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Measuring Hicksian Welfare Changes From Marshallian Demand Functions

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Abstract

A problem persists in measuring the welfare effects of simultaneous price and income changes because the Hicksian compensating variation (CV) and equivalent variation (EV), while unique, are based on unobservable (Hicksian) demand functions, and observable (Marshallian) demand functions do not necessarily yield a unique Marshallian consumer's surplus (CS). This paper proposes a solution by a Taylor series expansion of the expenditure function to approximate CV and EV by way of the Slutsky equation to transform Hicksian price effects into Marshallian price and income effects. The procedure is contrasted with McKenzie's "money metric" (MM) measure derived from a Taylor series expansion of the indirect utility function. MM requires a crucial assumption about the marginal utility of income to monetize changes in utility levels. No such assumption is required by the proposed procedure because the expenditure function is measured in money units. The expenditure approach can be used to approximate EV and CV while the MM is an approximation to EV. The EV and CV approximations are shown to be very accurate in numerical examples of two prices and income changing simultaneously, and are generally more accurate than MM.

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1. Introduction

Approximations to the Hicksian compensating variation (CV) and equivalent variation (EV) based on the Marshallian consumer's surplus (CS) (Willig, 1976; Shonkwiler, 1991) are fundamentally limited to single price change because CS is generally not unique when more than one price changes (Silberberg, 1972, 1978; Chipman & Moore, 1976, 1980; Just, Hueth & Schmitz, 1982).[1] For a single price change, however, these approximations may not be necessary as shown by Hausman (1981) since exact measures of CV and EV could be obtained in certain cases by recovery of a local indirect utility function from observed demand functions. Hausman's exact measure of CV and EV could in theory be extended to multiple price changes, but no one has done it to date.[2]

Thus, none of the above approximate or exact measures are practicable for measuring welfare changes in the more general case of multiple price changes. For the latter case, there remains a need for a "practical algorithm" sought by Chipman & Moore (1980) for a money measure of welfare change based on observable demand functions. This measure exists in theory as shown by Hurwicz & Uzawa (1971). One such algorithm is McKenzie's (1983) "money metric" (MM) measure of EV from a Taylor series approximation to the change in the indirect utility function. An alternative measure of EV and

[1]With more than one price changing at the same time, consumer's surplus is integrable or a unique measure of welfare change if and only if income elasticities are equal for the goods with the changing prices. This condition is equivalent to unitary income elasticities, i. e., homothetic preferences, for the case when **all** prices change simultaneously. Otherwise, with unequal income elasticities, consumer's surplus is not unique and will, therefore, result in misleading spproximations to welfare change given simultaneous price changes.

[2]His examples for a single price change when the demand function is linear or log-linear in own-price and income seem straightforward. However, his generalizations of these demand functions to include other prices yield exact welfare measures "when all other prices are constant" and "they cannot be used to analyze the welfare change when more than one price changes (except proportionately) without further analysis." (Hausman, 1980, p. 670). There is, however, no such further analysis in Hausman's paper.

of CV based on a Taylor series approximation to the change in the expenditure function is proposed in this paper. This alternative can also be computed from observable demand functions by means of the Slutsky equation, which links Hicksian price effects to Marshallian price and income effects. However, in comparison to MM, the alternative in this paper is less restrictive in theory because it does not involve assumptions about the marginal utility of income, hence does not impose restrictions on consumer preferences in addition to those in standard consumer theory.

This paper is organized as follows. Section 2 presents the welfare measures proposed in this paper based on a Taylor series approximation to the change in the expenditure function for the case of multiple price and income changes. It is shown to preserve the properties of the expenditure function, namely, linear homogeneity and concavity in prices. Two formulations of the proposed welfare measure are presented, one for EV and the other for CV. Section 3 presents McKenzie's money metric measure of welfare change from a Taylor series approximation to the change in the indirect utility function. This is reformulated to show that the third–order approximation has the properties of the indirect utility function, namely, non–decreasing in income, non–increasing in prices and homogeneous of degree zero in prices and income. Section 4 compares the two approximations relative to the true EV and true CV in an example of two goods where both prices and income change simultaneously. The proposed EV approximation is also compared to MM given the same changes in prices and income. Section 5 concludes this paper with a summary of findings.

2. Measuring Welfare Change From the Expenditure Function

The Hicksian measures of welfare change, namely, the equivalent variation (EV) and compensating variation (CV) are both based on the change in the value of the expenditure function when prices and/or income change. The difference between the two, however, is that EV uses the terminal level of utility and the original prices as bases for calculation while CV uses original utility and terminal prices. EV and CV are illustrated in Figure 1.

