rewritten as differential inclusions:

$$\dot{x} \in F(x(t), t, a(t))$$

An elementary discussion of the application of differential inclusions to optimal control is provided by Clarke (1983), a more technical treatment is provided by Kisielewicz (1991). In addition, Aubin (1991) has extended the theory of differential inclusions with a view to developing a general theory of the "evolution", i.e. dynamics of socioeconomic and biological systems.

Differential inclusions are particularly useful when the assumptions underlying Pontryagins maximum principle do not hold. although they appear to be a useful addition to the control theorists toolkit, they have rarely been used in applied work.

# 3 The Model

The farmers objective function is given by the discounted expected profit under the assumption of von liebigs law of the minimum with nonlinear mitscherlich responses of crops to soil nutrients.

$$\max_{a_i} \int_t^{\infty} e^{-rt} p \min\left\{m(1-k_1 e^{-\beta_1 r_1}), \ldots, m(1-k_n e^{-\beta_n r_n})\right\} - \sum_i^n c_i a_i$$

subject to the dynamics of soil nutrients (carryover) being given by

$$\dot{x}_i = \nu r_i$$

where $r_i = x_i + a_i$, i.e. the sum of soil nutrients plus applied fertilizer in addition fertilizer application is non-negative:

$$a_i \geq 0$$

( 1986) utilised this carryover function in his discussion of alternative rules for fertiulizer application. Note that we could have used a more realistic but complicated specification but have opted to use a simple form for expositionalpurposes. Need a nonlinear carryover function.

As it stands the farm objective function is non differentiable. This is a consequence of the limitational nature of von Liebigs law of the minimum.

A number of approaches could be used to address this issue. One might for example use generalised directional derivatives which do not require the usual smoothness properties of ordinary derivatives.

Alternatively, one can view the objective function as a set-valued map (correspondence) and formulate the optimization problem as an optimal control problem using differential inclusions.

In order to do this one proceeds as follows:

rearrange the state equation and substitute in to eliminate fertilizer application $a_i$ from the objective function.

the function $F(x, \dot{x}, t)$ is now a set-valued map

Now we wish to determine an optimal trajectory of $x$ such that $F(.)$ is maximised
Hence we define the multifunction

$$F(x, \dot{x}, t) := \{\nu(x_i + a_i), \lambda_i a_i, \mu_i a_i : a_i \in U(t) \forall i$$

$$, \sum_i^n c_i a_i - \nu \sum_i \lambda_i (x_i + a_i) - \sum_i \mu_i a_i - e^{-rt} p \min\left\{m(1-k_1 e^{-\beta_1 r_1}), \ldots, m(1-k_n e^{-\beta_n r_n}\right\} \leq \theta \leq M\right\}$$

where $U(t)$ is the control set, $M$ is a the maximal element that the objective function can take on.

Next we determine the Hamiltonian inclusion:

$$h(t, x_i, a_i, \lambda_i, \mu_i) := e^{-rt} p \min\left\{m(1 - k_1 e^{-\beta_1 r_1}), \ldots, m(1 - k_n e^{-\beta_n r_n})\right\}$$

$$- \sum_i^n c_i a_i + \nu \sum_i \lambda_i (x_i + a_i) + \sum_i \mu_i a_i$$

Note that the Hamiltonian inclusion implies the existence of a differential inclusion

$$(\dot{\lambda}_i, \dot{x}_i) \in \partial h$$

the solution of which gives the solution of the optimal control problem. Furthermore, $\mu$ is not time dependent as it represents the lagrange parameter associated with a non-negativity constraint.

