A Generalised Concept of Dominance in Linear Programming Models

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The notion of dominance most familiar to agricultural economists is perhaps the decision theoretic concept entailed in comparing one risky prospect to others. But dominance concepts are also relevant in the linear programming context, for example in identifying redundant constraints. In this note, the standard concept of dominance in linear programming is generalized by defining dominance with respect to differing levels of information about the programming problem.

1. Introduction

The decision theoretic concept of dominance refers to the situation in which a course of action in a decision problem is known to be not preferred to some other course(s) of action. Dominance is necessarily defined only with respect to a particular level of information about the decision problem. At one extreme is the case of the fully defined decision problem, where all acts are either optimal or non-optimal, and the non-optimal actions are dominated. But the more interesting cases arise when the decision problem is less-than-fully specified.

Various concepts of stochastic dominance have been developed for use in comparing risky prospects when the utility function expressing preferences is imprecisely known (Hanoch and Levy 1969; Fishburn 1974; Bawa 1982). The prospects are usually defined by known probability distributions of the attribute of interest (e.g. income), but concetps of dominance also apply when the probability distributions themselves are incompletely specified (e.g. Fishburn 1964; Kmietowicz and Pearman 1982).

Dominance concepts are not limited to risk situations. For example, the courses of action in a decision problem may have vector outcomes with the preference structure over the elements (attributes) known only to the extent of positive monotonicity. Concepts of Pareto efficiency have been developed for these cases. Particular applications include the identification of efficient sets in multiple-objectives decision making (Cohon

and Marks 1975) and the mean-variance approach to risk analysis (Markowitz 1959).

Analogous dominance concepts can be defined in the context of activity choice in linear programming (LP). The purpose of this note is to define some of the many dominance conditions that arise as the level of information assumed about the programming problem is varied. In the following section, sufficient conditions for an activity to be dominated are given. These conditions apply when the particular activity (and others) are fully known. The section is longer than strictly necessary given the availability of the results elsewhere in the linear programming literature. But there is little discussion of activity dominance in the agricultural economics literature, and the section therefore provides a useful background for the remainder of the note. In Section 3, a generalization of dominance is introduced. The level of problem information is reduced from the standard level to that of linear partial information about the activity coefficients. Sufficient conditions for dominance are developed.

2. Activity Dominance — Fully Known Activities

As with the general decision theoretic concepts, strong and weak forms of activity dominance can be defined. *Strong dominance* of an activity is the situation in which, for a given level of information about the programming problem, an activity is known never to be in an optimal basis for the LP problem at a positive level. Any such activity can be discarded without affecting the objective value attainable. *Weak dominance* of an activity can be defined as the situation in which it is known that an optimal solution always exists without the particular activity being in the basis at a positive level, though

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there may be other equally good solutions containing the activity at a positive level. The activity again can be discarded. Dominated activities have been called *extraneous variables* (e.g. Zionts 1974, p. 103). Non-dominated activites are said to be *efficient*.

Consider the following problem:

$$\max_{(X)} \ C^F{}'X$$

(1)   subject to $A^F X \leq B$

$$X \geq 0$$

where B is an (m x 1) vector of restraints, C^F and X are (n x 1) vectors of objective function coefficients and activity levels respectively, and A^F is an (m x n) matrix of input-output coefficients. It is assumed throughout that problem (1) has a finite, feasible solution.

Suppose that r of the activities are known, their objective function coefficients forming a vector C and their unit requirements for the m resources forming a matrix A with columns A_i, i = 1,2,...r. [1] The directions of optimization and of the inequalities are also assumed fixed as in problem (1). Other details of the problem are unknown. Standard activity dominance relates to the following question: given this information, under what conditions would activity A_i never be needed in an optimal solution to the full problem? [2]

First compare two activities A_i and A_j. This involves comparing two vectors and the familiar Pareto efficiency criterion applies. Given the objective and inequality directions, the higher an objective coefficient *ceteris paribus* the better, and the lower a resource requirement *ceteris paribus* the better. If with appropriate scaling, A_j never requires more of any resource and has an objective contribution at least as large as A_i, then A_i is clearly inferior to, and dominated by, A_j. Irrespective of what other activities may exist, or what the resource levels B are, A_i can never be the more attractive use of resources.

But most instances of activity dominance involve more than simple pair-wise comparison. For A_i to be dominated, it is sufficient that there exist some other use of resources which is always as good as A_i; and

this other use may involve a combination of the other known activities.