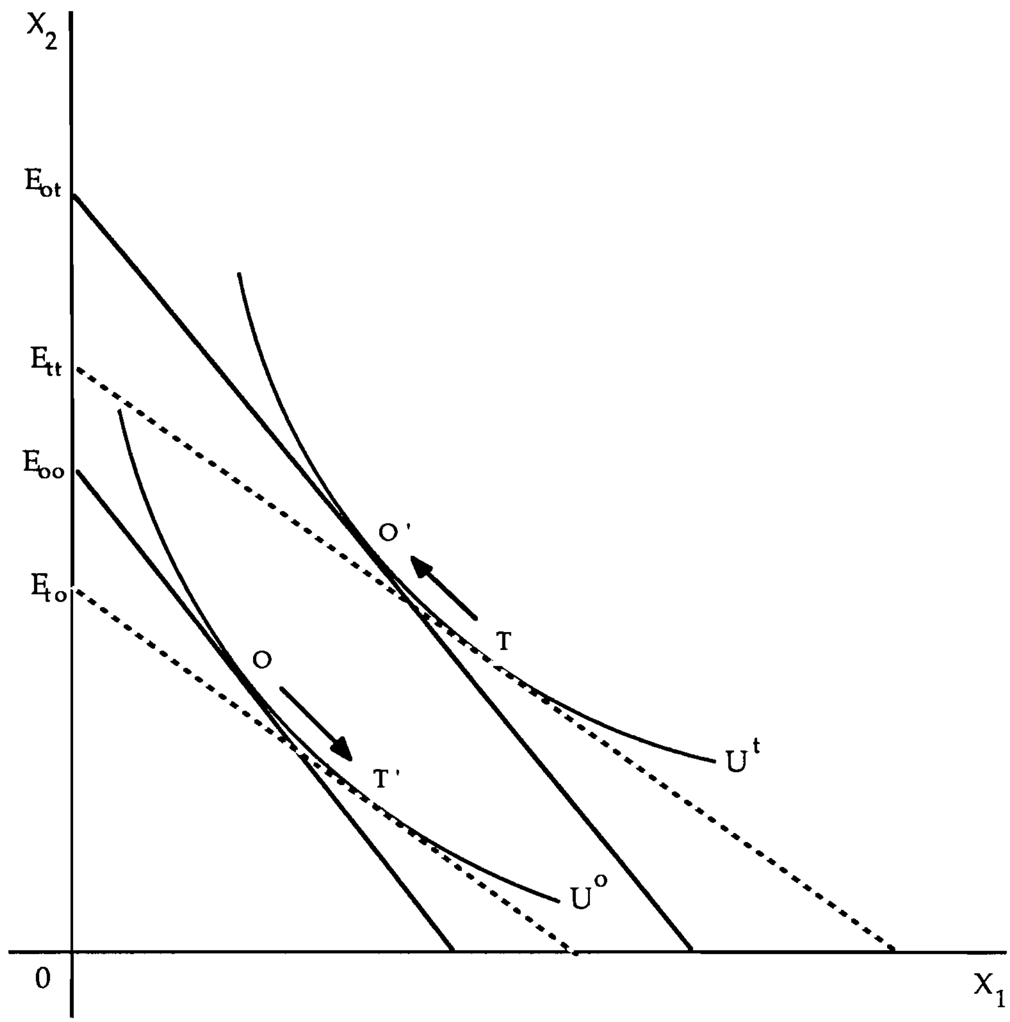

Figure 1. Hicksian Equivalent and Compensating Variations

The original situation is given by point O on the indifference curve U^o. Assume a change in prices and income such that the new or terminal situation is given by point T on the indifference curve U^t. For the move from O on U^o to T on U^t, EV requires pivoting from T to O′ on U^t such that at O′ prices are the same as the original prices at the starting point O. In this case, EV is equal to the change in the value of the expenditure function at the original prices that is required to maintain the terminal level of utility, which was achieved in the first place by a change in prices and income. Assuming in Figure 1 that X_2 is a numeraire good, EV = $E_{ot} - E_{oo}$, where E_{ot} is the minimum expenditure required to achieve the terminal utility level U^t at the original prices $P^o = \{P_1^o, P_2^o\}$ and, likewise, E_{oo} is the minimum expenditure to stay on the original level of utility U^o at these original prices. That is, EV is the change in expenditure for the parallel move between points O and O′. In contrast, CV is based on the original utility U^o and on the terminal prices. Thus, CV requires pivoting from point O to T′ on U^o and CV = $E_{tt} - E_{to}$, which is the change in expenditure for the parallel move between points T′ and T.

2.2 EV Approximation for Multiple Price and Income Changes

Let E(P, U) be the minimum expenditure at prices P to achieve utility U. This means that EV from above can be written as

$$EV = E_{ot} - E_{oo} = E(P^o, U^t) - E(P^o, U^o) \ . \qquad [1]$$

It is apparent from Figure 1 that the move from O to T involves not only a change in prices but also a change income, which is given by

$$\Delta I = E_{tt} - E_{oo} = E(P^t, U^t) - E(P^o, U^o) \ . \qquad [2]$$

Combining [1] and [2], EV can be decomposed (Boadway & Bruce, 1986) into price and income effects as

$$EV = E(P^o, U^t) - E(P^t, U^t) + \Delta I \ . \qquad [3]$$

In [3], EV is defined such that its sign is in the opposite direction to the change in prices but in the same direction as the change in income. That is, EV is positive (negative) when prices fall (rise) and positive (negative) when income rises (falls). This is adopted in this paper so that a positive (negative) EV implies a welfare improvement (deterioration).

The key idea to the proposed EV approximation is that point O′ on U^t is not an observable point. Only points O on U^o and T on U^t are observable for the simple reason that the consumer actually starts from O and ends up at T.[3]. The point O′ exists only as a theoretical construct by definition of EV. This realization has important implications in applied welfare analysis when the analyst has information only about the observable demand functions but not about the exact form of the underlying utility function. In this situation, EV for the move from the observable point O to the unobservable point O′ can only be approximated by going through the observable point T.

It follows from above that the minimum expenditure $E(P^o, U^t)$ at T′ is not known outright but the minimum expenditure $E(P^t, U^t)$ at T is in concept known. However, $E(P^o, U^t)$ can be obtained by a Taylor's series (Apostol, 1967; Chiang, 1984) expansion around $E(P^t, U^t)$, i. e.,

$$E(P^o, U^t) = E(P^t, U^t) + \sum_r \frac{1}{r!} d^r E(P, U^t) + R_r \qquad [4]$$

where r! is the factorial of r; $d^r E(P, U^t)$ is the rth–order total differential of the expenditure function; and R_r is the remainder term for a finite rth–order expansion. Because the expenditure function is continuous in prices, the change from $E(P^t, U^t)$ to $E(P^o, U^t)$ can be expressed as a line integral,

[3]Given a change in income, points O and T lie on different demand curves.

$$E(P^o, U^t) - E(P^t, U^t) = \int_{P^t}^{P^o} \sum_{i=1}^{n} \frac{\partial E(P, U^t)}{\partial P_i} \, dP_i \, . \qquad [5]$$

Combining [3], [4] and [5], it follows that EV can be expressed as

$$EV = \sum_r \frac{1}{r\,!} d^r E(P, U^t) + R_r + \Delta I = \int_{P^t}^{P^o} \sum_{i=1}^{n} \frac{\partial E(P, U^t)}{\partial P_i} \, dP_i + \Delta I \, . \qquad [6]$$

The value of the line integral is independent of the path of integration if and only if

$$\frac{\partial^2 E(P, U^t)}{\partial P_i \partial P_j} = \frac{\partial^2 E(P, U^t)}{\partial P_j \partial P_i} \Leftrightarrow \frac{\partial X_i^h}{\partial P_j} = \frac{\partial X_j^h}{\partial P_i} \qquad [7]$$

where the equivalence follows, first, by application of Shephard's lemma that the derivative of $E(P, U^t)$ with respect to price P_i is the Hicksian demand function $X_i^h = X_i^h\,(P, U^t)$ and, second, by Young's theorem. Because [7] is true, then EV is integrable or that its value is unique independently of the pattern of price changes. This implies that the unique value of EV can be obtained without regard to the order of the price change in the process of calculation, given the same set of prices that have changed at the same time. The uniqueness of EV remains true when it is calculated by the Taylor series expansion rather than by the line integral expression.

Ignoring the remainder term R_r, [6] yields a third–order (r = 3) Taylor series approximation, $EV_d \approx EV$,

$$EV_d = \sum_{i=1}^{n} \frac{\partial E(P, U^t)}{\partial P_i} \Delta P_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 E(P, U^t)}{\partial P_i \partial P_j} \Delta P_i \Delta P_j$$

$$+ \frac{1}{6} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^3 E(P, U^t)}{\partial P_i \partial P_j \partial P_k} \Delta P_i \Delta P_j \Delta P_k + \Delta I \, . \qquad [8]$$

All the partial derivatives in [8] should be evaluated at prices P^t since these serve as the base prices of the Taylor series expansion in [4]. In principle, EV_d is the approximate change in the value of the expenditure function as prices P^t change to P^o keeping utility constant at U^t, while the expenditure line pivots from tangency at T towards tangency point O′ in Figure 1. Therefore, the changes in prices in [8] should also be defined with prices P^t as the base values, i. e.,

$$\Delta P_i = P_i^o - P_i^t, \quad ; \quad i = 1, 2, \ldots, n. \tag{9}$$

It may now be shown that the approximation EV_d in [8] can be computed from observable demand functions. By Shephard's lemma,

$$\frac{\partial E(P, U^t)}{\partial P_i} = X_i^h = X_i^h (P, U^t) \tag{10}$$

which is the Hicksian demand function given the terminal utility level, U^t. Since all partial derivatives are evaluated at the observed point T on U^t at prices P_i^t, [10] takes the value $X_i^h (P, U^t) = X_i^h (P^t, U^t)$. The observed Marshallian quantities X_i^t demanded at the prices P^t can be substituted for the unobserved Hicksian quantities because point T corresponds to an intersection between Marshallian and Hicksian demand curves. That is, $X_i^h (P^t, U^t) = X_i^t (P^t, I^t)$ at point T where I^t is the level of income or expenditure required to attain utility level U^t given the prices P^t.

