Sufficient Conditions for Dominance of Simply Related Prospects

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Sufficient conditions for dominance of simply related prospects are developed for newly defined classes of limited-variation-in-risk-parameter utility functions. Necessary and sufficient conditions are given for classes of contant-risk-parameter utility functions. The latter include classes of quadratic, power and exponential utility functions. The conditions can be incorporated into easily implemented procedures for locating efficient prospects.

1. Introduction

In the context of decision theory, efficiency analysis aims to divide any set of feasible prospects into two mutually exclusive and exhaustive sets. All those prospects which are not considered optimal on the basis of any member of a particular class of utility functions are placed in one set (the dominated set), and the remaining prospects in the other (the efficient set). The classes of utility functions initially studied by efficiency analysts included the class of all Bernoullian utility functions, the class of all risk averse utility functions, and the class of decreasingly risk averse risk averters. Necessary and sufficient conditions for one prospect to dominate another, or equivalently, for one prospect to be dominated by another, for these classes are embodied in stochastic dominance ordering rules developed by Hanoch and Levy (1969), Hadar and Russell (1969, 1971) and Whitmore (1970). Readers of this *Review* will be acquainted with these results through the paper by Anderson (1974).

There have been subsequent developments. Drynan (1977), Fishburn (1978) and Meyer (1979) have noted that the (pure) dominance concept of one prospect being dominated by another involved in these rules is actually too narrow for the (mixed) dominance concept inherent in efficiency analysis. They have shown that the necessary and sufficient conditions for pure dominance are only sufficient conditions for determining if a prospect is dominated in the sense of efficiency analysis. Necessary and sufficient conditions for mixed dominance for the utility classes mentioned above have been given by these latter authors. Implementation of these conditions is relatively easy (Drynan 1977).

One cause for concern with efficiency analysis is the frequent finding that many prospects are not detected as dominated. There are two reasons. First, analysts have often used only sufficient rather than necessary and sufficient conditions for dominance. In particular, analysts have not routinely used the mixed dominance rules, opting instead for pure dominance rules. Whether this occurs through lack of knowledge of mixed dominance or because of the easier implementation of pure dominance conditions is not clear, though given the availability of computers for efficiency analysis, one suspects that ignorance of mixed dominance must be the primary explanation.

The second reason for large efficient sets is the use of classes of utility functions so broad, that diversity of optimal choice is inevitable. The class of risk averters, for example, includes near risk indifferent decision makers as well as extreme risk averters and many whose local risk aversion oscillates, or alters slowly, rapidly or uniformly between the extremes. The

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onus is on the decision analyst to ensure that the class of utility functions has been constrained as much as possible. One suspects that decision analysts have failed here, perhaps because of greater familiarity with, and the ease of implementation of, dominance conditions for the broader classes of utility functions. But also contributing to the tendence to study broad classes of utility functions is the analyst's desire to obtain strong, aesthetically appealing results: far better to be able to say that no risk averter would select some particular prospect rather than only that those who are not too risk averse would not select some prospect. From a practical viewpoint, the latter conclusion would be just as useful as the former if extreme risk averters did not exist.

Important advances in defining narrower classes of practically relevant utility functions, and in establishing conditions for dominance, have been achieved by Hammond (1974) and Meyer (1977a,b). Both defined classes of utility functions by quantitatively restricting the local risk aversion function rather than restricting the derivatives of the utility function in a qualitative way. In particular, Meyer (1977a) defined classes in which the local risk aversion function is constrained to lie between lower and upper bounding functions. Appropriate definition of these general risk constrained classes by bounding functions means that utility functions displaying extreme risk attitudes can be excluded. Meyer developed necessary and sufficient conditions for one prospect to dominate another. Implementation of the conditions, which are not in general in closed form, is tedious, though easy with computer facilities. When risk aversion is bounded from only one side, the conditions simplify (Meyer 1977b). Implementation of these conditions for "stochastic dominance with respect to a function" is similar to that for standard stochastic dominance rules.[1]

Drynan (1977, 1986) developed the necessary and sufficient conditions for mixed dominance for these risk constrained classes. Implementation in the case of classes bounded on only one side is straightforward, paralleling the procedures for determining second degree stochastic dominance. But in the absence of closed conditions for pure dominance for the more general risk constrained classes, no easy method for detecting mixed dominance has been developed.

Hammond (1974) reported more specific results for particular types of prospects, namely those which are simply related.[2] One important result (see Theorem 1 below) is that, for simply intertwined prospects, there exists a break-even value of local risk aversion such that a decision maker with this level of constant risk aversion is indifferent between the two prospects, decision makers more risk averse preferring one prospect, and all decision makers less risk averse preferring the other. This is a special case of a more general result obtained by Meyer (1977b) for any pair of prospects with distributions crossing a finite number of times: a break-even risk aversion function (not necessarily or generally constant) always exists dividing decision makers in this way.

These results represent a simple alternative to applying the Meyer (1977a) conditions for pure dominance, which apply for any pair of distributions, to simply related distributions. Unfortunately, corresponding simpler conditions for mixed dominance of simply related distributions are not apparent for general risk constrained classes.