Suppose a solution exists to the following problem of finding a vector W such that:

$$AW \leq A_i$$

(2) $$C'W \geq C_i$$

$$W_i = 0$$

$$W \geq 0$$

where W_i and C_i are the ith elements of the r-vectors W and C respectively. Then, whenever activity A_i is in a basis for problem (1), it can be replaced with the positive linear combination defined by W without causing infeasibility and without lowering the objective function. That is, A_i is weakly dominated.

A suitable W, if one exists, can be found by solving the following problem:

$$\max_{(W,W^i,W_0)} \ W_0$$

subject to

(3) $$AW - A_i \ W^i \leq 0$$

$$-C'W + C_i \ W^i + W_0 \leq 0$$

$$W^i \geq 1$$

$$W_i = 0$$

$$W, \ W^i, \ W_0 \geq 0$$

where W^i and W_i are scalars. Activity A_i is dominated if a feasible solution (unbounded) exists.

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1. The superscript F is used in problem (1) to refer to the "full" A and C matrices, that is when r = n.

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2. This question requires the weak dominance concept, *i.e.* the more general concept. Most of the discussion in the note refers to this case.

As an alternative approach to dominance, one can focus on the dual of problem (1):

$$\min_{(V)} B'V$$

$$(4) \quad \text{subject to} \quad A^{F}{}'V \geq C^{F}$$

$$V \geq 0$$

where V is an (m x 1) vector of resource shadow prices. When an activity is dominated, its corresponding dual constraint can never be active. Conversely, activity A_i is dominated if its dual constraint can never be active, that is, if its objective function coefficient is never sufficient to exceed the value of its resources used in other ways. Such never-limiting constraints in a programming problem are said to be *redundant*. Strong and weak forms of redundance can be defined (Thompson *et al.* 1966; Llewellyn 1964; Zionts 1974; Karwan *et al.* 1983).[3]

One dual-based LP model for detecting dominance is:

$$\underset{(Z, Z_0, H)}{\text{minimize}} \quad -H$$

subject to

$$a'Z - cZ_0 \geq 0$$

$$(5) \qquad A_i'Z - C_iZ_0 + H \leq 0$$

$$Z_0 \geq 1$$

$$Z, \ Z_0, \ H \geq 0$$

where Z_0 and H are scalar variables, and a and c the same as A and C except that the ith column and element respectively have been deleted. If problem (5) has no feasible solution, A_i is necessarily dominated.

The dual formulation for detecting dominance is perhaps the more instructive. An activity is dominated because no set of shadow prices that is possible, given the information on the activities, would justify the activity's inclusion in an optimal basis for problem (1).

Any additional information on the possible shadow prices can only serve to push an activity towards dominance. There are many ways in which such information might arise. The analyst may simply learn more about the activites of problem (1). Alternatively, information may be synthesized by the analyst in asking particular questions. Will A_i be dominated when a particular resource is known to be especially scarce, that is when it has a shadow price greater than some specified value? Will A_i have a role to play when a resource is not limiting?[4]

The information need not always relate only to one shadow price as in these cases. For example, the decision maker may believe that two resources have equal, though unknown, shadow prices.

Extra information on the shadow prices can be expressed as a further set of t constraints, $M'V \geq D$, on the shadow prices in (4), where both M (m x t) and D (t x 1) are known. Any such additional dual constraint information corresponds to additional primal activities. For example, by setting a shadow price to a fixed value V^*, the analyst is allowing for primal resource acquisition and selling activities at the price V^*.

As an illustration of dominance, consider three primal activities represented by the vectors (1 1.8 0.5), (5 9.5 2.35) and (10 20 4), where the first element in each case is an objective function coefficient. Inspection reveals that no activity is dominated by any one other activity. However, the second activity is dominated by, for example, the combination of 2.5 times the first and 0.25 times the third. The vector for this combination is (5 9.5 2.25).

In practice one would use more efficient algorithms for detecting dominance than the pedagogic procedures outlined. Karwan *et al.* (1983) surveyed the major methods and have given some guidance to relative performance.

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3. In earlier linear programming literature (*e.g.* Hadley 1962), the concept of redundancy was derived exactly from redundancy in linear equations. A redundant constraint was one which was implied by other constraints, not one which was as or more liberal than the others.

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4. For this case, the shadow price could be set to zero. But more efficiently, all mention of that variable could be dropped, that is, by omitting the resource known to be slack from problem (1).

Commercial linear programming packages usually include options to detect and eliminate redundant constraints (and dominated activities).

3. Activity Dominance: Linear Partial Information

Problem (1) could be less than fully specified because of lack of information either about activities or about constraints. In the "full information" analysis of Section 2, although only r activities were recognized, for each of these there was full information on their coefficients in the objective function and in the m constraints. The partial information analysis of this section can be seen as a case of less than full information about the constraints or the objective function.