By the Slutsky equation (Varian, 1984),

$$\frac{\partial^2 E(P, U^t)}{\partial P_i \partial P_j} = \frac{\partial X_i^h (P, U^t)}{\partial P_j} = \frac{\partial X_i^t}{\partial P_j} + X_j^t \frac{\partial X_i^t}{\partial I} \tag{11}$$

where X_i^t and X_j^t are Marshallian quantities noted above. It follows that

$$\frac{\partial^3 E(P, U^t)}{\partial P_i \partial P_j \partial P_k} = \frac{\partial^2 X_i^h (P, U^t)}{\partial P_j \partial P_k} = \frac{\partial^2 X_i^t}{\partial P_j \partial P_k} + X_j^t \frac{\partial^2 X_i^t}{\partial I \partial P_k} + \frac{\partial X_i^t}{\partial I} \frac{\partial X_j^t}{\partial P_k} . \qquad [12]$$

A third–order approximation accomodates to some degree non–linear demand since second–order derivatives with respect to prices and income would be taken into account. Thus, a third–order is proposed in this paper as given by [8].

2.3 Calculating the EV Approximation From Observed Demand

The Hicksian substitution effects can be computed from observable Marshallian demand functions by means of the Slutsky equation. Thus, by substitution of [11] and [12] into [8] and recalling that the Marshallian quantity X_i^t at price P_i^t can be substituted for the Hicksian quantity $X_i^h(P^t, U^t)$, the third–order EV_d approximation of this paper can be expressed as

$$EV_d = \Delta I + \sum_{i=1}^{n} X_i^t \Delta P_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial X_i^t}{\partial P_j} + X_j^t \frac{\partial X_i^t}{\partial I} \right) \Delta P_i \Delta P_j$$

$$+ \frac{1}{6} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\frac{\partial^2 X_i^t}{\partial P_j \partial P_k} + X_j^t \frac{\partial^2 X_i^t}{\partial I \partial P_k} + \frac{\partial X_i^t}{\partial I} \frac{\partial X_j^t}{\partial P_k} \right) \Delta P_i \Delta P_j \Delta P_k . \qquad [13]$$

In [13], the levels of the Marshallian quantities demanded, X_i^t, $i = 1, 2, \ldots, n$, as well as the derivatives of the corresponding demand functions are evaluated at the terminal prices $P^t = \{P_i^t\}$. Morever, the changes in prices are computed using P_i^t as bases, i. e., $\Delta P_i = P_i^o - P_i^t$ as defined earlier in [9].

2.4 CV Approximation for Multiple Price and Income Changes

In Figure 1,

$$CV = E_{tt} - E_{to} = E(P^t, U^t) - E(P^t, U^o) \ . \qquad [14]$$

Substituting for $E(P^t, U^t)$ from [2], CV can be expressed in terms of price and income effects as

$$CV = -\,[(E(P^t, U^o) - E(P^o, U^o)] + \Delta I \ . \qquad [15]$$

The key idea to the proposed CV approximation is that point T′ on U^o is not an observable point and, therefore, the expenditure $E(P^t, U^o)$ at T′ is not known outright. However, $E(P^t, U^o)$ can be approximated by a Taylor series expansion of the known expenditure $E(P^o, U^o)$ at the original (observable) point O. That is,

$$E(P^t, U^o) = E(P^o, U^o) + \sum_r \frac{1}{r\,!}\, d^r E(P, U^o) + R_r \ . \qquad [16]$$

Ignoring the remainder term R_r, [15] and [16] yield a third–order (r = 3) Taylor series approximation, $CV_d \approx CV$,

$$CV_d = \Delta I - \sum_{i=1}^{n} \frac{\partial E(P, U^o)}{\partial P_i} \Delta P_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 E(P, U^o)}{\partial P_i \partial P_j} \Delta P_i \Delta P_j$$

$$- \frac{1}{6} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^3 E(P, U^o)}{\partial P_i \partial P_j \partial P_k} \Delta P_i \Delta P_j \Delta P_k \ . \qquad [17]$$

All the partial derivatives in [17] are evaluated at prices P^o since these serve as the base prices of the Taylor series expansion in [16]. CV_d is the approximate change in the value of the expenditure function as prices P^o change to P^t keeping utility constant at U^o, while the expenditure line pivots from tangency at O towards tangency point T′ in Figure 1. Hence, the changes in prices in [17] should be defined with prices P^o as the base values, i. e.,

$$\Delta P_i = P_i^t - P_i^o, \quad ; \quad i = 1, 2, \ldots, n. \tag{18}$$

Following earlier analysis,

$$\frac{\partial E(P, U^o)}{\partial P_i} = X_i^h = X_i^h(P, U^o) \tag{19}$$

where $X_i^h(P^o, U^o) = X_i^o(P^o, I^o)$ at point O where I^o is the level of income or expenditure required to attain utility level U^o given the prices P^o. Moreover,

$$\frac{\partial^2 E(P, U^o)}{\partial P_i \partial P_j} = \frac{\partial X_i^h(P, U^o)}{\partial P_j} = \frac{\partial X_i^o}{\partial P_j} + X_j^o \frac{\partial X_i^o}{\partial I} \tag{20}$$

where X_i^o and X_j^o are Marshallian quantities demanded at the original prices. It follows from [20] that

$$\frac{\partial^3 E(P, U^o)}{\partial P_i \partial P_j \partial P_k} = \frac{\partial^2 X_i^h(P, U^o)}{\partial P_j \partial P_k} = \frac{\partial^2 X_i^o}{\partial P_j \partial P_k} + X_j^o \frac{\partial^2 X_i^o}{\partial I \partial P_k} + \frac{\partial X_i^o}{\partial I} \frac{\partial X_j^o}{\partial P_k} \; . \tag{21}$$

2.5 Calculating the CV Approximation From Observed Demand

Substituting [19], [20] and [21] into [17] and substituting the Marshallian quantity X_i^o demanded at price P_i^o for the Hicksian quantity $X_i^h(P^o, U^o)$, the third–order CV_d approximation can be expressed as

$$CV_d = \Delta I - \sum_{i=1}^{n} X_i^o \Delta P_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial X_i^o}{\partial P_j} + X_j^o \frac{\partial X_i^o}{\partial I} \right) \Delta P_i \Delta P_j$$
$$- \frac{1}{6} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\frac{\partial^2 X_i^o}{\partial P_j \partial P_k} + X_j^o \frac{\partial^2 X_i^o}{\partial I \partial P_k} + \frac{\partial X_i^o}{\partial I} \frac{\partial X_j^o}{\partial P_k} \right) \Delta P_i \Delta P_j \Delta P_k \; . \tag{22}$$

In [22], the levels of X_i^o, $i = 1, 2, \ldots, n$, as well as the derivatives are evaluated at the original prices $P^o = \{P_i^o\}$ and the changes in prices are computed using P_i^o

as bases, i. e., $\Delta P_i = P_i^t - P_i^o$ as defined earlier in [18]. Moreover, $\Delta I = I^t - I^o$.

