A Dynamic Nutrient Carryover Model for Pastoral Soils and its Application to Optimising Fertiliser Allocation to Several Blocks with a Cost Constraint

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A dynamical model of soil phosphorus carryover in grazed pasture provides the basis for bioeconomic optimization of fertiliser rates. A linear constraint is introduced to optimise the allocation of limited funds to fertiliser on several farm blocks, each of which represents a different land and/or stock class. The optimal constrained maintenance application of fertiliser to each block is calculated, and an heuristic approach to this equilibrium is suggested. In some cases this involves withholding of fertiliser from unresponsive blocks. The dynamical model and economic optimisation method have been implemented in a commercial fertiliser planning decision support tool Outlook™

1. Introduction

Fertiliser application is the single most significant input to pastoral farms in New Zealand. Farmers must attempt to maximise the benefits of increased production over the costs of fertiliser application. Minimising fertiliser run-off into waterways is an important additional consideration. However, the problem of optimising fertiliser economics is complicated because some proportion of soil nutrient is carried over from one year to the next, so that investment in fertiliser at a point in time returns a stream of benefits for several years into the future (Godden and Helyar). In addition, farm economic planning may have to consider the costs and contributions from several blocks with different soil and/or stock characteristics. Finance available to be spent on fertiliser in any given year may also not be unlimited. All of these factors make the budgeting of fertiliser expenditure an important and complicated task, and one which has received much attention from economists (Kennedy *et al.,* Kennedy 1981, 1986a,b, Godden and Helyar).

Kennedy *et al.,* used dynamic programming to calculate the optimal long-term fertiliser application in a cropping system, assuming that a constant proportion of fertiliser was carried over into the following year. Other models have extended this to consider the carryover of nutrients for several years (Godden and Helyar, Kennedy 1986a,b). These studies have focused on the economic aspects of revenue and cost prices instead of the soil processes affecting annual cycling of nutrients, and the empirical carryover functions they have used are a considerable simplification of the soil processes affecting retention and loss of fertiliser nutrients. Since many of the processes affecting nutrient cycling in fertilised soils are well understood, a mechanistic approach to modelling carryover is preferred.

This paper describes a method to calculate optimal fertiliser application to several grazed blocks where nutrient cycling on each block is described by a simple dynamical model (which is due to Metherell). The bio-physical and economic models are first described. The optimal fertility level with a financial constraint is then calculated using a discrete optimal control formulation (Clark). An heuristic is proposed to provide fertiliser recommendations for the initial years until the optimal level is reached. This is then illustrated with an example.

2. A Phosphate Cycling Model for Grazed Pasture

The use of dynamical models in describing the dynamics of biological and bio-physical systems is well established (Edelstein-Keshet, Clark, Woodward).

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They offer a number of advantages. Firstly, dynamical models allow mechanistic modelling of the interactions between the components of a system. In the case of fertiliser economics, for example, this means that an understanding of soil system mechanisms may be used to predict fertiliser carryover. Secondly, the mathematical analysis of dynamical systems models yields insights into the stability properties of the system (Edelstein-Keshet, Woodward). Thirdly, the mathematical formulation facilitates bioeconomic optimization.

As part of the development of a commercial fertiliser decision support program called Outlook™, a mechanistic model describing the dynamics of soil phosphate (P) cycling in New Zealand pastoral soils has been developed (Metherell, Metherell *et al.*). Phosphorus has been chosen because of its relatively high cost, wide use, and great quantity applied, but it is intended that future models will incorporate sulphur and potassium, the other two main long term nutrients. Metherell's model describes the carryover of soil phosphate on block i from year t to year t+1 using the difference equation:

$$(1)\quad \underbrace{P_i(t+1) - P_i(t)}_{\text{Soil P Accumulation}} = \underbrace{P_{Si}}_{\text{Slow Release P}} + \underbrace{F_i(t)}_{\text{Fertiliser}} - \underbrace{\beta_i\,(P_i(t) + F_i(t))}_{\text{Soil Losses}} - \underbrace{\gamma_i\, s_i\left(1 - e^{-k_i (P_i(t) + F_i(t))}\right)}_{\text{Animal Losses}}$$

where

$P_i(t)$ is the soil P (kgP/ha) in block i at the start of year t, prior to fertiliser application,

P_{Si} is the annual rate of release of slow release P in the soil (kgP/ha),

$F_i(t)$ is the rate of fertiliser application (kgP/ha) on block i at the start of year t,

β_i is the proportion of soil P lost annually to soil processes (the "soil loss"),

γ_i is the rate of soil P lost per stock unit per year (the "animal loss"),

s_i is the potential stocking rate on block i, and

k_i is the calibration coefficient for the relative yield to soil P response curve.