Now consider the case where we have only two types of fertilizer, e.g. nitrogen and phosporus, so that the Hamiltonioan inclusion becomes:

$$e^{-rt} p \min\left\{m(1 - k_N e^{-\beta_N (x_N + a_N)}), m(1 - k_P e^{-\beta_P (x_P + a_P)})\right\}$$

$$-c_N a_N - c_P a_P + \nu(\lambda_N (x_N + a_N) + \lambda_P (x_P + a_P)) + \mu_N a_N + \mu_P a_P$$

Then we can distinguish at four maxima:

$$\text{Case I} \;\; \phi = a_N^* = \frac{log\left(\frac{(\nu\lambda_N + \mu_N) e^{rt} - c_N}{p m k_N \beta_N}\right)}{\beta_N} - x_N, a_P = P_{\max}$$

$$\text{Case II} \;\; \phi = a_N^* = \frac{log\left(\frac{(\nu\lambda_N + \mu_N) e^{rt} - c_N}{p m k_N \beta_N}\right)}{\beta_N} - x_N, a_P = 0$$

$$\text{Case III} \;\; a_N = N_{\max}, \psi = a_P^* = \frac{log\left(\frac{(\nu\lambda_P + \mu_P) e^{rt} - c_P}{p m k_P \beta_P}\right)}{\beta_P} - x_P$$

$$\text{Case IV} \;\; a_N = 0, \psi = a_P^* = \frac{log\left(\frac{(\nu\lambda_P + \mu_P) e^{rt} - c_P}{p m k_P \beta_P}\right)}{\beta_P} - x_P$$

Case I and Case II are nitrogen limited and Case III and Case IV are phosphorus limited. The folowing conditions characterise these two situations:

$$log(k_N) - log(k_P) > \beta_N (x_n + a_N) - \beta_P (x_P + a_P)$$

implies plant growth is limited by the availability of phosphorus and

$$log(k_N) - log(k_P) < \beta_N(x_n + a_N) - \beta_P(x_P + a_P)$$

implies growth is limited by the avialability of nitrogen.

Substituting each case into the the Hamiltonian inclusion gives the following values of the Hamiltonian:

$$h = \max\left\{e^{-rt}pm(1 - k_N e^{-\beta_N(x_N+\phi)})\right.$$

$$-c_N\phi - c_P P_{max}) + \nu(\lambda_N(x_N + \phi) + \lambda_P(x_P + P_{max})) + \mu_N\phi + \mu_P P_{max}$$

$$, e^{-rt}(pm(1 - k_N e^{-\beta_N(x_N+\phi)})$$

$$-c_N\phi) + \nu(\lambda_N(x_N + \phi) + \lambda_P x_P) + \mu_N\phi$$

$$, e^{-rt}(pm(1 - k_P e^{-\beta_P(x_P+\psi)})$$

$$-c_N N_{max} - c_P\psi) + \nu(\lambda_N(x_N + N_{max}) + \lambda_P(x_P + \psi)) + \mu_N N_{max} + \mu_P\psi,$$

$$e^{-rt}(pm(1 - k_P e^{-\beta_P(x_P+\psi)})$$

$$-c_P\psi) + \nu(\lambda_N x_N + \lambda_P(x_P + \psi)) + \mu_P\psi\}$$

The derivatives of these $\partial h_P$ and $\partial h_N$ in $x_N, x_P, \lambda_n, \lambda_P$ define multifunctions which contain the differential equations for $-\dot{\lambda}_N, -\dot{\lambda}_P, \dot{x}_N, \dot{x}_P$. these must be determined for each of the four possible values of $h$ at the edge of the control set. Thus we have eight cases obtained by dividing the state space into a number of regions within which a system of differentiable equations is defined. The solution of each of these systems of differential equations solves the differential inclusion proiblem and hence the control problem within each region. The differential equations for the costate variable must be solved backwards in time as only the terminal condition of the costate variable is known, e.g. $\lambda(T) = 0$. The state equation on the otherhand must be solved forwards in time as only the initial condition is known. This can be done simultaneously in a spreadsheet so the method is not overly complicated.