2.6 Properties of the EV and CV Approximations

The Taylor series expansions for EV in [4] and for CV in [16] have similar mathematical constructs, except for the differences in base prices and base utility levels. Hence, in general, they can be expressed as the Taylor series expansion of an expenditure function E(P′, U) about a given expenditure level E(P, U) when the original price vector P changes to the terminal price vector P′ holding utility fixed at the level U.

It may be shown that EV and CV are homogeneous of degree one in prices, which is a property of the expenditure function. This property pertains only to a change in prices. Thus, $\Delta I = 0$ is assumed. Hence, the third–order Taylor series expansion of E(P′, U) around E(P, U) can be expressed as

$$E(P', U) = E(P, U) + \sum_{i=1}^{n} X_i^h \Delta P_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial X_i^h}{\partial P_j} \Delta P_i \Delta P_j$$

$$+ \frac{1}{6} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 X_i^h}{\partial P_j \, \partial P_k} \Delta P_i \Delta P_j \Delta P_k + R_3 \ . \qquad [23]$$

Since the base prices are given by the vector $P = \{P_i\}$ and terminal prices by the vector $P' = \{P'_i\}$, then the changes in prices are defined by $\Delta P_i = P'_i - P_i$. Suppose that all prices change by the same proportion δ, i. e., $P'_i = \delta P_i$. It follows in this case that

$$\frac{\Delta P_i}{P_i} = \frac{P_i' - P_i}{P_i} = \delta - 1 \ . \qquad [24]$$

Substituting [24] into [23],

$$E(P', U) = E(P, U) + (\delta - 1) \sum_{i=1}^{n} X_i^h P_i + \frac{1}{2} (\delta - 1)^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial X_i^h}{\partial P_j} P_i P_j$$

$$+ \frac{1}{6} (\delta - 1)^3 \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 X_i^h}{\partial P_j \partial P_k} P_i P_j P_k + R_3 \ . \qquad [25]$$

To evaluate [25], consider that X_i^h is determined at price P_i given the utility level U. Therefore, by definition of an expenditure function,

$$\sum_{i=1}^{n} X_i^h P_i = E(P, U) \ . \qquad [26]$$

Moreover,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial X_i^h}{\partial P_j} P_i P_j = \sum_{i=1}^{n} P_i \left(\sum_{j=1}^{n} \frac{\partial X_i^h}{\partial P_j} P_j \right) = 0 \qquad [27]$$

since, by Euler's theorem,

$$\sum_{j=1}^{n} \frac{\partial X_i^h}{\partial P_j} P_j = 0 \quad ; \quad i = 1, 2, \dots, n \qquad [28]$$

because Hicksian demand functions are homogeneous of degree zero in prices. Differentiate [28] with respect to P_k, then post–multiply by P_k and, finally, sum over all k to obtain

$$\sum_{k=1}^{n} \left[\sum_{j=1}^{n} \left(\frac{\partial^2 X_i^h}{\partial P_j \partial P_k} P_j + \frac{\partial X_i^h}{\partial P_j} \frac{\partial P_j}{\partial P_k} \right) \right] P_k = 0 \ . \qquad [29]$$

Consider the fact that the derivative $\partial P_j / \partial P_k$ equals zero when $j \neq k$ but equals one when $j = k$. Therefore, [29] implies that

$$\sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 X_i^h}{\partial P_j \partial P_k} P_j P_k + \sum_{j=1}^{n} \frac{\partial X_i^h}{\partial P_j} P_j = 0 \qquad [30]$$

and by virtue of [28]

$$\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 X_i^h}{\partial P_j\,\partial P_k}P_j\,P_k = 0\ . \qquad [31]$$

The results in [28] and [31] imply that the sum expressions in the second and third terms of [25] are equal to zero. Moreover, since these results generalize to any rth–order Taylor series expansion, the remainder term R_r must reduce to zero when prices change in the same proportion.

It follows from [26] to [31] that [25] simplifies to

$$E(P', U) = \delta\, E(P, U) \qquad [32]$$

given that $P_i' = \delta\, P_i$ for all $i = 1, 2, \ldots, n$. That is, if all prices change δ times the original prices, the minimum expenditure required to attain the same level of utility at the new prices will be δ times the minimum expenditure required to attain the same level of utility at the original prices. This shows that EV and CV preserve the linear homogeneity in prices property of the expenditure function.

It may be shown that EV and CV embody also the expenditure function property of concavity in prices. For this purpose, consider the fact that the Hicks–Slutsky substitution matrix is the Hessian matrix S of the second–order price derivatives of the expenditure function,

$$S = \{s_{ij}\} \quad ; \quad s_{ij} \equiv \frac{\partial X_i^h}{\partial P_j} \quad ; \quad i, j = 1, 2, \ldots, n \qquad [33]$$

which is symmetric by Young's theorem. The expenditure function is concave in prices if and only if the matrix S is negative semi–definite or has non–positive eigenvalues. One eigenvalue is zero because [28] implies that S is singular. Therefore, if S is symmetric and negative semi–definite, then [23] embodies the concavity in prices property of the expenditure function.

3. Measuring Welfare Change From the Indirect Utility Function

The indirect utility function can be written as

$$U(P, I) = U[X_1(P,I), X_2(P,I), \ldots, X_n(P,I)] \qquad [34]$$

where $(P, I) = (P_1, P_2, \ldots, P_n; I)$ is the vector of prices and income and $X_i(P, I)$ is the Marshallian demand for the ith good. Referring back to Figure 1, the value of $U(P, I)$ at the original point O may be represented by $U(P^o, I^o)$ and by $U(P^t, I^t)$ at the terminal point T. Because the indirect utility function is continuous in prices and income, then the change in its value from point O to point T may be expressed as a Taylor series expansion,

$$\Delta U = U(P^t, I^t) - U(P^o, I^o) = \sum_q \frac{1}{q!} d^q U(P, I) + R_q \qquad [35]$$

or as a line integral,

$$\Delta U = \int_{P_i^o}^{P_i^t} \sum_{i=1}^{n} \frac{\partial U}{\partial P_1} dP_i + \int_{I^o}^{I^t} \frac{\partial U}{\partial I} dI . \qquad [36]$$

In [35], $d^q U(P, I)$ is the qth–order total differential of [34] and R_q is the remainder term. The first–order total differential can be shown to be,

$$dU = \sum_{i=1}^{n} \frac{\partial U}{\partial P_1} dP_1 + \frac{\partial U}{\partial I} dI = \sum_{i=1}^{n} (-\lambda X_i) dP_i + \lambda dI \qquad [37]$$

where λ is the marginal utility of income. For a given change in prices and income, [36] can therefore be expressed (Just, Hueth and Schmitz, 1982) as,

$$\Delta U = \int_{P_i^o}^{P_i^t} \sum_{i=1}^{n} (-\lambda X_i) dP_i + \int_{I^o}^{I^t} \lambda dI \qquad [38]$$

where (P_i^o, P_i^t) comprise the elements of the original and terminal price vectors and I^o and I^t are the original and terminal income levels.