In this model, soil P represents the pool of phosphorous compounds in the soil which are available for use by plants, rather than the total soil P content, and all processes are assumed to operate on this pool of plant-available phosphorous. At the beginning of year t we have $P_i(t)$ units (kgP/ha) of "plant-available" phosphorus in the soil (soil P), to which $F_i(t)$ units (kgP/ha) of fertiliser phosphorous are then added. From this new level of soil P the relative yield for year t is calculated,

$$(2)\quad RY_i(t) = 1 - e^{-k_i (P_i(t) + F_i(t))}$$

as is the loss of P to soil processes (such as immobilisation by soil organisms and roots) in year t,

$$(3)\quad \beta_i\,(P_i(t) + F_i(t))$$

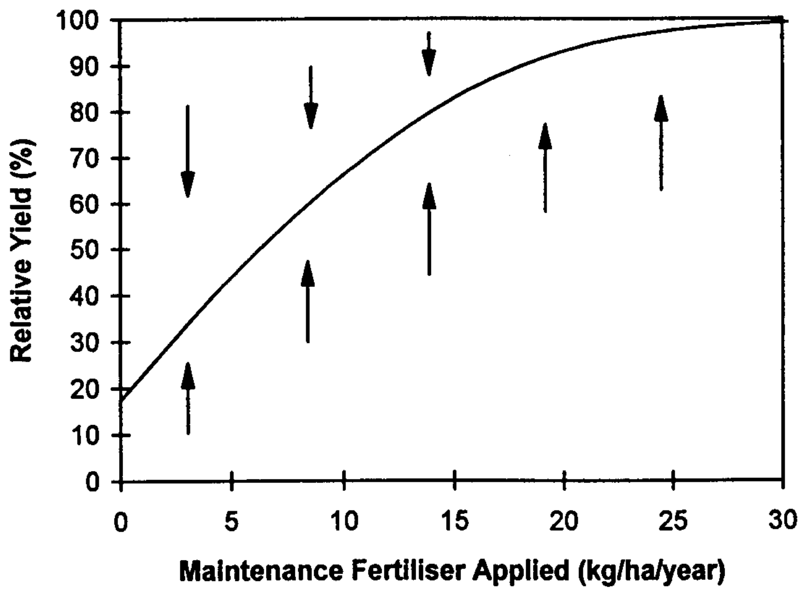
Relative yield (Equation 2) is a fraction which describes the depression of pasture production below potential due to a limiting availability of plant-available soil P, and follows a Mitscherlich diminishing returns curve. The stocking rate (stock units per hectare) and the pasture yield (and thus relative yield) are assumed to be in constant ratio. This represents a fixed efficiency of utilisation of herbage on each block. Therefore as soil P increases, and relative yield increases, so does stocking rate:

$$(4)\quad SR_i(t) = \frac{SR_i(0)}{RY_i(0)} RY_i(t) = s_i\left(1 - e^{-k_i (P_i(t) + F_i(t))}\right)$$

where the ratio of initial stocking rate $SR_i(0)$ to initial relative yield $RY_i(0)$ is the potential stocking rate on that block and is denoted by s_i. Losses of soil P in the model due to animals (through transfer and through incorporation into products such as meat, wool, or milk) are assumed to be proportional to the stocking rate with proportionality constant γ_i (see Equation 1). Equation 1 also includes a constant term, P_{Si}, which represents release of phosphorus into the soil from non-plant-available sources such as apatite (phosphate bearing rocks) and chemical mineralisation of organic phosphorous compounds. This dynamic model (Equation 1) is assumed to hold across a range of sites, each block of land on a farm being characterised by a different set of soil and animal parameters.

Thus the carryover of soil nutrient into the following year is a non-linear function of the initial soil P and the

Figure 1: Steady State Relative Yield for Different Rates of Fertiliser Applied to Block 1 (Table 1)



fertiliser applied. Figure 1 shows the stability analysis for this model (see Edelstein-Keshet) following the example of Godden and Helyar. For a given maintenance application of fertiliser F_i there is a "steady state" (equilibrium) level of soil P (and thus relative yield) which makes the right-hand side of Equation 1 equal to zero. The arrows on Figure 1 indicate the direction of year-to-year nutrient accumulation or loss if the current relative yield is not at the equilibrium level. Equally, this graph shows the maintenance fertiliser requirement at any particular level of relative yield.