First we split the state space into regions corresponding to each solution for the control variable, e.g.

$$h = e^{-rt}(pm(1 - k_N e^{-\beta_N(x_N+\phi)})$$

$$-c_N\phi - c_P P_{max}) + \nu(\lambda_N(x_N + \phi) + \lambda_P(x_P + P_{max})) + \mu_N\phi + \mu_P P_{max}$$

hence the differential inclusion is given by

$$(-\dot{\lambda}_N, -\dot{\lambda}_P, \dot{x}_N, \dot{x}_P) \in (e^{-rt}c_N + \nu\lambda_N - \mu_N, \nu\lambda_P$$

$$, e^{-rt}(pmk_N\beta_N\phi_{\lambda_N}e^{-\beta_N(x_N+\phi)} - c_N\phi_{\lambda_N}) + \nu\lambda_N\phi_{\lambda_N} + \mu_N\phi_{\lambda_N}$$

$$, \nu x_P)$$

$$h = e^{-rt}(pm(1 - k_N e^{-\beta_N(x_N+\phi)})$$

$$-c_N\phi) + \nu(\lambda_N(x_N + \phi) + \lambda_P x_P) + \mu_N\phi$$

hence

$$(-\dot\lambda_N, -\dot\lambda_P, \dot x_N, \dot x_P) \in (e^{-rt}c_N - \mu_N, \nu\lambda_P$$
$$, e^{-rt}(pmk_N\beta_N\phi_{\lambda_N}e^{-\beta_N(x_N+\phi)} - c_N\phi_{\lambda_N}+$$
$$\nu(x_N + \phi + \lambda_N\phi_{\lambda_N}) + \mu_N\phi_{\lambda_N}, \nu x_P)$$

$$h = e^{-rt}(pm(1 - k_P e^{-\beta_P(x_P+\psi)})$$

$$-c_N N_{max} - c_P\psi) + \nu(\lambda_N(x_N + N_{max}) + \lambda_P(x_P + \psi)) + \mu_N N_{max} + \mu_P\psi$$

hence

$$(-\dot\lambda_N, -\dot\lambda_P, \dot x_N, \dot x_P) \in (\nu\lambda_N, e^{-rt}c_P - \mu_P, \nu(x_n + N_{max},$$
$$e^{-rt}pmk_P\beta_P\psi_{\lambda_P}e^{-\beta_P(x_P+\psi)} - c_P\psi_{\lambda_P})$$
$$+\nu(x_P + \lambda_P\psi_{\lambda_P} + \psi) + \mu_P\psi_{\lambda_P})$$

$$h = e^{-rt}pm(1 - k_P e^{-\beta_P(x_P+\psi)})$$

$$-c_P\psi) + \nu(\lambda_N x_N + \lambda_P(x_P + \psi)) + \mu_P\psi$$

hence

$$(-\dot\lambda_N, -\dot\lambda_P, \dot x_N, \dot x_P) \in (\nu\lambda_N, e^{-rt}c_P - \mu_P, \nu x_N,$$
$$e^{-rt}(pmk_P\beta_P\psi_{\lambda_P}e^{-\beta_P(x_P+\psi)} - c_P\psi_{\lambda_P})$$
$$+\nu(\lambda_P\psi_{\lambda_P} + x_P + \psi) - \mu_P\psi_{\lambda_P})$$

From this one obtains the following system of differential equations for the case in wich phosphorus is limiting:

First solution:

$$\dot\lambda_N = -e^{-rt}c_N - \nu\lambda_N + \mu_N$$
$$\dot\lambda_P = -\nu\lambda_P$$
$$\dot x_N = e^{-rt}(pmk_N\beta_N\phi_{\lambda_N}e^{-\beta_N(x_N+\phi)} - c_N\phi_{\lambda_N}) + \nu\lambda_N\phi_{\lambda_N} + \mu_N\phi_{\lambda_N}$$
$$\dot x_P = \nu x_P$$