At the extreme of no information about the objective function or a constraint (other than its existence), each activity necessarily would have to be judged efficient since each might have the most desirable coefficient value. As some information is learned about the coefficients, situations of dominance begin to emerge.

Suppose activities A_i and A_j are represented by the vectors $(C_i\ 6\ 8)$ and $(C_j\ 5\ 5)$. With no information on C_i and C_j, neither activity A_i nor A_j is dominated. With full information on C_i and C_j, either A_i or A_j could be detected as dominated depending on C_i and C_j. But full information is not essential to detect dominance. For example, if the only information about the objective coefficients was that $C_i \leq C_j$, A_i could never be as desirable as A_j, and hence would be dominated.

This pair-wise comparison indicates the nature of linear partial information and how such information may be sufficient to establish dominance. More generally and formally, suppose that r activities, or the coefficients of A and C, are known only to the extent of satisfying the following inequalities:

$$(6) \qquad E^S\ A^S\ \leq\ Q^S$$

$$for\ \ s\ =\ 1,2,\ldots m,$$

$$E^C\ C\ \leq\ Q^C$$

where A^s is the sth row of A, E^s and E^C are known matrices, and Q^s and Q^C are known

vectors. It is assumed that this information is consistent, that is that it defines a set of feasible coefficient values.

For activity A_i to be dominated, there must be, for each possible set of A and C coefficients, a vector $w = (w_1\ w_2\ \ldots w_r)$ such that problem (3) is satisfied. It is *a fortiori* sufficient if there is one w such that problem (3) is satisfied for all possible coefficient values. Such a w exists if there is a feasible solution to the following problem:

$$(7) \qquad maximize\ w_0$$

$$(w,\ w^i,\ w_0)$$

subject to

$$\left\{ \begin{array}{l} max \qquad (A^S w - A_{si}\ w^i) \\ (A^S |w,\ w^i,\ w_0) \\ subject\ to\ E^S\ A^S \leq Q^S \end{array} \right\}$$

$$\leq\ 0,\quad s\ =\ 1,2\ \ldots\ m,$$

$$\left\{ \begin{array}{l} max \qquad (-C'w + C_i w^i + w_0) \\ (C|w,\ w^i,\ w_0) \\ subject\ to\ E^C\ C \leq Q^C \end{array} \right\}$$

$$\leq\ 0$$

$$w^i\ \geq\ 1$$

$$w_i\ =\ 0$$

$$w,\ w^i,\ w_0\ \geq\ 0$$

where A_{si} is the ith element in this s^{th} row of A. Problem (7) is simply a modified version of problem (3) in which the simple linear functions of W in problem (3) have been replaced by their "worst" possible values,

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5. If the coefficients in a row were known exactly, then the constraint would remain as the linear constraint in problem (3).

namely their maxima over possible coefficient values.[5] If problem (7) has a feasible solution with $w_0 > 0$, activity A_i is weakly dominated. If problem (7) has no feasible solution, then A_i may still be dominated, but this has not been established. That is, a feasible solution to problem (7) is a sufficient, but not a necessary, condition for dominance.

Suppose each maximization problem in the constraints of problem (7) is replaced by its dual. For the sth row of A this would be:

(8)
$$\min Q^{s\prime} v^s$$

$$(v^s)$$

$$\text{subject to } E^{s\prime} v^s = w - w^i t$$

$$v^s \geq 0$$

where t is an (r x 1) vector containing zeros except for the ith position which contains the unit value. Now for any feasible solution to the constraints of problem (8), $Q^{s\prime} v^s \geq \min Q^{s\prime} v^s$. Since the latter is equal to max ($A^s w - A_{si} w^i$), then the constraint for row s in problem (7) is a fortiori satisfied if the following constraint is satisfied:

(9)
$$\begin{cases} \text{Any } Q^{s\prime} v^s \\ \text{subject to} \\ \\ \quad E^{s\prime} v^s = w - w^i t \\ \quad v^s \geq 0 \end{cases} \leq 0 \; .$$

A similar constraint can be written for the objective function coefficients. Using all these constraints to replace those in problem (7), it follows that problem (7) has a feasible solution, and A_i is dominated, if the following problem has a feasible solution:

(10)

$$\max w_0$$

$$(w, \; w^i, \; w_0, \; v^s, \; v^C)$$

subject to

$$\begin{cases} \quad\quad Q^{s\prime} v^s \leq 0 \\ E^{s\prime} v^s - I(w - w^i t) = 0 \end{cases}$$

$$\text{for } s = 1, 2, \dots m,$$

$$Q^{C\prime} v^C + w_0 \leq 0$$

$$E^{C\prime} v^C + I(w - w^i t) = 0$$

$$w_i = 0$$

$$w^i \geq 1$$

$$w, \; w^i, \; w_0, \; v^s, \; v^C \geq 0$$

where I is the identity matrix of order r. Note that problem (3) is the special case of problem (10) when there is full information on the coefficients.