Since the indirect utility function in [34] is non–increasing in prices and non–decreasing in income, the change in indirect utility in [38] is in the opposite direction to the change in prices but in the same direction to the change in income. Therefore, ΔU is positive (negative) when prices fall (rise) and/or when income rises (falls). Thus, welfare improves (deteriorates) when ΔU is positive (negative).

3.1 Money Metric Measure of Welfare Change

The "money metric" proposed by McKenzie (1983) assumes the same situation envisaged in this paper. That is, there is information only about observable demand functions and the exact form of the underlying utility function is unknown. Thus, the money metric proceeds from a finite order (truncated) Taylor series expansion of ΔU in [35] around the original value of the indirect utility function, $U(P^o, I^o)$, i. e., ΔU is evaluated at the original prices, $P^o = \{P_i^o\}$, $i = 1, 2. \ldots, n$, and income I^o. By Roy's identity, the derivatives of the indirect utility function in the Taylor series expansion can be expressed in terms of the parameters of observable demand functions.

The line integral in [38] implies that ΔU is integrable or a unique value if and only if,

$$\frac{\partial(-\lambda X_i)}{\partial P_j} = \frac{\partial(-\lambda X_j)}{\partial P_i} \quad ; \quad \frac{\partial(-\lambda X_i)}{\partial I} = \frac{\partial \lambda}{\partial P_i} \quad ; \quad \frac{\partial(-\lambda X_j)}{\partial I} = \frac{\partial \lambda}{\partial P_j} \ . \qquad [39]$$

Thus, the terms in the money metric Taylor series expansion of ΔU are expressed in accordance with [39], which implies that the marginal utility of income, λ, varies with income and/or prices.

The indirect utility function in [34] is measured in units of "utility".

This can be monetized to obtain "money metric" utility by the following monotonic transformation. From [37],

$$\frac{\partial U(P, I)}{\partial P_i} = -\lambda X_i \ ; \ \frac{\partial U(P, I)}{\partial I} = \lambda \qquad [40]$$

where λ is the marginal utility of income. Thus, the reciprocal $1/\lambda$ is the marginal cost of utility. To obtain money metric utility, choose the value of λ evaluated at the original prices and income, i. e., $\lambda^o = \lambda(P^o, I^o)$, as the scalar for the transformation. In this case, the money metric transformation of indirect utility may be denoted by M(P, I),

$$M(P, I) = \frac{U(P, I)}{\lambda^o} \ . \qquad [41]$$

M(P, I) represents exactly the same preferences as U(P, I). However, because M(P, I) is measured in money, it has the property that the marginal utility of income equals one at (P^o, I^o). Denoting the money metric marginal utility of income by λ_m,

$$\lambda_m (P, I) = \frac{\partial M(P, I)}{\partial I} = \frac{1}{\lambda^o} \frac{\partial U(P, I)}{\partial I} = \frac{\lambda(P, I)}{\lambda^o} \qquad [42]$$

$$\lambda_m (P^o, I^o) = \frac{\lambda(P^o, I^o)}{\lambda^o} = 1 \ .$$

McKenzie's money metric measure of welfare change from changes in prices and income is a Taylor series approximation to the change in M(P, I) evaluated at (P^o, I^o). That is, the change being measured is

$$\Delta M = \frac{1}{\lambda^o} \Delta U(P, I) = \frac{1}{\lambda^o} [U(P^t, I^t) - U(P^o, I^o)] \qquad [43]$$

where $\Delta U(P, I)$ is defined in [35] to [38]. In view of the money metric property in [42], McKenzie's approximation assumes that λ_m is constant with respect to income at that (P^o, I^o). That is,

$$\frac{\partial^s \lambda_m (P^o, I^o)}{\partial I^s} = 0 \quad , \quad s \geq 1 \qquad [44]$$

which means that the derivative of λ_m of all orders with respect to income equals zero. This implies from [42] that

$$\frac{\partial^s \lambda (P^o, I^o)}{\partial I^s} = 0 \quad , \quad s \geq 1 \ . \qquad [45]$$

The assumption in [45] is the key to factoring out λ from the right–hand side of the Taylor series expansion of $\Delta U(P, I)$ in order to yield the change in money metric utility defined by [43] when evaluated at original prices and income (P^o, I^o).

For a third–order expansion, dU is given by [37] from which

$$d^2 U = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 U}{\partial P_i \partial P_j} dP_i dP_j + 2 \sum_{i=1}^{n} \frac{\partial^2 U}{\partial P_i \partial I} dP_i dI + \frac{\partial^2 U}{\partial I^2} (dI)^2 \ ; \qquad [46]$$

$$d^3 U = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^3 U}{\partial P_i \partial P_j \partial P_k} dP_i dP_j dP_k \qquad [47]$$

$$+ 3 \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{\partial^3 U}{\partial P_i \partial P_k \partial I} dP_i dP_k dI$$

$$+ 3 \sum_{i=1}^{n} \frac{\partial^3 U}{\partial P_i \partial I^2} dP_i (dI)^2 + \frac{\partial^3 U}{\partial I^3} (dI)^3 \ .$$

All the partial derivative terms are obtained subject to the integrability conditions in [39] and are evaluated at (P^o, I^o).

3.2 Properties of the Third–Order Money Metric Approximation

The third–order money metric measure of welfare change will be denoted by MM_3 where, from [43], $MM_3 \approx \Delta M = (1/\lambda^o)\ \Delta U(P, I)$. Dumagan (1989, 1991) reformulated McKenzie's third–order money metric to facilitate showing that it has the properties of an indirect utility function. This reformulation is

$$\frac{MM_3}{I} = \frac{\Delta I}{I} - \sum_{i=1}^{n} w_i \frac{\Delta P_i}{P_i} - \sum_{i=1}^{n} w_i E_{iI} \frac{\Delta P_i}{P_i}\frac{\Delta I}{I} \qquad [48]$$

$$+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} (w_i w_j E_{jI} - w_i E_{ij}) \frac{\Delta P_i}{P_i}\frac{\Delta P_j}{P_j}$$

$$+ \frac{1}{6}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \left(w_i P_j P_k \frac{\partial^2 X_j}{\partial I\, \partial P_k} + w_i w_j E_{ik} E_{jI} \right) \frac{\Delta P_i}{P_i}\frac{\Delta P_j}{P_j}\frac{\Delta P_k}{P_k}$$

$$- \frac{1}{6}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \left(\frac{P_i P_j P_k}{I} \frac{\partial^2 X_i}{\partial P_j\, \partial P_k} + I\, w_i w_j P_k \frac{\partial^2 X_k}{\partial I^2} \right) \frac{\Delta P_i}{P_i}\frac{\Delta P_j}{P_j}\frac{\Delta P_k}{P_k}$$