Second solution:

$$\dot{\lambda}_N = -e^{-rt}c_N + \mu_N$$
$$\dot{\lambda}_P = -\nu\lambda_P$$
$$\dot{x}_N = e^{-rt}(pmk_N\beta_N\phi_{\lambda_N}e^{-\beta_N(x_N+\phi)} - c_N\phi_{\lambda_N})+$$
$$\nu(x_N + \phi + \lambda_N\phi_{\lambda_N}) + \mu_N\phi_{\lambda_N}$$
$$\dot{x}_P = \nu x_P$$

For the case in which nitrogen is limiting the following system is obtained:

$$\dot{\lambda}_N = -\nu\lambda_N$$
$$\dot{\lambda}_P = -e^{-rt}c_P + \mu_P$$
$$\dot{x}_N = \nu(x_n + N_{max})$$
$$\dot{x}_P = e^{-rt}pmk_P\beta_P\psi_{\lambda_P}e^{-\beta_P(x_P+\psi)} - c_P\psi_{\lambda_P})$$
$$+\nu(x_P + \lambda_P\psi_{\lambda_P} + \psi) + \mu_P\psi_{\lambda_P}$$

$$\dot{\lambda}_N = -\nu\lambda_N$$
$$\dot{\lambda}_P = -e^{-rt}c_P + \mu_P$$
$$\dot{x}_N = \nu x_N$$
$$\dot{x}_P = e^{-rt}(pmk_P\beta_P\psi_{\lambda_P}e^{-\beta_P(x_P+\psi)} - c_P\psi_{\lambda_P})$$
$$+\nu(\lambda_P\psi_{\lambda_P} + x_P + \psi) - \mu_P\psi_{\lambda_P}$$

Solving these equations along with the appropriate control values for cases I-IV gives the solution of the control problem.

# 4 Interpretation of Solution

This system was solved numerically using a first-order Euler scheme. Studying the solution curves of each of these equations gives some indication of when the system switches from one regime to another.

The results are ofcourse specific to the chosen parameter values. The values chosen were purely for illustrative purposes and not actual estimates. The results are quite sensitive to price variatons which appear to have a large impact on soil nitrogen and phosphorus levels.

The graph show declining soil nutrient values it should be noted that in the early stages nitrogen is limiting and later phosphorus and that the optimal allocation rule will accordingly change. Which equilibrium occurs for the factor that is not limited depends on a switching function similar to that used in bang-bang control. I have not evaluated this function in this paper but it involves a simple extension to the analysis.