In this paper, necessary and sufficient conditions are defined for mixed dominance of simply related distributions for a number of more narrowly defined, single risk parameter classes of utility

[1] The "function" is not the bounding function itself, but the marginal utility function corresponding to it.

[2] Simply related distributions have cumulative distribution functions which intersect no more than once. Those which do intersect once are said to be simply intertwined. That prospect which has a distribution function which first increases from 0 to exceed that of another is said to be the more prone to low outcomes of the two.

functions. These classes include the class of constant risk aversion functions, the class of quadratic utility functions, and the class of power functions. Sufficient conditions are also given for somewhat more general classes of limited-variation-in-risk-parameter utility functions. All these classes are subclasses of Meyer's general risk constrained classes, differing most notably in eliminating utility functions displaying widely varying local risk aversion. Means of implementation are outlined in all cases.

These results are necessarily of limited practical relevance since, in general, distribution functions would cross more than once. As well, the classes of utility function may be too restrictive. Nevertheless, when these restrictions are met, or judged to be reasonable approximations, the conditions provide an added weapon in the analyst's arsenal.

In an attempt to develop results that should prove more useful more often, the class of utility functions is enlarged by defining classes on the basis of two risk parameters. Necessary and sufficient conditions are again given. The paper concludes with a simple illustration of the ability of the rules to detect dominated prospects that are not detected with the pure dominance rules.

2. Utility Classes

The general class of utility functions considered in this paper is the class of limited-variation-in-risk-aversion utility functions defined by:

$$U_{rv} = \{u(w) : r_L \leq r(w) \leq r_U, \text{ and } \max_w r(w) - \min_w r(w) \leq r_v\}$$

where $a < w < b$, $u(w)$ is a Bernoullian utility function of monetary consequences w, $r(w)$ $(= -u''(w)/u'(w))$ is the local risk aversion function, and a and b are the tightest lower and upper bounds on the domain of monetary consequences encompassing all possible consequences of the feasible prospects. The class includes all utility functions such that local risk aversion never exceeds lower (r_L) and upper (r_U) limits, and never varies more than a specified amount (r_v) over the domain of monetary consequences.

Specific classes can be defined by selecting particular bounding values and values for r_v. First when $r_v = 0$, U_{rv} reduces to a class U_{rc} of constant risk aversion utility functions. As noted earlier, Hammond (1974) obtained a number of useful results for comparing one prospect to another for U_{rc} classes, but he gives no mixed dominance results. A second special case occurs when $r_v = r_U - r_L$, in which case the only constraints on the class of utility functions are the upper and lower bounds on risk aversion. The U_{rv} class is then a member of the general risk constrained classes studied by Meyer (1977a), and in fact a special case of his classes since he permitted r_L and r_U to be bounding functions (of w) rather than set constants. By appropriate choice or r_L and r_U, this case includes the class of all Bernoullian utility functions, and the class of risk averse utility functions. As already noted, necessary and sufficient conditions for dominance for general risk constrained classes are available in Drynan (1977). Necessary and sufficient conditions for dominance for U_{rv} classes have not been published.

Other specific classes can be developed by defining bounds on risk aversion implicitly. In particular, classes of limited-variation-in-risk-parameter utility functions can be defined as follows:

$$U_{pv} = \{u(w) : p_L \leq p(w) \leq p_U, \text{ and } \max_w p(w) - \min_w p(w) \leq p_v\}$$

where $a < w < b$, $p(w)$ is a risk parameter function such that the utility function $u(w)$ has a local risk aversion function $r(w)$ $(= -u''(w)/u'(w)) = r^*(w,p(w))$ which is assumed to be bounded given the risk parameter function $p(w)$, $\partial r^*(w,p)/\partial p$ is continuous and positive, and a and b are the tightest lower and upper bounds on the domain of monetary consequences encompassing all possible consequences of the feasible prospects. When $p_v = 0$ (the class U_{pc}), each utility function has a constant risk parameter for all w levels. Local risk aversion may, however, still vary. By appropriate choice of $p(w)$, risk aversion for any w can be made arbitrarily small or large.[3] Clearly, the higher is $p(w)$, the more risk averse the decision maker at that w.

Examples of risk aversion functions based on a constant risk parameter are given in Drynan (1981). The classes of quadratic utility functions, constant absolute risk aversion utility functions, power utility functions and constant partial risk aversion functions, amongst others, can be represented in this way.

Comparing the U_{pv} and U_{rv} classes, it is clear that when $r^*(w, p(w)) = p(w)$ the two are identical. Although there is always an $r(w)$ corresponding to a $p(w)$, a U_{pv} class cannot generally be re-expressed as a U_{rv} class because the lower and upper bounds on $r(w)$ would generally be functions of w rather than constants. On the other hand, all U_{rv} classes can be expressed as U_{pv} classes.