$$- \frac{1}{6}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} (w_i w_j w_k E_{jI} E_{kI} - w_i w_k E_{ij} E_{kI}) \frac{\Delta P_i}{P_i}\frac{\Delta P_j}{P_j}\frac{\Delta P_k}{P_k}$$

$$+ \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} \left(I\, w_i P_k \frac{\partial^2 X_k}{\partial I^2} - P_i P_k \frac{\partial^2 X_i}{\partial P_k\, \partial I} \right) \frac{\Delta P_i}{P_i}\frac{\Delta P_k}{P_k}\frac{\Delta I}{I}$$

$$+ \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} (w_i w_k E_{iI} E_{kI}) \frac{\Delta P_i}{P_i}\frac{\Delta P_k}{P_k}\frac{\Delta I}{I} - \frac{1}{2}\sum_{i=1}^{n} I\, P_i \frac{\partial^2 X_i}{\partial I^2}\frac{\Delta P_i}{P_i}\left(\frac{\Delta I}{I}\right)^2$$

where

$$I = \sum_{i=1}^{n} P_i X_i \quad ; \quad w_i = \frac{P_i X_i}{I} \quad ; \quad E_{ij} = \frac{\partial X_i}{\partial P_j}\frac{P_j}{X_i} \quad ; \quad E_{iI} = \frac{\partial X_i}{\partial I}\frac{I}{X_i}\ . \qquad [49]$$

In view of [35] and [43], the changes in prices and income in the money metric measure are defined using the original levels as bases, i. e.,

$$\frac{\Delta P_i}{P_i} = \frac{P_i^t - P_i^o}{P_i^o} \quad ; \quad i = 1, 2, \ldots, n \quad ; \quad \Delta I = I^t - I^o \ . \qquad [50]$$

Moreover, all terms in [48] and [49] are evaluated at (P^o, I^o). Thus, MM_3 has the same sign as ΔU, which is positive (negative) when prices fall (rise) and/or when income rises (falls). This sign is in the same direction as the change in utility.

MM_3 has the properties of the indirect utility function $U(P, I)$. When all prices are constant, [48] yields

$$\frac{MM_3}{I} = \frac{\Delta I}{I} \quad \text{if} \quad \frac{\Delta P_i}{P_i} = 0 \quad ; \quad i = 1, 2, \ldots, n \qquad [51]$$

which means that the third–order money metric is non–decreasing in income. To show that it is non–increasing in prices and homogeneous of degree zero in prices and income, the following budget constraint identities are useful:

$$\sum_{j=1}^{n} P_j \frac{\partial X_j}{\partial P_i} = -X_i \ ; \qquad [52]$$

$$\sum_{i=1}^{n} P_i \frac{\partial X_i}{\partial I} = \sum_{i=1}^{n} w_i E_{iI} = 1 \ ;$$

$$\sum_{i=1}^{n} P_i \frac{\partial^2 X_i}{\partial I^2} = 0 \ ;$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} P_i P_j \frac{\partial X_j}{\partial P_i} = -\sum_{i=1}^{n} P_i X_i = -I \ ;$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{P_i P_j}{I}\frac{\partial X_j}{\partial P_i}=\sum_{i=1}^{n}\sum_{j=1}^{n} w_j\, E_{ji}=-1\ ;$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} P_i P_j \frac{\partial^2 X_j}{\partial P_i \partial I}=-1\ ;$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{P_i P_j P_k}{I}\frac{\partial^2 X_j}{\partial P_i \partial P_k}=2\ .$$

Consider the case where income remains the same but all prices change in the same proportion given by

$$\rho=\frac{\Delta P_i}{P_i}\quad ;\quad i=1,2,\ldots,n\ . \tag{53}$$

Substituting [53] into [48] and making use of the budget identities in [52], it can be shown that

$$\frac{MM_3}{I}=-\rho+\rho^2-\rho^3\ . \tag{54}$$

This result is consistent with the property of being non–increasing in prices since it is clear that MM_3 is positive (negative) if ρ is negative (positive).

Finally, if all prices and income change in the same proportion given by

$$\tau=\frac{\Delta I}{I}=\frac{\Delta P_i}{P_i}\quad ;\quad i=1,2,\ldots,n \tag{55}$$

it can be verified that by substitution of [55] into [48] and using the budget identities in [52],

$$\frac{MM_3}{I}=0 \quad \text{for all} \quad \tau\ . \tag{56}$$

This result implies that [48] is homogeneous of degree zero, i.e. welfare is unchanged by a proportional change in prices and income.

The properties of the money metric of being non–decreasing in income and homogeneous of degree zero in income and prices are not dependent on the order of the Taylor series approximation, at least up to a third–order as in the case of [48]. However, the property of being non–increasing in prices is not held by a second–order approximation.

The first–order approximation includes only the first two terms on the right–hand side of [48] It is straightforward to show that this is non–decreasing in income, non–increasing in prices and homogeneous of degree zero in income and prices. The second–order approximation consists of the first four terms of [48]. Once again, the expression is non–decreasing in income and is zero degree homogeneous in prices and income. However, if all prices change proportionally (with income constant) by a factor ρ, then the second–order money metric is

$$\frac{MM_2}{I} = -\rho + \rho^2 \ . \tag{[57]}$$

This is obviously a perverse result. It is sufficient to point out that if prices increase where ρ is greater than one, that MM_2 /I is positive. This means that it is possible for welfare to improve when prices rise while income remains the same, which is nonsense.

The conclusion is that the money metric from a finite Taylor series approximation is not per se a welfare indicator for it depends on the order of the approximation. For example, a second–order approximation is not a welfare indicator since it is not non–increasing in prices. However, the third–order money metric in [48] is a welfare indicator in that it is non–decreasing in income, non–increasing in prices and homogeneous of degree zero in income and prices.

4. Numerical Examples

Consider the following example adopted from Silberberg (1978), which he used to demonstrate the lack of uniqueness of the Marshallian consumer's surplus for multiple price changes.[4] The utility function is given by

$$U = \ln X_1 + X_2 \ . \qquad [58]$$

By maximizing [58] subject to the budget constraint

$$I = P_1 X_1 + P_2 X_2 \qquad [59]$$

the resulting Marshallian demand functions are

$$X_1 = \frac{P_2}{P_1} \quad \text{and} \qquad [60]$$

$$X_2 = \left(\frac{I}{P_2} - 1 \right) . \qquad [61]$$

From [58], [60] and [61], the indirect utility function is

$$U = \ln \left(\frac{P_2}{P_1} \right) + \frac{I}{P_2} - 1. \qquad [62]$$

This utility function will be used in the following numerical examples to demonstrate the accuracy of the EV and CV approximations proposed in this paper as well as to compare this EV approximation to McKenzie's money metric EV approximation.

[4]This example may be found in pages. 354–5. Note our differences in notation, which is potentially confusing. For example, Silberberg uses "log" to mean a natural logarithm as he explains in page 45. However, we use the usual notation "ln" in place of his "log" to mean natural logarithm. Also, we use "I" in place of his "M" to denote income. Finally, we use "A" in place of his "W" to denote the Marshallian consumer's surplus.