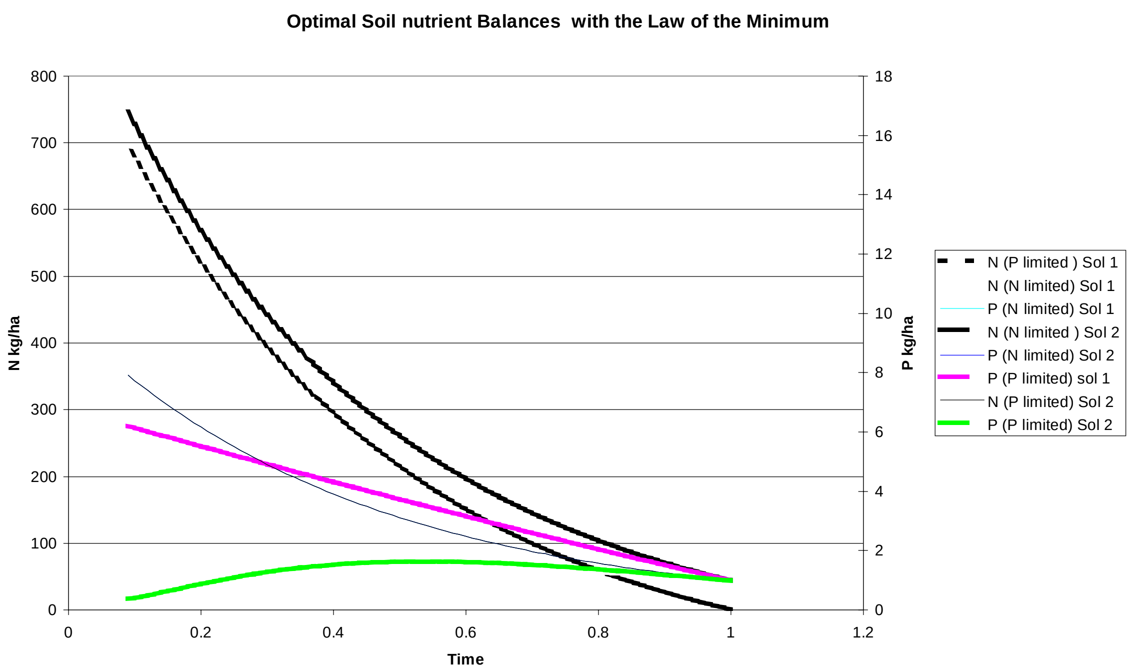


Figure 1: Optimal Soil Nutrient Balance

## 5 Possible Extensions IViability vs. Sustainability

How can soil resources be managed in a lasting way is question that is playing an increasingly important role in agricultural practice. One way of examining this question is to view the problem in terms of sustainability, e.g. maintenance of the stock of natural capital.

An alternative to the notion of sustainability is the concept of viability. Instead of managing resources sustainably it might be preferable to manage soil resources viably. "Viability theory is a mathematical theory that offers metaphors of evolution of macrosystems arising in biology, economics, cognitive sciences, games, and similar areas, as well as in nonlinear systems of control theory"( 1991, p. vii). Viability can be viewed as an offshoot of control theory based on the theory of differential inclusions. s in addition to the question as to what the optimal level of fertilizer application and carryover should be. The fertilizer carryover can be posed differently from the perspective of viability theory. The basic idea behind viability theory is that not all trajectories of the system are possible but subject to constraints - hence only some trajectories are viable. Secondly, controls, e.g. fertilizer application should be kept constant over-time unless the viability of the system is at stake. this is known as the *inertia principle.*

Viability theory should be contrasted with the differential inclusion approach to optimal control in that the approach is not based on optimization. Two main solution concepts are used in viability theory - smooth viable solutions and heavy viable solutions.

A system is said to be viable if there is asubset of $F$ such that the trajectory always remains within this subset.

Analysis of sustainability and farm viability could be carried out by extendsing the differential inclusion approach used in this paper to a setting involving viability theory.

## 6 Possible Extensions II: Variable Rate Application Technology and Precision Farming

Variable rate application technology, sometimes known as precision farming invlolves application of fertilizer at different rates at different locations depending on variations in soil chemistry at these different locations. Thus this approach to fertilizer application requires taking spatial heterogeneities in soil chemistry into account. How can the fertilizer carryover problem be addressed under these circumstances?

A spatial model could be introduced by representing fertilizer carryover in terms of a partial differential equation in time and space. The optimal fertilizer carryover problem would then involve optimal control of a partial differential equation. The differential inclusion approach used here would then need to be modified to incorporate partial differential inclusions. This would be the case regardless of whether an optimization or a viability approach is taken.

Whilst the analysis would be more complicated, the fact that the approach presented here can conceivably be extended to the spatial setting is testament to the strength and flexibility of the differential inclusion approach to analysing control problems. The implementation of this approach will be left to a later paper.

# 7 Conclusion

In this paper I have attempted to draw together to strands of the literature on the impact of fertilizer application on crop yields and to develop unified framework for the analysis of the economics of agricultural production in a dynamic setting based on the differential inclusion approach to optimal control theory. The approach is relatively intuitive involving only a few steps and capable of being implemented simply in a spreadsheet using commonly employed numerical methods. The results obtained here are only preliminary and it would seem desirable to make a comparison of a number of different fertilizer carryover functions. Nevertheless the simplicity of the model presented facilitates the presentation of a "new technique" for the solution of problems that would otherwise be considered intractable.

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