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A Note on the Use of a Logarithmic Time Trend

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It is shown that parameter estimates in a regression with a logarithmic time trend are not invariant to the choice of the starting point of that time trend. Possible solutions to this problem are discussed and applied to data from a recent article in this *Review* where a logarithmic time trend was included. It is suggested that the appropriate course is either to estimate the starting date directly or to adopt a functional form invariant to the choice of starting date.

1. Introduction

In applied economic analysis, a time trend is often included in a regression specification as a proxy variable to reflect changes in technology through time or as a catch-all for all omitted variables that have a time component but cannot be easily measured, such as changes in tastes or educational standards.

While a linear time trend in a regression model causes few estimation problems, it is shown here that parameters estimated in a model with a logarithmic time trend are not invariant to the choice of the starting point of the time trend. That is, if a variable $\ln(t)$ is included in a regression model, where t is a time trend, then the estimates of the coefficients of all other variables will be different if t is defined as $1, 2, 3, \ldots, T$ rather than $1961, 1962, 1963, \ldots, 1960 + T$ for instance. The invariance property for coefficients other than the constant and that of the time trend is possessed only by linear and exponential time trends, and by a linear combination of the two. A proof of this is given in section 2 of the paper.

The logarithmic time trend arises quite naturally in the context of the translog functional form with time series data. This approach has recently gained popularity for the estimation of cost and production functions because a particular functional form is not imposed. In a recent article in this *Review*, McKay, Lawrence and Vlastuin (1980) (hereafter MLV) estimate input share demand functions, derived from a translog cost function with a logarithmic time trend (see also McKay, Lawrence and Vlastuin 1982). In section 3 of this paper, the effect of the invariance property is demonstrated using the model and data in MLV and a number of solutions to the problem are discussed and applied.

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2. The Invariance Property

Consider first a regression model of the following form:

(1) $Y_t = \alpha + \beta' X_t + \gamma f(T_t) + u_t$ where:

Y_t is the dependent variable

X_t is a row vector of observations on the independent variables

T_t is a linear time trend and $f(T_t)$ is some transformation

u_t is a random error term with zero mean and constant variance

α, β' and γ are parameters to be estimated.

Since the choice of starting date for T_t is normally arbitrary, it is clearly desirable that the coefficients should be independent of this choice. The choice of starting date should only affect coefficients α and γ.

Taking $T_t = t + \delta$, it may be easily shown that the invariance condition will be satisfied in general if and only if:

(2) $f(t + \delta) = vf(t) + w$

for some v and w, that is, if a change in start date is equivalent to a linear transformation of the time trend variable. A non-linear transformation of the time trend variable will normally affect all of the estimated coefficients.

Equation (2) may be regarded as a difference equation with the general solution:

(3) $f(t) = \exp(t(\log a - \log \delta)) + tb/\delta + c$

for any a, b and c. See Allen (1959, ch. 6).

Thus, the invariance property is possessed only by linear time trends (i.e. $a = \delta$) and exponential time trends, (i.e. $b = 0$) and by linear combinations of the two (i.e. $a \neq \delta, b \neq 0$). Furthermore, it may be observed that for the exponential time trend, the coefficient β' in (1) is also invariant with respect to the starting date.

The main advantage which may be claimed for the choice of a logarithmic trend with starting date 1 relates to the behaviour of dependent variables which should lie within the interval (0,1) because, with other factors held constant, these variables may fall outside the interval (0,1) for sufficiently large T. Since a logarithmic trend will ultimately be dominated by either a linear or an exponential trend, it may be argued that this formulation is in some sense "better behaved".

There are a number of possible responses to this. First, it may be noted that while a linear trend will ultimately dominate a logarithmic one, the question of which will leave the interval (0,1) first depends on the coefficient estimates. More importantly, it must be recognized that the logarithmic trend breaks down completely in back projection since $\log(0) = -\infty$. Back projection is an important means of testing the validity of estimated models of this kind.

Finally, it may be noted that, if this issue is of major concern, it would seem preferable to use a transformation such as the logistic, which yields a variable which always lies in the interval (0,1).