Kennedy (1986a) showed that, dependent on the type of crop response function, including fertiliser carryover in the economic analysis can have a significant impact on the estimated optimum rate of application, and on net profitability. Therefore, carryover is an important aspect of the fertiliser problem. Previous economic analyses of fertiliser carryover have focused on crop farming, and have used empirical single- or multi- period carryover functions (e.g. Godden and Helyar, Kennedy 1986a), the simplest of which assumes that a proportion V_i of fertiliser carries over from year t to year t+1 in the soil (Kennedy *et al.*), so that:

(5) $P_i(t+1) = V_i(t)\,[P_i(t) + F_i(t)]$

Equation 5 is an empirical model of fertiliser carryover which plays the same role in the economic analysis as Metherell's more mechanistic description of soil phosphorus dynamics (Equation 1). The key differences are: firstly, that the Metherell model explicitly accounts for the fate of all nutrient entering the system; secondly, it includes a contribution of nutrient from slow release soil processes (the parameter P_{Si}); and thirdly, the Metherell model also considers additional losses due to the level of production (i.e. stocking rate) through the parameter γ_i. By modelling the soil mechanisms which effect nutrient cycling, we eliminate the need for a non-mechanistic carryover function. We will show how the dynamic model may be used to carry out an economic cost-benefit analysis of fertiliser application with carryover into subsequent years.

3. The Discounted Profit Function for N Blocks

The aim in simple one-year fertiliser analysis is to maximise the excess of revenue from animal products and sales over the fertiliser costs in that year. This becomes more complicated when the farm consists of several blocks and because fertiliser applied in the present carries over to benefit revenue for several years into the future (Godden and Helyar).

One method of estimating the present value of future prices paid and received is to apply a time discounting factor (Kennedy *et al.*). We assume that fertiliser expenditure is deducted at the beginning of each year, and that revenue from animal sales and products is received at the end of that year. When summed across

N blocks and over T years, the net present value (NPV) is:

$$(6)\quad NPV = \sum_{t=1}^{T} \alpha^{t-1} \sum_{i=1}^{N} \Big[\overbrace{\alpha r_i A_i s_i \left(1 - e^{-k_i (P_i(t) + F_i(t))}\right)}^{\text{Revenue}} - \overbrace{c_i A_i F_i(t)}^{\text{Cost}} \Big]$$

where

T is the number of time periods (years),

α is the discrete time preference discount factor, = 1/(1+discount rate),

N is the number of blocks on the farm,

r_i is the gross margin per stock unit per year ($/SU/yr),

A_i is the land area of block i (hectares), and

c_i is the cost per kgP of fertiliser applied to block i ($/kgP).

Note that as well as possibly different soil and animal characteristics, blocks also may have different costs and gross margins, reflecting the different costs of fertiliser application on those blocks and the different animal classes being run. The gross margin value in Equation 6 is assumed to consist of the gross revenue per stock unit minus the variable costs, which include interest paid on stock capital.

When used in conjunction with the bio-physical model for fertiliser carryover (Equation 1), Equation 6 provides the NPV of adopting a given fertiliser strategy.

4. Constrained Optimisation

Our aim is to choose the fertiliser application rates $F_i(t)$ in each block in each year to maximise the NPV (Equation 6), subject to certain constraints on the fertiliser application rates. For instance, calculations may suggest a high expenditure on fertiliser to maximise profits. However, other factors such as weather and changing markets may mitigate against the full return being received from this investment. A simple method for managing this risk is for the farmer to specify a "cap" on the amount of money he or she is willing to invest in fertiliser each year. The size of this cap will be determined by the farmer's available credit and cash, as well as by his or her attitude to risk. In many cases we find that almost maximal profit may be still obtained with a significantly lower outlay of investment (Kennedy 1986a). The cap constraint across N blocks is:

$$(7)\quad D - \sum_{i=1}^{N} c_i A_i F_i \geq 0$$

where D is the maximum expenditure on fertiliser in any one year ($).