4.1 Accuracy of the EV Approximation

From [62], the new level of utility (U^t) can be expressed as

$$U^t = \ln\left(\frac{P_2^o}{P_1^o}\right) + \frac{E(P^o, U^t)}{P_2^o} - 1 \qquad [63]$$

where $E(P^o, U^t)$ is the minimum expenditure to attain U^t at the original prices given by $P^o = \{P_1^o, P_2^o\}$. Alternatively, U^t can be attained at the new income I^t and prices $P^t = \{P_1^t, P_2^t\}$, i. e.,

$$U^t = \ln\left(\frac{P_2^t}{P_1^t}\right) + \frac{I^t}{P_2^t} - 1 \ . \qquad [64]$$

It follows from [63] and [64] that

$$E(P^o, U^t) = \left[\ln\left(\frac{P_2^t}{P_1^t}\right) + \frac{I^t}{P_2^t} - \ln\left(\frac{P_2^o}{P_1^o}\right)\right] P_2^o \ . \qquad [65]$$

By duality between utility maximization and expenditure minimization, it is true that $I^o = E(P^o, U^o)$ and $I^t = E(P^t, U^t)$. Therefore, letting $EV(\Delta P, \Delta I, U^t)$ be the EV for attaining U^t, it follows from [1] and [65] that

$$EV(\Delta P, \Delta I, U^t) = \left[\ln\left(\frac{P_2^t}{P_1^t}\right) + \frac{I^t}{P_2^t} - \ln\left(\frac{P_2^o}{P_1^o}\right)\right] P_2^o - I^o \ . \qquad [66]$$

Since $I^t = I^o + \Delta I$, $EV(\Delta P, \Delta I, U^t)$ can also be written to conform to [3],

$$EV(\Delta P, \Delta I, U^t) = \left[\ln\left(\frac{P_2^t}{P_1^t}\right) + \frac{I^t}{P_2^t} - \ln\left(\frac{P_2^o}{P_1^o}\right)\right] P_2^o - I^t + \Delta I \ . \qquad [67]$$

Suppose that the original prices and income are $P^o = \{P_1^o, P_2^o\} = \{2, 2\}$ and $I^o = 50$. Let the original prices change between 1.25 and 2.75 and let $\Delta I = 2.5$. In this case, the values of $EV(\Delta P, \Delta I, U^t)$ in [67] are tabulated in Table 1.

Table 1. **Values of $EV(\Delta P, \Delta I, U^t)$ at Alternative Prices**

$P_1^t, P_2^t \rightarrow$ $\downarrow$	1.25	1.50	1.75	2.00	2.25	2.50	2.75
1.25	34.00	20.36	10.67	3.44	− 2.16	− 6.61	− 10.24
1.50	33.64	20.00	10.31	3.08	− 2.52	− 6.98	− 10.61
1.75	33.33	19.69	10.00	2.77	− 2.83	− 7.29	− 10.91
2.00	33.06	19.42	9.73	2.50	− 3.10	− 7.55	− 11.18
2.25	32.82	19.19	9.50	2.26	− 3.33	− 7.79	− 11.42
2.50	32.61	18.98	9.29	2.05	− 3.54	− 8.00	− 11.63
2.75	32.42	18.79	9.10	1.86	− 3.73	− 8.19	− 11.82

For the same cases in Table 1, the values of the proposed EV approximation (EV_d) in [13] have also been computed. In all cases, the sign of EV_d agrees with the sign of EV. The values of EV_d are reported in Table 2 as proportions of the corresponding values of the true EV from [67] in order to highlight the accuracy of EV_d.

Table 2. **Percent Ratios of EV_d to $EV(\Delta P, \Delta I, U^t)$**

$P_1^t, P_2^t \rightarrow$ $\downarrow$	1.25	1.50	1.75	2.00	2.25	2.50	2.75
1.25	100.00	100.05	100.24	101.28	97.13	98.78	99.03
1.50	100.02	100.00	100.02	100.16	99.68	99.84	99.85
1.75	100.03	100.01	100.00	100.01	99.98	99.99	99.98
2.00	100.03	100.01	100.00	100.00	100.00	99.99	99.99
2.25	100.03	100.01	100.00	100.00	100.00	100.00	99.99
2.50	100.04	100.02	100.02	100.05	99.99	100.00	100.00
2.75	100.06	100.04	100.06	100.19	99.95	99.99	100.00

It is amply evident from Table 2 that EV_d is very accurate. Except in three cases when $P_1^t = 1.25$ and P_2^t is equal to 2.00, 2.25 or 2.50, the absolute error of EV_d is less than 1% of the true value of EV.

4.2 Accuracy of the CV Approximation

From [62], the original level of indirect utility (U^o) can be expressed as

$$U^o = \ln\left(\frac{P_2^t}{P_1^t}\right) + \frac{E(P^t, U^o)}{P_2^t} - 1 \qquad [68]$$

where $E(P^t, U^o)$ is the minimum expenditure to attain U^o at the new prices given by $P^t = \{P_1^t, P_2^t\}$. Alternatively, U^o can be attained at I^o and $P^o = \{P_1^o, P_2^o\}$, i. e.,

$$U^o = \ln\left(\frac{P_2^o}{P_1^o}\right) + \frac{I^o}{P_2^o} - 1 \ . \qquad [69]$$

From [15], the CV for a change in both prices and income can be written as $CV(\Delta P, \Delta I, U^o) = -[E(P^t, U^o) - E(P^o, U^o)] + \Delta I$ where $I^o = E(P^o, U^o)$. The term $-[E(P^t, U^o) - E(P^o, U^o)]$ is the change in the value of the expenditure function as the expenditure line pivots from tangency at point O to T′ on U^o in Figure 1. Therefore, solving for $E(P^t, U^o)$ from [68] and [69], it follows that

$$CV(\Delta P, \Delta I, U^o) = I^o - \left[\ln\left(\frac{P_2^o}{P_1^o}\right) + \frac{I^o}{P_2^o} - \ln\left(\frac{P_2^t}{P_1^t}\right)\right] P_2^t + \Delta I \ . \qquad [70]$$

Suppose that $P^o = \{P_1^o, P_2^o\} = \{2, 2\}$, $I^o = 50$. Let the original prices change in the range between 1.25 and 2.75 and let $\Delta I = 2.5$. Table 3 gives the values of $CV(\Delta P, \Delta I, U^o)$ from [70].

Table 3 **Values of $CV(\Delta P, \Delta I, U^o)$ at Alternative Prices**

$P_1^t, P_2^t \rightarrow$ $\downarrow$	1.25	1.50	1.75	2.00	2.25	2.50	2.75
1.25	21.25	15.27	9.34	3.44	– 2.43	– 8.27	– 14.08
1.50	21.02	15.00	9.02	3.08	– 2.84	– 8.72	– 14.58
1.75	20.83	14.77	8.75	2.77	– 3.18	– 9.11	– 15.01
2.00	20.66	14.57	8.52	2.50	– 3.48	– 9.44	– 15.37
2.25	20.52	14.39	8.31	2.26	– 3.75	– 9.74	– 15.70
2.50	20.38	14.23	8.13	2.05	– 3.99	– 10.00	– 15.99
2.75	20.26	14.09	7.96	1.86	– 4.20	– 10.24	– 16.25

For the same cases in Table 3, the values of the proposed CV approximation (CV_d) in [22] have also been computed. The signs of CV_d are all in agreement with those of true CV. The values are reported in Table 4 as proportions of the corresponding values of the true CV from [70] in order to highlight the accuracy of CV_d.