Finally, broader utility classes can be defined by introducing additional risk parameter functions. Here attention is confined to classes of constant risk parameter utility functions. These classes can be defined as follows:

$$U_{mpc} = \{u(w) : p_{kL} \leq p_k \leq p_{kU};\ \text{for } k = 1, \ldots K,\}$$

where $a < w < b$, p_k, $k = 1, 2, \ldots K$, are risk parameters such that the utility function $u(w)$ has a local risk aversion function $r(w)$ $(= -u''(w)/u'(w)) = r^*(w, p_1, p_2, \ldots)$ which is assumed to be bounded given the risk parameters, $\partial r^*(w,p_k)/\partial p_k$ is continous and positive for all k, and a and b are the tightest lower and upper bounds on the domain of monetary consequences encompassing all possible consequences of the feasible prospects. This class is clearly a generalization of the single constant risk parameter classes defined earlier. Again, the higher is any p_k, the more risk averse the decision maker.

In practice, analysis with more than two parameters may prove intractable. Example risk aversion functions with two risk parameters include linear risk aversion functions (*i.e.*, $r(w) = p_1 + p_2 w$), power functions (*i.e.*, $r(w) = p_1 w^{p_2}$), inverse functions such as those associated with constant relative risk aversion (*i.e.*, $r(w) = p_1/(w-p_2)$) and functions formed as the sum of two single risk parameter risk aversion functions, namely $r(w) = r_1(w,p_1) + r_2(w,p_2)$.

3. Sufficient Conditions for Dominance

In proving the subsequent results, the following dominance relationship between two simply intertwined prospects will be used.[4]

Theorem 1: Let the prospects a_i and a_j be simply intertwined, with a_i being more prone to low outcomes than a_j. Then there exists a breakeven risk aversion level r_{ij} such that, for all utility functions with local risk aversion everywhere less than r_{ij}, a_i is preferred to a_j; and for all utility functions with local risk aversion everywhere greater than r_{ij}, a_j is preferred to a_i.[5]

[3] Strictly, some qualifications are needed. For example, If $u(w) = w - pw^2$ and if $u'(w)$ is required to be positive, then p cannot be selected to achieve arbitrarily high levels of local risk aversion. See also, Theorem 7.

[4] If simply related prospects are not simply intertwined, then either the distributions are identical or one distribution stochastically dominates the other in the first degree (and hence for all classes of utility functions). Consequently, throughout the text, all distributions are assumed to be different. Only trivial modifications are needed to include identical distributions. As well attention is confined to simply intertwined prospects because the dominance question is solved if the distribution functions do not intersect.

[5] Note the convention that the order of subscripts in r_{ij} (and later p_{ij}) reflects the fact that a_i is preferred by more risk preferring decision makers, and a_j is preferred by those more risk averse. The notation is subsequently extended to $p_{1ij}|p_2$ to indicate a conditional breakeven value for p_1 given p_2.

Proof: This theorem follows from Hammond (1974), Theorems 1 and 3.

Theorem 2: Let the prospects a_i and a_j be simply intertwined, with a_i being more prone to low outcomes than a_j. Then there exists a breakeven parameter value p_{ij} such that, for all utility functions with bounded local risk aversion $r(w,p)$ where $\partial r(w,p)/\partial p$ is continuous and positive, a_i is preferred to a_j when $p < p_{ij}$; and a_j is preferred to a_i when $p > p_{ij}$.

Proof: Given the monotonic relationship between p and local risk aversion, it follows from Theorem 1 that if for some value of p the decision maker is indifferent between prospects a_i and a_j, then for all higher p a_j is preferred and for all lower p a_i is preferred. A breakeven value of p will always exist for simply intertwined prospects since expected utility is continuous in p and, from Theorem 1, for sufficiently large p, prospect a_j is preferred, and for sufficiently small p, prospect a_i is preferred. This completes the proof.

Theorem 3: For the class of utility functions U_{pv}, and for a set of simply intertwined feasible prospects, prospect a_j is dominated:

(a) if there exists a prospect a_i more prone to low outcomes than a_j, and $p_{ij} > p_U$; or

(b) if there exists a prospect a_k less prone to low outcomes than a_j, and $p_{jk} < p_L$; or

(c) if there exist two prospects, a_i and a_k, with a_i more and a_k less prone to low outcomes than a_j, and such that $p_{ij} - p_{jk} > p_v > 0$.

Proof: It follows from Theorem 2 that a_j will be dominated by a_i if $p(w) < p_{ij}$. If (a) holds, then $p(w) < p_U < p_{ij}$, so a_j will be dominated. Following a similar argument, a_j will be dominated (by a_k) if (b) holds. Suppose that (c) holds. Then no utility function in U_{pv} can have both a minimum $p(w) < p_{jk}$ and a maximum $p(w) > p_{ij}$. Each utility function either has risk parameter levels everywhere greater than p_{jk} (and so a_k is preferred to a_j), or has risk parameter levels everywhere less than p_{ij} (and so a_i is preferred to a_j). Hence, for U_{pv}, when (c) holds, a_j is dominated since either a_i or a_k is preferred to a_j. This completes the proof.