A logarithmic time trend would be appropriate in cases where the series had a natural start date, for example when the series modelled the diffusion of a particular invention. In cases of this kind, back projection is clearly inappropriate. However, in most cases, the time period over which a relationship is modelled depends on data availability rather than on the inherent properties of the relationship in question.

In the specific case of a translog functional form, the general specification is:

$$(4)\ \ln y = a + \sum_{i=1}^{n} b_i \ln X_i + b_t f(t) + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} \ln X_i \ln X_j + \sum_{i=1}^{n} b_{it} \ln X_i f(t) + \tfrac{1}{2} b_{tt} (f(t))^2$$

For this equation, the variance argument above must be modified to take account ot the cross-product terms $b_{it} \ln X_i f(t)$. Because of these terms, the coefficients b_i and b_{ij} will be invariant with respect to the starting date only for some v:

$$(5)\ f(t + \delta) = v\ t$$

i.e. if an exponential trend is used. If a linear trend is used, the coefficients b_{ij} will be independent of the choice of starting date but the coefficients b_i will not be.

3. An Example and Discussion

In MLV, a set of input share demand equations derived from a transcendental logarithmic cost function for the Australian sheep industry was estimated. The translog cost function was derived as:

$$(6)\ \ln C\,(w, y, T) = b_o + b_y \ln y + \sum_{i=1}^{n} b_{io} \ln w_i + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} \ln w_i \ln w_j + \sum_{i=1}^{n} b_{it} \ln w_i \ln T + b_t \ln T + b_{ty} \ln T \ln y + \tfrac{1}{2} b_{yy} (\ln y)^2 + \tfrac{1}{2} b_{tt} (\ln T)^2$$

where

w_i = input prices, $i = 1, \ldots, n$

y = output quantity

T = index of technology

and b's are parameter coefficients.

After deriving input share demand equations and imposing restrictions on the parameters to satisfy both linear homogeneity of the cost function and symmetry, MLV obtained this system of equations:

$$(7)\quad S_L = b_{LO} + b_{LL} \ln \frac{w_L}{w_M} + b_{LN} \ln \frac{w_N}{w_M} + b_{LV} \ln \frac{w_V}{w_M} + b_{LC} \ln \frac{w_C}{w_M} + b_{Lt} \ln T + e_1$$

$$(8)\quad S_N = b_{NO} + b_{LN} \ln \frac{w_L}{w_M} + b_{NN} \ln \frac{w_N}{w_M} + b_{NV} \ln \frac{w_V}{w_M} + b_{NC} \ln \frac{w_C}{w_M} + b_{Nt} \ln T + e_2$$

$$(9)\quad S_V = b_{VO} + b_{LV} \ln \frac{w_L}{w_M} + b_{VN} \ln \frac{w_N}{w_M} + b_{VV} \ln \frac{w_V}{w_M} + b_{VC} \ln \frac{w_C}{w_M} + b_{Vt} \ln T + e_3$$

$$(10)\quad S_C = b_{CO} + b_{LC} \ln \frac{w_L}{w_M} + b_{CN} \ln \frac{w_N}{w_M} + b_{CV} \ln \frac{w_V}{w_M} + b_{CC} \ln \frac{w_C}{w_M} + b_{Ct} \ln T + e_4$$

where subscripts, L, N, V, C and M refer to the inputs of labour, land, livestock, capital and materials and services respectively, and the S_i are the share of input i in total cost.

In order to estimate this system of equations, MLV used a time trend as a proxy for technology where $T = 1, 2, 3, \ldots, 25$. However, as MLV note (1982, p. 11) the parameter estimates are not invariant to the starting point of this time trend. In order to assess the dependence of estimates on the choice of starting date, the MLV model will be used as an example.

This system of equations was estimated using an iterative Zellner's procedure applied to MLV's data. The results for the labour shares equation (7)[1] under different specifications of the technology variable (T) are given in Table 1. Specification I has the technology variable defined as $T = 1, 2, 3, \ldots, 25$ whereas specification II has it defined as $T = 152, 153, \ldots, 176$ and specification III has it defined as $T = 1952, 1953, \ldots, 1976$. Specifications IV and V which are included in Table 1 are discussed below. As can be seen, parameter estimates[2] for specifications I to III differ quite markedly depending on the starting point of the time trend.