Table 4 **Percent Ratios of CV_d to $CV(\Delta P, \Delta I, U^o)$**

$P_1^t, P_2^t \rightarrow$ $\downarrow$	1.25	1.50	1.75	2.00	2.25	2.50	2.75
1.25	100.00	99.98	99.91	98.59	100.84	100.33	100.25
1.50	99.99	100.00	99.99	99.92	100.14	100.07	100.07
1.75	99.98	100.00	100.00	100.00	100.01	100.01	100.02
2.00	99.98	99.99	100.00	100.00	100.00	100.01	100.02
2.25	99.98	99.99	100.00	100.00	100.00	100.00	100.02
2.50	99.95	99.97	99.97	99.92	100.01	100.00	100.01
2.75	99.89	99.89	99.86	99.59	100.10	100.01	100.00

The precision of CV_d is very high in Table 4. The approximation error is less than 1% in all cases but one when $P_1^t = 1.25$ and $P_2^t = 2.00$ for which the error is largest, equal to 1.41% of the true CV.

4.3 Accuracy of the Money Metric Approximation

Following the money metric definition in [43], the normalized indirect utility function in [62] yields the money metric utility function with $\partial M/\partial I = 1$ at the initial set of prices,

$$M(P, I) = P_2^o\left[\ln\left(\frac{P_2}{P_1}\right) + \frac{I}{P_2} - 1\right]. \qquad [71]$$

This implies that the change in utility in money metric terms from a change in both prices and income, $\Delta M(\Delta P, \Delta I)$, can be written as

$$\Delta M(\Delta P, \Delta I) = M(P^t, I^t) - M(P^o, I^o) \qquad [72]$$

$$= P_2^o\left[\ln\left(\frac{P_2^t}{P_1^t}\right) + \left(\frac{I^t}{P_2^t}\right) - \ln\left(\frac{P_2^o}{P_1^o}\right) - \frac{I^o}{P_2^o}\right]$$

where $I^t = I^o + \Delta I$.

Since the utility function in [71] is linear in income, EV in [66] equals ΔM in [72]. Thus, Table 1 represents both $EV(\Delta P, \Delta I, U^t)$ and $\Delta M(\Delta P, \Delta I)$, which are the true values of the equivalent variation and of the change in money metric utility from changes in both prices and income. In this case, the accuracy of the proposed EV approximation (EV_d) in [13] can be compared directly to McKenzie's money metric approximation (MM_3) in [48]. The results of the comparison are summarized in Table 5.

Table 5 **Percent Ratios of EV_d to EV and of MM_3 to MM When EV = ΔM**

P_1^t ↓	P_2^t →	1.25	1.50	1.75	2.00	2.25	2.50	2.75
1.25	$\frac{EV_d}{EV}$	100.00	100.05	100.24	101.28	97.13	98.78	99.03
1.25	$\frac{MM_3}{\Delta M}$	94.73	98.41	99.68	**99.59**	**100.96**	102.08	106.15
1.50	$\frac{EV_d}{EV}$	100.02	100.00	100.02	100.16	99.68	99.84	99.85
1.50	$\frac{MM_3}{\Delta M}$	94.70	98.44	99.79	**99.92**	100.35	101.80	105.83
1.75	$\frac{EV_d}{EV}$	100.03	100.01	100.00	100.01	99.98	99.99	99.98
1.75	$\frac{MM_3}{\Delta M}$	94.66	98.42	99.80	**100.00**	100.23	101.69	105.64
2.00	$\frac{EV_d}{EV}$	100.03	100.01	100.00	100.00	100.00	99.99	99.99
2.00	$\frac{MM_3}{\Delta M}$	94.62	98.40	99.80	100.00	100.21	101.63	105.51
2.25	$\frac{EV_d}{EV}$	100.03	100.01	100.00	100.00	100.00	100.00	99.99
2.25	$\frac{MM_3}{\Delta M}$	94.58	98.38	99.79	100.00	100.20	101.59	105.39
2.50	$\frac{EV_d}{EV}$	100.04	100.02	100.02	100.05	99.99	100.00	100.00
2.50	$\frac{MM_3}{\Delta M}$	94.54	98.36	99.77	99.92	100.23	101.56	105.31
2.75	$\frac{EV_d}{EV}$	100.06	100.04	100.06	100.19	99.95	99.99	100.00
2.75	$\frac{MM_3}{\Delta M}$	94.49	98.31	99.70	99.59	100.38	101.60	105.27

Table 5 shows that the computed values of EV_d and of MM_3 as proportions of the exact values of $EV = \Delta M$. In all cases, the sign of MM_3 agree with the sign of EV, like EV_d. Out of 49 cases, there are only four, namely, $(P_1^t, P_2^t) = \{(1.25, 2.00), (1.25, 2.25), (1.50, 2.00), (1.75, 2.00)\}$ (highlighted in bold) for which the absolute percentage errors of MM_3 are smaller than those of EV_d. This demonstrates that EV_d is a generally more precise approximation to EV than MM_3 for the example presented.

5. Conclusion

Approximations to EV and CV based on the Marshallian consumer's surplus are generally limited to measuring the welfare effects of a single price change. With more than one price changing at the same time, consumer's surplus is a unique measure if and only if income elasticities are equal for the goods with changing prices. For most realistic situations, consumer's surplus will result in misleading approximations to welfare change given simultaneous price and income changes.

This paper presents the procedures that provide the "practical algorithm" sought by Chipman and Moore (1980) for a welfare measure based on observed demand behavior. This measure exists in theory as shown by Hurwicz and Uzawa (1971). One procedure is McKenzie's "money metric" approximation to the change in the indirect utility function (McKenzie, 1983; Dumagan, 1989 & 1991) and the other proposed in this paper uses approximations to the expenditure function. McKenzie's method can be viewed as an approximation to EV, while the other method can approximate either EV or CV. All three of these approximations can be computed from observed Marshallian demand functions, and consequently, can be used for situations in which an explicit utility function is not specified.

For the utility function used by Silberberg (1978) to demonstrate the lack of uniqueness of consumer's surplus, the three approximations are shown to work well. In all cases considered, the correct direction of change in welfare is predicted by all three measures, and the level of accuracy is very high (less than 1% error). The EV approximation from the expenditure function (EV_d) is a closer approximation than the money metric (MM_3) for 45 out of 49 cases. Furthermore, because the money metric measure is derived from the indirect utility function, it requires a restrictive assumption about the marginal utility of income in order to monetize changes in utility levels. The procedures proposed in this paper impose no such restriction because the marginal utility of income is simply not involved in EV and CV approximations from the expenditure function.

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