This theorem, composed essentially of known dominance relations between two prospects and of pure dominance concepts, provides sufficient conditions for determining mixed dominance for U_{pv}. That the conditions are not generally necessary is easily shown by counter-example (Drynan 1977). Stronger sufficient conditions can be developed using the mixed dominance results for general risk constrained classes. Thus if there is no utility function with risk aversion within the boundaries implied by a risk parameter function within the limits defined by $(p_{ij} - p_v)$ and $(p_{kj} + p_v)$, then a_j is dominated. The difficulty with this stronger condition lies in the lack of efficient methods for determining mixed dominance for general risk constrained classes.

Necessity can be proved for one important case, namely, for the class of constant risk parameter utility functions, U_{pc}, that is for a class of utility functions each member of which has a risk parameter function $p(w) = p$, a constant.

Theorem 4: For the class U_{pc} of constant risk parameter utility functions, and for a set of simply intertwined feasible prospects, prospect a_j is dominated if and only if one or more of the following conditions hold:

(a) there exists a prospect a_i more prone to low outcomes than a_j, and $p_{ij} > p_U$; or

(b) there exists a prospect a_k less prone to low outcomes than a_j, and $p_{jk} < p_L$; or

(c) there exist two prospects, a_i and a_k, with a_i more and a_k less prone to low outcomes than a_j, and such that $p_{ij} > p_{jk}$.

Proof: Sufficiency follows immediately from Theorem 3 using $p_v = 0$. To prove necessity, suppose a_j is dominated but that no prospects exist satisfying (a), (b) or (c). Hence, any prospect, say a_k, less prone to low outcomes than a_j produces a break-even value of $p_{jk} > p_L$; and any prospect, say a_i more prone to low outcomes than a_j produces a break-even value $p_{ij} < p_U$. If either an a_i or a_k does not exist, a_j clearly is preferred for utility functions with local risk parameter levels at one end of the risk parameter domain, contradicting the assumption of dominance. Suppose then that both a_i and a_k exist. Since (a), (b) and (c) are untrue, $p_L \leq p_{ij} \leq p_U$, and there will exist in U_{pc}, a utility function with constant risk parameter p_*, $p_{ij} \leq p_* \leq p_{jk}$, for which neither a_i nor a_k is preferred to a_j. This again contradicts the assumption of dominance, and some prospects must exist satisfying (a), (b) or (c). This completes the proof.

Similar conditions to those of Theorems 3 and 4 for U_{pv} and U_{pc} can be stated for limited-variation-in-risk-aversion classes of utility functions. Proofs parallel those in Theorems 3 and 4.

Theorem 5: For the class of utility functions U_{rv}, and for a set of simply intertwined feasible prospects, prospect a_j is dominated:

(a) if there exists a prospect a_i more prone to low outcomes than a_j, and $r_{ij} > r_U$; or
(b) if there exists a prospect a_k less prone to low outcomes than a_j, and $r_{jk} < r_L$; or
(c) if there exist two prospects, a_i and a_k, with a_i more and a_k less prone to low outcomes than a_j, and such that $r_{ij} - r_{jk} > r_v > 0$.

Theorem 6. For the class U_{rc} of constant risk aversion utility functions, and for a set of simply intertwined feasible prospects, prospect a_j is dominated if and only if one or more of the following conditions hold:

(a) there exists a prospect a_i more prone to low outcomes than a_j, and $r_{ij} > r_U$; or
(b) there exists a prospect a_k less prone to low outcomes than a_j, and $r_{jk} < r_L$; or
(c) there exist two prospects, a_i and a_k, with a_i more and a_k less prone to low outcomes than a_j, and such that $r_{ij} > r_{jk}$.

Another constant risk parameter class that may be of interest is the class of quadratic utility functions, $u(w) = w - pw^2$ with $p_L < p < p_U$. But in this case conditions for pure and mixed dominance are quite simple, not just for simply related distributions, but for any set of distributions. For example, the "E–V rule" is a sufficient condition for pure dominance for any class with $p_L > 0$. Necessary and sufficient conditions for pure dominance have been established by Hanoch and Levy (1970) for the case $p_L = 0$. Necessary and sufficient conditions for the more general case and for mixed dominance are established below.

Theorem 7: For the class of utility functions with members $u(w) = w - pw^2$, where $p_L \leq p \leq p_U$, and $u'(w) > 0$ (and hence $p < 1/2b$ where b is the tightest upper bound on the domain of monetary consequences), a_j is dominated by a_i if and only if:

$$M_i - M_j > p_U (V_i - V_j)/[1 - p_U (M_i + M_j)] \quad \text{if } V_j + M_j^2 < V_i + M_i^2,$$

$$M_i - M_j > 0 \quad \text{if } V_j + M_j^2 = V_i + M_i^2,$$

$$M_i - M_j > p_L (V_i - V_j)/[1 - p_L (M_i + M_j)] \quad \text{if } V_j + M_j^2 > V_i + M_i^2,$$

where M and V denote mean and variance respectively.