Experiments studying pasture responses at different rates of fertiliser have tended to examine fertiliser rates in common use. For this reason it is rare for extremes of fertiliser rate to be applied and the yield responses in this situation are therefore less well understood. Furthermore, recent emphasis on the polluting effects of excess fertiliser running off into waterways has shown that the proportion of nutrients lost in run-off rises rapidly with the rate of application. Therefore, we wish to specify an upper limit to the fertiliser rate, beyond which the P cycling model (Equation 1) may not be valid and where runoff may be considered excessive. This upper limit is around 120 kgP per hectare (1.3 tonnes of superphosphate). The rate constraint is expressed mathematically as (recalling that $F_i(t)$ must also be non-negative):

$$(8)\quad 0 \leq F_i(t) \leq F_i^{max}$$

Thus there are both rate and financial constraints imposed on the amount of fertiliser applied in each block in each year.

Most previous analyses have used the method of dynamic programming to find the optimal fertiliser rate in each period and the long term optimal equilibrium level (Kennedy *et al.,* Kennedy 1981, 1986a,b. Godden and Helyar). Here, however, as we intend to use a dynamic model (Equation 1) to calculate the cycling of nutrient from year to year, it is more appropriate to use an optimal control formulation (Clark), which concerns the optimal strategy for controlling a dynamically evolving system where there is some regular input ("control variable"). In this case we wish to use fertiliser application to control the evolution of the soil nutrient system so as to maximise NPV over the whole farm. Nevertheless, as Kennedy (1986b) points out, the methods of dynamic programming and optimal control are mathematically equivalent in this application.

The present value Hamiltonian for this optimal control problem (see Clark) is constructed by appending the state equation (Equation 1) to the year-t net revenue, using adjoint multipliers $\lambda_i(t)$ (for conciseness we omit the year if no ambiguity exists, e.g. $P_i = P_i(t)$):

$$H(t) = \sum_{i=1}^{N} \alpha^{t-1} \left[\alpha\, r_i\, A_i\, s_i \left(1 - e^{-k_i (P_i + F_i)}\right) - c_i\, A_i\, F_i \right]$$

$$(9) \qquad + \lambda_i \left[P_{Si} + F_i - \beta_i (P_i + F_i) - \gamma_i\, s_i \left(1 - e^{-k_i (P_i + F_i)}\right) \right]$$

The Pontryagin maximum principle (see Clark) states that to maximise the objective function (NPV) we must choose the control variables $F_i(t)$, subject to the cap and rate constraints (Equations 7 and 8), to maximise H(t) at every point in time. In addition we must satisfy the discrete "adjoint equation" (Clark p237):

$$(10) \qquad -\frac{\partial H(t)}{\partial P_i(t)} = \lambda_i(t) - \lambda_i(t-1)$$

When the rate constraint (Equation 8) is "slack" so that $0 < F_i < F_i^{max}$, the Hamiltonian is maximised at a local maximum, constrained by the cap constraint. This can be achieved by forming a Lagrangian, so that the constrained local maximum point satisfies

$$(11) \qquad \frac{\partial}{\partial F_i(t)} \left\{ H(t) + \mu \alpha^{t-1} \left(D - \sum_i c_i\, A_i\, F_i \right) \right\} = 0$$

in each year, where the cap constraint has been appended to the Hamiltonian using a Lagrange multiplier (μ). The value of μ is interpreted as the (undiscounted) marginal value of increasing the financial constraint D by one unit. A special case occurs when $\mu=0$, where spending an extra dollar earns no extra profit —this indicates that the cap constraint is slack and we are at the *unconstrained* optimum.

In the case where the unconstrained optimum is infeasible, the value of μ must be found so that the total fertiliser cost is equal to D. If, for example, $\mu=0.5$ at this point, then for every extra dollar spent on fertiliser (over and above D) in year t we expect the NPV to increase by \$0.50 times α^{t-1}.

5. Equilibrium Solution

In the long term we expect the optimum soil P level on each block to approach some equilibrium level (Figure 1), which may be constrained by the cap constraint (Equation 7). At equilibrium $P_i(t+1)=P_i$ in Equation 9 and the maximum principle and adjoint equations (Equations 11 and 10 respectively) may be solved simultaneously to give the optimal post-application fertility level $\{P_i+F_i\}$ (see Appendix for details):

$$(12) \qquad \{P_i + F_i\}^{optimal} = -\frac{1}{k_i} \ln \left[\frac{\beta_i - 1 + \frac{1}{\alpha}}{s_i\, k_i \left(\frac{r_i}{c_i (1+\mu)} - \gamma_i \right)} \right]$$

Setting $\mu=0$ in Equation 12 gives the unconstrained optimal soil P level on block i to maximise the net present value on that block. In the absence of constraints on the annual fertiliser application rate, the optimal strategy is to apply a capital application of fertiliser to achieve this equilibrium level in as short a time as possible (i.e. after one year) and to maintain that level thereafter.