It should be noted that the sufficient condition for dominance is limited to cases where the linear partial information for C and for each row of A is treated as independent of that for other rows. Any dependence between the coefficients in different rows, which may mean that an activity is dominated, is not exploited in the above sufficient condition.[6]

As an illustration of dominance with linear partial information, consider the following three activities, $A_1 = (C_1 \; 1.8 \; 0.5)$, $A_2 = (C_2 \; 20 \; 4)$ and $A_3 = (C_3 \; 9.5 \; 2.35)$. With no information on C, each activity is potentially more desirable than the others. For example, if C = (1 10 6), each activity could be in an optimal basis and is efficient. Suppose,

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6. Stronger sufficient conditions, but which still do not account for dependence between rows, can be defined by subdividing the feasible space for the coefficients and searching for a series of w vectors, each of which serves for a particular sub-region of the feasible space.

however, that the objective function coefficients are known to be constrained as follows:

$$1 \leq c_1 \leq 1$$

$$(11) \qquad 9 \leq c_2 \leq 13$$

$$4 \leq c_3 \leq 6$$

$$c_3 - 0.4 c_2 \leq 0.$$

To determine the dominance of, say, A_3, the following problem must be solved:

(12)

$$\max w_0$$

subject to

$$v_1 - v_2 + 13 v_3 - 9 v_4 +$$

$$6 v_5 - 4 v_6 + w_0 \leq 0$$

$$v_1 - v_2 - w_3 = 0$$

$$v_3 - v_4 - 0.4 v_7 + w_2 = 0$$

$$v_5 - v_6 + v_7 + w_3 - w^3 = 0$$

$$1.8 w_1 + 20 w_2 +$$

$$9.5 w_3 - 9.5 w^3 \leq 0$$

$$0.5 w_1 + 4 w_2 +$$

$$2.35 w_3 - 2.35 w^3 \leq 0$$

$$w^3 \geq 1$$

$$w_3 = 0$$

$$v_i, \ w_i, \ w^3, \ w_0 \geq 0$$

$$\text{for all } i.$$

The existence of feasible solutions to this problem (for example, $w_1 = 0.833$, $w_2 = 0.4$, $w_3 = 0$, $w^3 = 1$) indicates that A_3 is

dominated. Once A_3 is eliminated, a pair-wise comparison of A_1 and A_2 indicates that neither is dominated.

4. Concluding Comments

This note has recalled the standard concept of "full information" activity dominance in linear programming, and developed a generalized concept of dominance for the case of linear partial information. It is not the algebra of dominance which is important. Neither is it the ability to detect dominance, for it has to be recognized that dominance and constraint redundancy are of little consequence in solving LP problems (Karwan et al. 1983). They serve mainly to reduce computational efficiency marginally, a matter of little concern with today's computing power. Further, it is the net saving after allowing for the time required to detect dominance that is relevant. Only if a model is to be solved repeatedly with some minor variations would there be value in detecting dominance.

The practical relevance of dominance concepts is in guiding LP activity specification, in particular in avoiding wasting time on specifying and budgeting activites that are doomed never to be in an optimal basis. For example, in specifying alternative activities for a crop, if the activity vectors would differ only in their gross margins (perhaps because of differing fertilizer rates or because of differing varietal yields), then only the activity corresponding to the fertilizer rate or variety that maximizes gross margin is efficient and necessary in the model.

But care is necessary when using dominance to discard alternatives from further analysis. For example, in any decision problem involving the possibility of collecting more precise information on uncertain parameters, dominated alternatives on existing information may become non-dominated in the new information scenario. Similarly, if decision analysis were to involve sensitivity analysis on parameters of the decision problem, a dominated alternative in the original problem may not be dominated in the modified problem. In the programming context, these possibilities arise when parametric or other post-optimality analysis is intended.

The real significance of dominance, however, lies in the concept itself and the

insight into other problems it offers. For example, the identification and measurement of the technical efficiency of production has been approached in an activity dominance framework (*e.g.* Farrell 1957). The question of technical efficiency and its link to allocative efficiency, and others, are pursued further in Drynan (1987).

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