Proof: The expected utility of a_i is given by:

$$EU_i = M_i - p (M_i^2 + V_i);$$

and similarly for a_j. Note that any increase in the mean of a prospect can only serve to increase its expected utility. Expected utility is clearly a negatively sloping linear function of p, and there will exist a breakeven p_{ij} value defined by:

$$M_i - p_{ij} (M_i^2 + V_i) = M_j - p_{ij} (M_j^2 + V_j)$$

or equivalently when:

$$M_i - M_j = p_{ij}(V_i - V_j)/[1 - p_{ij}(M_i + M_j)].$$

For smaller p values the prospect with the larger $(M^2 + V)$ is preferred, and for larger p the one with the smaller $(M^2 + V)$ is preferred. To prove sufficiency, suppose $V_j + M_j^2 < V_i + M_i^2$. Suppose that for p_U, a_i is preferred, that is that the first condition of the theorem holds. Then a_i must be preferred for all utility functions in the class. Suppose $V_j + M_j^2 = V_i + M_i^2$ and that a_i is preferred (second condition). It must then have the larger mean, and be preferred for all quadratic utility functions since p does not affect expected utility. Suppose $V_j + M_j^2 > V_i + M_i^2$. If a_i is preferred for p_L (third condition), it must be preferred for all utility functions with higher p and hence for all in the given class.

To prove necessity, one need only to note that, if the conditions of the theorem do not apply, then there is at least one quadratic function for which a_i is not preferred to a_j. This completes the proof.

If $p_L = 0$, that is the class contains all risk averse quadratic utility functions, the conditions correspond to the Hanoch and Levy (1970) rule. If the conditions are applied with $p_L = 0$ and p_U infinite, they reduce to the "E–V rule". This is sufficient for dominance but not necessary because the upper bound is needlessly loose since no p can exceed $1/2b$ if $u'(w)$ is required to be positive.

Necessary and sufficient conditions for mixed dominance are embodied in the following theorem.

Theorem 8: For the class of quadratic utility functions $u(w) = w - pw^2$, where $p_L \leq p \leq p_U$ and $u'(w) > 0$, prospect a_j is dominated if and only if one of the following holds:

$$M_z - M_j > p_U(V_z - V_j)/[1 - p_U(M_z + M_j)] \quad \text{if } V_j + M_j^2 < V_z + M_z^2,$$

$$M_z - M_j > 0 \text{ if } V_j + M_j^2 = V_z + M_z^2,$$

$$M_z - M_j > p_L(V_z - V_j)/[1 - p_L(M_z + M_j)] \quad \text{if } V_j + M_j^2 > V_z + M_z^2,$$

where M and V denote mean and variance, respectively, and the subscript z denotes some mixed prospect formed by selecting available prospects other than a_j with defined probabilities.

Proof: This theorem is a particular application of the necessary and sufficient conditions for mixed dominance for general classes of utility functions as developed in Drynan (1977, 1986).

However, from a practical viewpoint, one requires the means of establishing if a suitable mixed prospect exists. It is easy to search for that mixed prospect which maximizes expected utility for the quadratic with $p = p_L$ but which has $M_z^2 + V_z \geq M_j^2 + V_j$. Similarly, one can locate that mixed prospect which maximizes expected utility for the quadratic with $p = p_U$ but which has $M_z^2 + V_z \leq M_j^2 + V_j$. The required probabilities are determined as the solution to the following linear programming problem:

$$\max \Sigma_t P_{t*}(M_t + p_*(M_t^2 + V_t))$$

$$\text{subject to } \Sigma_t P_{t*} = 1,$$

$$\Sigma P_{t*}(M_t^2 + V_t) \{\geq/\leq\} M_j^2 + V_j \quad \text{and } P_{t*} \geq 0,$$

where * is either L or U and the inequality is $\geq$ or $\leq$, respectively. If either solution satisfies one of the conditions, then a_j is dominated. If neither of these solutions satisfies one of the three conditions, the conditions can never be satisfied since the mixed prospects best available to satisfy them cannot do so. Hence the necessary and sufficient conditions for dominance cannot be met, and a_j is not dominated.

To complete the set of results, Theorem 4 for a single risk parameter class U_{pc} is generalized to the double risk parameter class U_{mpc}.

Theorem 9: For a class U_{mpc} of utility functions with two constant risk parameters, and for a set of simply intertwined feasible prospects, prospect a_j is dominated if and only if one or more of the following conditions hold: for every p_2 within the limits of p_{2L} and p_{2U},

(a) there exists a prospect a_i more prone to low outcomes than a_j, and $p_{1ij} \mid p_2 > p_{1U}$; or

(b) there exists a prospect a_k less prone to low outcomes than a_j, and $p_{1jk} \mid p_2 < p_{1L}$; or

(c) there exists two prospects, a_i and a_k, with a_i more and a_k less prone to low outcomes than a_j, and such that $p_{1ij} \mid p_2 > p_{1jk} \mid p_2$; where the prospects a_i and a_k may change for different levels of p_2.

Proof: This follows immediately since the conditions amount to the application of Theorem 4 for all levels of parameter p_2.

An alternative to the necessary and sufficient conditions of Theorem 9 is the following sufficient condition for dominance.