The capital fertiliser application required to reach this optimal level in one year is found by subtracting the current soil P level:

$$(13) \qquad F_i^{capital} = \{P_i + F_i\}^{optimal} - P_i^{current}$$

The annual maintenance fertiliser application then needed to maintain this optimal level is (rearranging Equation 1 when P_i is in equilibrium, and substituting the optimal soil P level from Equation 12):

$$(14) \qquad F_i^{maintenance} = -P_{Si} + \beta_i \{P_i + F_i\}^{optimal} + \gamma_i\, s_i \left(1 - e^{-k_i (P_i + F_i)^{optimal}}\right)$$

However, this is of no use to us if the financial or rate constraints prevent us from applying the capital or maintenance rates of fertiliser specified in Equations 13 and 14.

The constrained equilibrium is not difficult to calculate numerically by increasing μ in Equation 12 from zero until the recommended maintenance applications (Equation 14) satisfy the financial constraint exactly — i.e. to find μ so that:

$$(15) \qquad D - \sum_{i=1}^{N} c_i\, A_i\, F_i^{maintenance} = 0$$

The optimal strategy of capital applications for reaching this constrained optimum starting from given initial conditions of soil P on each block $P_i(1)$ is more difficult to calculate. The question is: how may we raise the fertility level over a period of several years while still honouring the financial constraint? We also wish to do this in the most profitable manner. This problem is called finding the optimal approach path (to the optimal equilibrium).

6. An Approximately Optimal Approach Path

The optimal approach path is calculated from the Hamiltonian (Equation 9) by solving Equations 11 and 10 (i.e. the maximum principle and the adjoint equation). However this produces a condition at the start (because the $P_i(1)$ are given) and another condition at the end (the $\lambda_i(T)$ must go to zero), so that finding the optimal approach path is a (discrete) two-point boundary value problem. Although there are numerical methods for tackling this kind of problem (see Atkinson), we wish to find a simpler method.

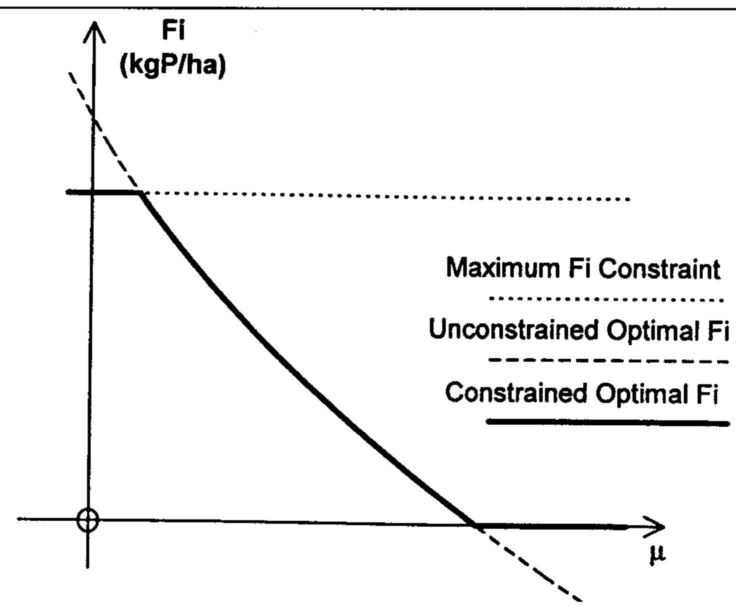
An approximately optimal approach path to the long run optimum works as follows: the total cost of moving from the known current soil P_i to some constrained optimal equilibrium level is (combining Equations 12 and 13):

The portion in the braces {} is the capital fertiliser application on each block. This must be adjusted to satisfy both the rate and financial constraints. The rate constraint (Equation 8) is satisfied by truncating the value in the braces to lie in the range $0<F_i<F_i^{max}$, as illustrated in Figure 2. If the unconstrained optimum is infeasible (i.e. the fertiliser cost is greater than D when $\mu = 0$) then the financial cap constraint (Equation 7) must also be satisfied by numerically adjusting μ until the right hand side of Equation 16 exactly equals D, i.e. the constrained "optimal" soil P value comes down to a level which we can afford to move to in one year with a single capital fertiliser application. The value in the braces in Equation 16 is then taken for $F_i(t)$.