Theorem 10: For a class U_{mpc} of utility functions with two constant risk parameters, and for a set of simply intertwined feasible prospects, prospect a_j is dominated if one or more of the following conditions hold:

(a) there exists a prospect a_i more prone to low outcomes than a_j, and $p_{1ij}|p_{2U} > p_1$; or

(b) there exists a prospect a_k less prone to low outcomes than a_j, and $p_{1jk}|p_{2L} < p_{1L}$; or

(c) there exist two prospects, a_i and a_k, with a_i more and a_k less prone to low outcomes than a_j, and such that $p_{1ij}|p_2 > p_{1jk}|p_2$ for all $p_{2L} \leq p_2 \leq p_{2U}$.

Proof: Consider (a). For any value of p_2 there exists a break-even value of p_1 $(p_{1ij} \mid p_2)$ such that for those utility functions with greater risk parameter than $p_{1ij} \mid p_2$, a_j is preferred, and for those with smaller p_1 values, a_i is preferred (Theorem 2). Since $r(w)$ is continuous and monotonic in both p_1 and p_2, the break-even values for different p_2 values form a negatively sloped locus in $p_1 \times p_2$ space. For any utility function with risk parameters below this locus, a_i is preferred. Hence if the co-ordinates (p_{1U}, p_{2U}) lie under the locus, a_i is preferred for all utility functions in the class. If the latter point lies above the break-even locus, a_j is preferred to a_i for some utility functions.

A similar argument is used for (b). When condition (c) applies, it ensures that one break-even locus (for a_j and a_k) lies below the other break-even locus (for a_i and a_j) throughout the relevant p_2 domain, and hence, whenever a_j is preferred to a_i, a_j is not preferred to a_k. This completes the proof.

4. Discussion

Few decision makers probably display constant risk parameter behaviour over all possible monetary outcomes. However, it may often be reasonable to approximate utility functions over the relevant domain of monetary consequences by such functions. Except where normal distributions have been assumed, users of the "E–V rule" have been prepared to accept this. Those involved in more recent attempts to estimate farmers' risk attitudes have also used this type of approximation, at least locally. King and Robison (1981) and Wilson and Eidman (1983) assumed constant absolute risk aversion in establishing upper and lower bounds on an individual's risk aversion over a narrow monetary domain. Binswanger (1980) assumed constant partial risk aversion in classifying farmers by risk aversity.[6] The quality of these approximations remains an empirical question. In all three cases, the evidence indicates that, over wider domains, the particular assumed constant risk parameter is inappropriate. Apart from indicating that an alternative single (or double) risk parameter model may be necessary, it also implies that the local risk assumptions used in these studies were inappropriate.

If this level of approximation is unacceptable, Theorems 3 and 5 give sufficient conditions for prospect dominance for more general classes of

[6] Binswanger's work is also interesting in that the farmers were given choice sets of simply related prospects some of which were dominated (in the mixed sense). A number of respondents preferred the dominated prospects.

utility functions. The important empirical points here concern how much variation in the risk parameter must be acknowledged for realism and whether one can more successfully detect dominance by working with sufficient conditions for these limited-variation-in-risk-parameter classes than by applying necessary and sufficient conditions for the broad classes associated with the traditional stochastic dominance concepts.[7]

Even larger classes of utility functions can be defined as the union of any two or more of the limited-variation-in-risk-parameter classes. A necessary and sufficient condition for dominance for the union is that a prospect be dominated for every member class of the union.

When Theorems 3 and 5 are invoked for U_{pv} and U_{rv} respectively, it may still be useful to examine the application of Theorems 4 and 6 to the corresponding U_{pc} and U_{rc} classes. If it is found that Theorem 4 or 6 yields the same reduced set as Theorem 3 or 5, that set is also the efficient set for U_{pv} or U_{rv} respectively. Hence, in some cases, even though necessary and sufficient conditions for dominance for U_{pv} and U_{rv} have not been given, it is possible to recognize the efficient set.

5. Implementation

Under the assumption of simply intertwined distributions with a known distributional form, the break-even value of a single risk parameter can be found relatively easily. At this point, the expected utility of the two prospects, a_i and a_j say, must be equal. Letting F_i and F_j be distribution functions, expected utilities are equal when:

$$_a\int^b[F_i - F_j]\, u'(w)\, dw = 0.$$

Expressing $u(w)$ in terms of $r(w)$, with $r(w) = r^*(w,p)$, the break-even value p_{ij} is defined by:

$$_a\int^b[F_i - F_j]\, \exp(— r^*(x,p_{ij})dx)\, dw = 0.$$

Generally, numerical or iterative search methods would have to be used. In some cases, the task is simpler. For example, for a constant risk aversion utility function with risk aversion p $(p \neq 0)$, $u(w) = -(1/p \exp(-pw)$, and the break-even value is defined by:

$$E_i[-1/r_{ij}) \exp(-r_{ij}w)] = E_j[-(1/r_{ij}) \exp(-r_{ij}w)],$$

where E_i and E_j denote expectations with respect to the distributions for prospects a_i and a_j, respectively. These expectations have the form of moment generating functions, and can be written in terms of the parameters of the distributions and the constant risk aversion level r_{ij}. The latter break-even value often has an explicit analytical solution. For example, in the case of normal distributions,

$$r_{ij} = 2\,(M_i - M_j)/(V_i - V_j).$$

But even for constant risk aversion functions explicit solutions do not always exist (*e.g.*, for uniform distributions), although solutions are easily obtained iteratively using the convenient expressions for expectations.

For U_{pc} (and similarly for U_{rc}), the efficient set contains only those prospects which are preferred for at least one value of the risk parameter in the prescribed range. An easy way to locate the efficient set is to search over the risk parameter values noting which prospects are preferred. As a first step, the feasible prospects should be arranged in order from that most prone to low outcomes to that least prone to low outcomes (descending order of variance if the distributions are normal). Locate the prospect with the largest expected utility for $p = p_L$ (that with maximum $(M - 0.5r_L V^{0.5})$ value if normality and constant risk aversion is assumed). From Theorem 2, it follows that this prospect will be preferred on the basis of all utility functions with risk parameter greater than

[7] There is theoretical supporting evidence in the results of Meyer (1977a). For any general risk constrained class of utility functions, the function which will first prevent dominance (as a class is widened) is one with risk aversion varying from the lower to the upper risk aversion limits.

p_L and less than p_{ik}, where p_{ik} is the minimum p_{it} value, $t > i$. It is easily shown that the sequence of preferred prospects as the risk parameter increases shows decreasing proneness to low outcomes; hence, in determining p_{ik}, only prospects which are less prone to low outcomes than prospect a_i need be considered. At p_{ik}, ignoring the possibility of identical break-even values, the preferred prospect changes from a_i to a_k. Prospect a_k will be the preferred prospect for risk parameter levels to p_{ks}, where p_{ks} is the smallest p_{kt} value, $t > k$. At p_{ks}, a_s becomes the preferred prospect. The efficient set can be isolated by continuing in this way either until the smallest break-even value is larger than p_U, or until no more prospects less prone to low outcomes than the current preferred prospect remain. If two prospects yield the same minimum break-even value, both prospects belong in the efficient set. As the risk paramater is further decreased, the one with the higher expected utility would be chosen as the next preferred prospect; and the procedure continues as outlined.

The preceding procedure is designed for locating the efficient set. Where the question is simply whether a given prospect is dominated, one can either use this same procedure, or one can compute all break-even values for that prospect and examine them to see if any of the conditions of Theorem 4 hold.

Dominance analysis for a U_{pv} (and similarly U_{rv}) class is a little more difficult, and in essence, each prospect must be examined in turn. Some computational efficiencies can be made. Again, arrange the prospects in order of decreasing proneness to low outcomes. Then locate the preferred prospects for risk parameter levels of p_L and p_U, say a_L and a_U. It follows from Theorem 2, and (a) and (b) of Theorem 3, that all prospects which are more prone to low outcomes than a_L or less prone to low outcomes than a_U are dominated either by a_L or a_U, and can be discarded. Prospects a_L and a_U are necessarily efficient and need not be examined for dominance. For prospect a_j, compute the break-even values with the other remaining prospects. Find max p_{kj}, $k < j$, and min p_{js}, $s > j$, and test for conditions (a), (b) and (c) of Theorem 3. If any holds, discard a_j. Continue with other prospects until all have been examined for dominance, leaving a reduced set of all efficient, and possibly other, prospects.

Finally, for the double risk parameter utility functions, the procedure essentially involves repeating the single parameter analysis. Once the prospects have been ordered, dominance on the basis of p_1 values is assessed conditional on each of a series of p_2 values. The finer the p_2 grid, the more precisely are the conditions of Theorem 9 applied. The sufficient conditions of Theorem 10 (a) and (b) require only a single parameter search each and are therefore readily examined.

6. An Illustration

The limited-variation-in-risk-parameter classes of utility functions defined in this paper differ significantly from other classes defined in the literature in that a host of seemingly unrealistic utility functions, each displaying wide variation in the risk parameter over the relevant domain of monetary consequences, are now excluded. The potential for reducing the set of feasible prospects is illustrated in Figure 1 under the assumption of normally distributed prospects and U_{rv} classes.

Consider prospects a_i and a_k, with means M_i and M_k, and variances V_i and V_k respectively. If the necessary and sufficient conditions of second degree stochastic dominance (the "E–V rule" under normality) were applied to all pairs of prospects, all prospects in the dotted area would be eliminated since they are dominated by a_i or a_k.

Suppose that the relevant class of utility functions is that of all risk averse, constant risk aversion utility functions. From Theorem 6, conditions (a) and (b) yield the same dotted area as for second degree stochastic dominance. Condition (c) says