This procedure is repeated each year, and the new soil P level, $P_i(t+1)$, is calculated from the dynamics equation (Equation 1). It may take several years to approach the constrained optimal equilibrium in this manner. However, the financial constraint is satisfied in every year, and the allocation of funds for fertiliser in every year is approximately optimal, as we shall show by an example.

$$(16)\ \sum_{i=1}^{N} c_i A_i F_i = \sum_{i=1}^{N} c_i A_i \left\{ -\frac{1}{k_i} \ln\left[\frac{\beta_i - 1 + \frac{1}{\alpha}}{s_i k_i \left(\frac{r_i}{c_i (1+\mu)} - \gamma_i \right)} \right] - P_i^{current} \right\}$$

Figure 2: Incorporating the Rate Constraint $0 \le F_i \le F_i^{max}$



Example

Table 1 contains details of the three land blocks of a hill country sheep-beef farm in the Waikato region of New Zealand. The block characteristics include slope, soil type, initial soil P, stocking rate and relative yield, gross margins and fertiliser costs. Additional information needed to calculate the constrained optimum fertiliser policy are the time preference discount factor α and the maximum annual fertiliser expenditure D. A discount rate of 10 per cent per annum gives $\alpha=1/(1+0.1)=0.91$ for this example.

Using the initial soil P and stocking rate from Table 1 and rearranging Equation 1 enables us to calculate the maintenance strategy, which is defined as applying sufficient fertiliser to maintain the existing fertility level on each block:

$$(17) \quad F_i^{\text{maintenance}} = \frac{-Ps_i + \beta_i P_i(1) + \gamma_i SR_i(0)}{1-\beta_i}$$

The maintenance policy for this farm consists of annual applications of 17, 20, and 17 kg P per hectare respectively for the three blocks and costs \$7,241 per annum. The annual net revenue over fertiliser costs at maintenance is \$58,400. The discounted NPV (Equation 6) for the maintenance strategy over the first ten years (T=10) is \$309,902.

The unconstrained optimal solution (using Equation 12 with μ =0) consists of a capital application to improve the fertility in blocks 2 and 3 in the first year, but almost two years withholding from block 1, and an application of 15, 23, and 21 kg P per hectare thereafter to maintain the new equilibrium fertility level, the "optimal maintenance" (Table 2). After the initial period of adjustment in the first two years, this strategy costs \$7,787 per year, and earns an undiscounted net revenue over costs of \$60,653 per annum. The NPV over ten years is \$311,790.

Setting the cap on annual fertiliser expenditure at D=\$7,241, the same expenditure as under the maintenance strategy, allows us use the approximately optimal method to assess the opportunity to achieve greater returns while spending the same amount of money as required for maintenance. In the constrained solution shown in Table 3, fertiliser is withheld from block 1 for the first three years, with small increases in application to blocks 2 and 3. Table 3 also shows the values of μ required (see comments on Equation 11) to meet the constraint (Equation 7). The equilibrium levels reached after the initial four years of adjustment require 13, 21, and 20 kg P per hectare respectively to maintain, and earn around \$59,100 net per annum. The ten-year NPV is \$313,832.

Table 1: Parameters for a Typical Sheep/Beef Farm Consisting of 3 Blocks of Land

	Parameter	Block 1	Block 2	Block 3	Units
Area	A_i	50	30	50	ha
Slope		Steep	Easy	Easy	
Soil Type		Sedimentary	Sedimentary	Volcanic	
Soil Loss	β_i	0.04	0.04	0.05	yr^{-1}
Initial Olsen P		8	9	9	
Initial Soil P	$P_i(1)$	200	214	176	$kgP\ ha^{-1}$
Slow Release P	Ps_i	3	3	3	$kgP\ ha^{-1}\ yr^{-1}$
Initial Stocking Rate	$SR_i(0)$	10	18	14	$SU\ ha^{-1}$
Initial Relative Yield	$RY_i(0)$	87%	89%	84%	
$SR_i(0)$: $RY_i(0)$ Ratio	s_i	11.5	20.2	16.7	$SU\ ha^{-1}$
Animal Loss	γ_i	1.08	0.75	0.75	$kgP\ SU^{-1}\ yr^{-1}$
Soil Response Calibration	k_i	0.0095	0.0095	0.0095	$ha\ kgP^{-1}$
Gross Margin	r_i	30	35	35	$\$\ SU^{-1}\ yr^{-1}$
Fertiliser Cost	c_i	3.50	3.00	3.00	$\$\ kgP^{-1}$
Maximum Fertiliser Rate	F_i^{max}	120	120	120	$kgP\ ha^{-1}$