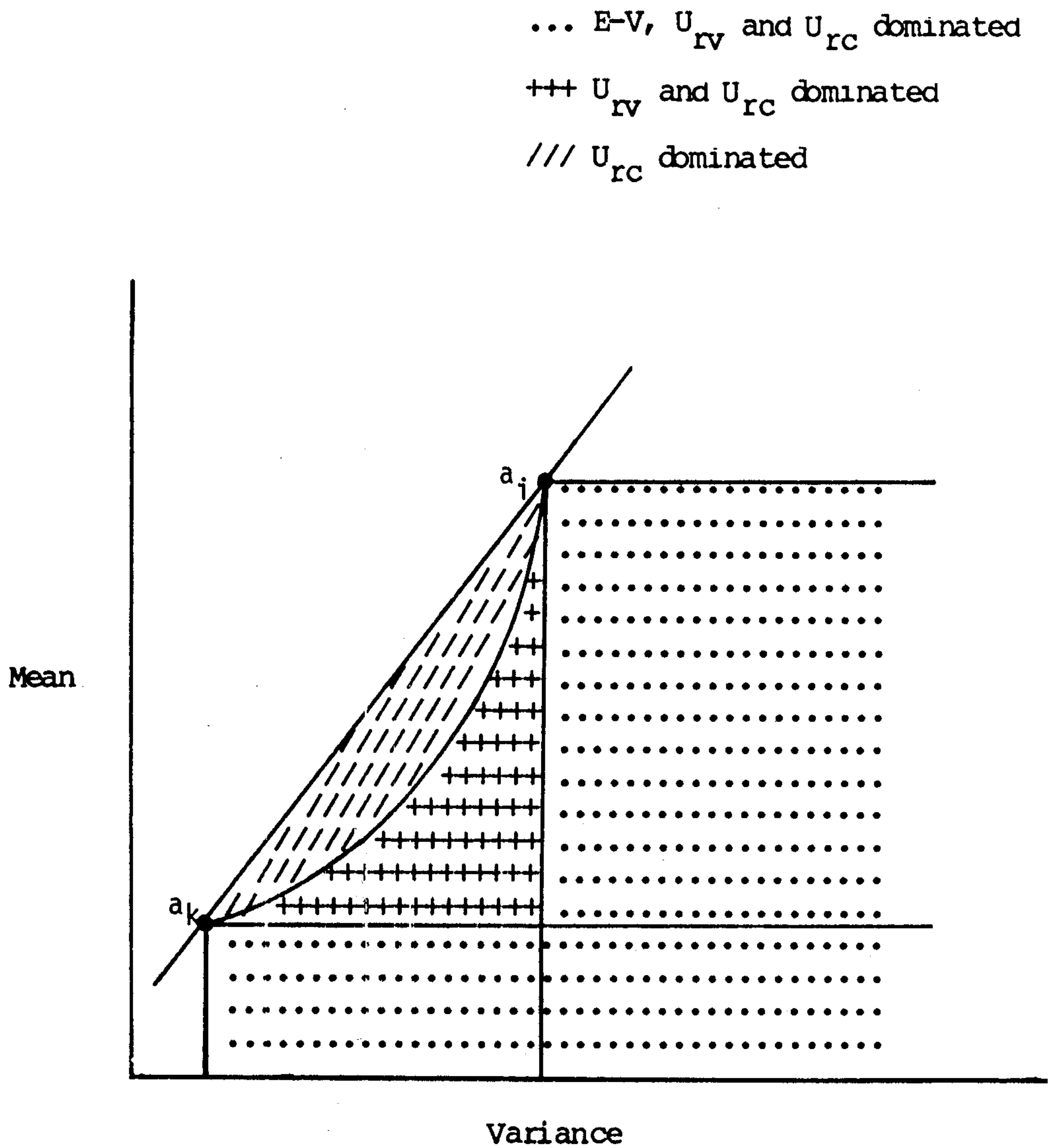

Figure 1: Areas of dominance for example utility classes

that a prospect a_j is dominated if $r_{ij} > r_{jk}$, where it is assumed that variance $V_i > V_j > V_k$. Using the expression for break-even risk aversion for normality, a_j is dominated if:

$$2\,(M_i - M_j)/(V_i - V_j) > 2\,(M_j - M_k)/(V_j - V_k).$$

In terms of Figure 1, any prospect a_j lying to the lower right side of a straight line joining a_i and a_k is dominated. Hence the hatched and crossed areas are to be added to the dotted area of dominance. This result reflects the fact that iso-utility lines in mean-variance space under the assumptions of normality and constant risk aversion are linear, implying that the efficient set of prospects must lie on a convex frontier.

Finally, consider the class of all risk averse utility functions in which local risk aversion for any particular utility function cannot vary more than r_v. Conditions (a) and (b) of Theorem 5 imply that any prospects in the dotted area will be dominated. Condition (c) reduces to:

$$2\,(M_i - M_j)/(V_i - V_j) - 2\,(M_j - M_k)/(V_j - V_k) > r_v.$$

That is, if the slope of the line from a_j to a_i exceeds the slope from a_k to a_j by $r_v/2$, then a_j is dominated. Consequently, any prospects which fall in the crossed area in Figure 1 will be dominated. For a U_{rv} class, the area of dominated prospects includes the dotted and the crossed area.

7. Concluding Remarks

This paper has presented new results in efficiency analysis by defining new classes of utility functions. The utility classes proposed here are likely to be of more relevance to an efficiency analysis which relates to a group of decision makers than to an analysis for a single decision maker whose utility function has not been precisely defined. In the latter case, it is difficult to see situations in which more would be known about the variation in a risk parameter than is contained in the upper and lower risk bounds. But for the former case, one may frequently be confident of the type and variation of the risk parameter function for each decision maker, yet may need to acknowledge the differences between decision makers.

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