Table 2: Optimal Unconstrained Ten-year Strategy for the Farm Specified in Table 1

Year	Block 1 kgP/ha	Block 2 kgP/ha	Block 3 kgP/ha	Total Fert Cost ($)	Revenue ($)
1	-	71	81	18,555	61,031
2	1	23	21	5,410	60,653
3	15	23	21	7,787	60,653
4	15	23	21	7,787	60,653
5	15	23	21	7,787	60,653
6	15	23	21	7,787	60,653
7	15	23	21	7,787	60,653
8	15	23	21	7,787	60,653
9	15	23	21	7,787	60,653
10	15	23	21	7,787	60,653

Table 3: Optimal Constrained Ten-year Strategy with a Fertiliser Expenditure Cap of $7,241 per annum

Year	Block 1 kgP/ha	Block 2 kgP/ha	Block 3 kgP/ha	μ	Revenue ($)
1	-	24	34	0.51	58,800
2	-	31	30	0.37	59,015
3	-	31	30	0.25	59,100
4	7	26	24	0.20	59,098
5	13	21	20	0.20	59,095
6	13	21	20	0.20	59,091
7	13	21	20	0.20	59,088
8	13	21	20	0.20	59,086
9	13	21	20	0.20	59,083
10	13	21	20	0.20	59,081

The ten-year NPV values are similar for all three strategies. The improvement made by the constrained $7,241 strategy over the maintenance strategy suggests that less money should be spent on block 1 of the farm, since the other two blocks are more responsive. Nevertheless, the differences in NPV between the strategies are small, reflecting the high initial fertility of the farm.

In the first ten years, the unconstrained policy achieves an NPV only 0.6 per cent better than that of the maintenance strategy. This initially mediocre performance is due to the high cost of the initial capital fertiliser application, and in the long-term, the unconstrained optimal strategy will be superior to every alternative strategy. However, a ten-year planning horizon is likely to be more realistic from a farmer's point of view, and incurring the cost of large capital fertiliser applications in a single year may also be undesirable. Therefore it is of interest to know whether almost-optimal NPV may be achieved from fertiliser policies that do not require such high initial financial outlay.

From Table 2, the cost of maintaining the unconstrained optimal equilibrium fertility level is $7,787 per annum, and the unconstrained capital application to reach this equilibrium is $18,555. Therefore, since D is the maximum expenditure in any year, if $7,787<D<$18,555 then the approach to the optimal equilibrium will be constrained, and several years of capital applications will be required to reach the optimal equilibrium. Furthermore, if D<$7,787 then it will not be possible to maintain the optimal equilibrium at all, even if it were possible to reach it.

Figure 3: Long-term (100 years) Constrained Optimum NPV for Different Values of the Cap Constraint

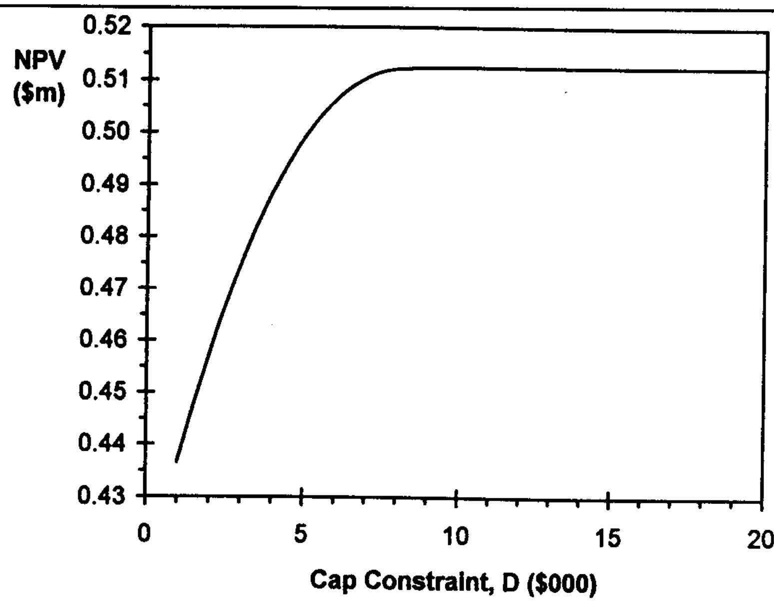


Figure 3 shows how the long-term (100 year) optimal NPV varies in response to the cap constraint. When $7,787<D<$18,555, although capital applications of fertiliser are constrained, NPV is not noticeably affected. When D<$7,787, insufficient finance is available to apply the optimal maintenance, and in this case, NPV declines sharply. The graph implies that constraining the approach path to this equilibrium has very little effect on the long term NPV, so any approach path will be equally good, including the approximately optimal approach we have suggested in this paper. However, if annual expenditure is constrained to such an extent that the optimal equilibrium cannot be maintained, this will significantly decrease long term NPV. Therefore, calculating the optimal equilibrium is the key to maximising NPV in the long term. Furthermore, spending less money on capital fertiliser applications avoids the risk of overcapitalising.

8. Software Implementation

The biological model (Equation 1) has been developed to be the engine of a decision support computer programme Outlook™, which is designed to assist farmers' fertiliser planning by predicting the long-term (10 years plus) biological and economic outcomes of different fertiliser policies based on the current scientific knowledge of phosphorus dynamics in New Zealand soils. Outlook™ gives non-technical users access to this information and to sophisticated mathematical analyses, and also mediates the process by selecting appropriate data from a scientific database, performing the mathematical analyses, and displaying the results in a practical format.

The constrained optimisation calculation described here is an important part of this software both because fertiliser is relatively expensive to apply and because farmers are often constrained by cash flow. Specifying a maximum expenditure in each year also provides a simple means for introducing risk management into fertiliser planning.

9. Conclusion

This paper illustrates the use of optimal control theory to control a biological system in order to maximise NPV when there are constraints. This required a dynamical model of phosphorus cycling in pastoral soils. These methods are very general and are readily extended to optimal control of other agricultural systems.

The solutions for an example farm system of three blocks show that it is sometimes optimal to withhold fertiliser on part of the farm in the short term, because other blocks may give higher returns. In addition, it has been shown that it is the long-term maintenance fertiliser level that determines long-term profitability, and that the initial pattern of capital applications is not significant in the long term. However, this analysis considered only phosphate fertiliser economics; other inputs (such as re-sowing) may also be important. As

Outlook™ is upgraded, similar dynamical models of sulphur and potassium cycling will be developed for finding optimal applications of multi-nutrient fertilisers for pastoral farming systems.

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Appendix: Calculation of Equilibrium Solution

At equilibrium, P_i and F_i are constant from year to year. Since F_i is constant, the undiscounted marginal value of the constraint should be constant too. That is, μ is constant with t. When $0<F_i<F_i max$, the maximum principle (incorporating the cap constraint, see Equation 11) becomes:

$$\text{(A1)}\qquad \alpha^{t} r_i A_i s_i k_i e^{-k_i(P_i+F_i)} - \alpha^{t-1} c_i A_i + \lambda_i - \lambda_i \beta_i - \lambda_i \gamma_i s_i k_i e^{-k_i(P_i+F_i)} - \mu \alpha^{t-1} c_i A_i = 0$$

Rearranging Equation A1 gives an expression for $\lambda_i(t)$,

$$\text{(A2)}\qquad \lambda_i(t) = \alpha^{t-1} \left\{ c_i A_i \frac{(1+\mu) - \alpha \frac{r_i}{c_i} s_i k_i e^{-k_i(P_i+F_i)}}{1 - \beta_i - \gamma_i s_i k_i e^{-k_i(P_i+F_i)}} \right\}$$

At equilibrium the portion in the braces { } in Equation A2 is constant, and it is only the factor α^{t-1} that changes with time. So

$$\text{(A3)}\qquad \lambda_i(t-1) = \frac{\lambda_i(t)}{\alpha}$$

Substituting this into the discrete adjoint equation (Equation 10) and evaluating the partial derivative gives,

$$\text{(A4)}\qquad -\alpha^{t} r_i A_i s_i k_i e^{-k_i(P_i+F_i)} + \lambda_i \beta_i + \lambda_i \gamma_i s_i k_i e^{-k_i(P_i+F_i)} = \lambda_i(t) - \frac{\lambda_i(t)}{\alpha}$$

and then after substituting in Equation A2 for $\lambda_i(t)$ we obtain an expression for the optimal value of $\{P_i+F_i\}$:

$$\text{(A5)}\qquad \{P_i + F_i\}^{\text{optimal}} = -\frac{1}{k_i} \ln \left[\frac{\beta_i - 1 + \frac{1}{\alpha}}{s_i k_i \left(\frac{r_i}{c_i(1+\mu)} \right) - \gamma_i} \right]$$