This problem arises in this case because the time trend does not have a natural "zero" or base. Other variables, for instance prices, do have a natural zero. Thus one solution to this problem would be to use an index of technology. Presumably if such an index existed it would have been used. Such an approach was adopted in a paper by White and Havlicek (1982), where an index based on research and extension expenditures was constructed for the U.S.

An alternative approach would be to estimate, as part of the model, an appropriate starting point for the time trend. That is, in equations (7) to (10) above, T is specified as $(t + t_0)$ where t_0 is to be jointly estimated and $t = 1, 2, 3, \ldots, 25$. When this was attempted for this system, no stable solution was obtained. The estimates obtained for a range of different starting values gave different solutions. This occurred because the logarithmic function is not sensitive to small changes in the argument as the value of the argument gets large.

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[1] The full set of parameter estimates are available from the authors. Other share equations give generally similar results.

[2] Our estimates for specification I differ slightly from those given in MLV due to different estimation procedures, associated with a change in computer packages. The differences are not significant.

A different approach is to choose a transformation of the time trend that is invariant to the starting point of the time trend. As shown in section 2, linear and exponential functions of time, or linear combinations of the two, are invariant with respect to the starting point.

The system of equations was re-estimated with T in equations (7) to (10) specified as $\exp(t)$ and $\exp(\exp(t-16))$ where $t = 1, 2, 3, \ldots, 25$. The results for the labour equation are given in Table 1 as specifications IV and V, respectively. Note that specification IV is equivalent to the inclusion of a linear time trend in equations (7) to (10) above[3], while specification V is equivalent to the inclusion of an exponential time trend, i.e. $\ln(T)$ has been replaced by t and $\exp(t)$, respectively.

It is also shown in section 2 that complete invariance to the choice of starting values only occurs when an exponential time trend is included. However, for many applications (e.g. the calculation of elasticities) only the coefficients b_{ij} and the estimated shares S_i are required. Both of these estimates are independent of the starting date for a linear trend.

Table 1: Restricted Estimates of the Parameter of the Labour Share Demand Function by Specification of the Technology Variable

Specification[c]	Parameters[a,b]						
	b_{LL}	b_{LN}	b_{LV}	b_{LC}	b_{LM}	b_{Lt}	b_{L0}
I	0.015	-0.042	-0.052	0.007	0.073	0.025	0.302
	(0.4)	(3.0)	(4.0)	(0.2)	(2.3)	(3.5)	(20.6)
II	0.176	-0.022	0.061	0.080	-0.173	1.30	6.82
	(5.9)	(1.5)	(-6.2)	(3.3)	(4.8)	(-8.4)	(8.8)
III	0.185	-0.023	-0.061	0.083	-0.184	15.7	119.9
	(6.0)	(-1.6)	(6.2)	(3.3)	(4.8)	(8.3)	(8.3)
IV	0.186	-0.024	-0.061	0.083	0.185	-0.008	0.328
	(6.0)	(-1.6)	(6.2)	(3.3)	(4.8)	(8.2)	(35.7)
V	0.121	0.090	-0.030	0.067	0.067	0.000002	0.255
	(3.1)	(6.5)	(-2.8)	(-3.1)	(1.0)	(0.5)	(79.4)

[a] The letters L, N, V, C and M refer to labour, land, livestock, capital and materials and services, respectively.

[b] Parametric restrictions imposed

$$\sum_{i=1}^{n} b_{i0} = 1,\ \sum_{j=1}^{n} b_{ij} = 0,\ b_{ij} = b_{ji},\ \sum_{i=1}^{n} b_{it} = 0 \text{ for all } i, j.$$

[c] Specifications for technology T in equations (7) to (10) are, for $t = 1, 2, 3, \ldots, 25$.

I t

II $t + 151$

III $t + 1951$

IV $\exp(t)$ i.e. $\ln T$ replaced by t

V $\exp(\exp(t-16))$, i.e. $\ln T$ replaced by $\exp(t-16)$

Note: Figures in parentheses are t-statistics.

[3] The linear time trend rather than a logarithmic time trend has been used before in a translog approximation by May and Denny (1979). They offer no explanation for this choice.

An examination of Table 1 shows that estimates for coefficients under specifications III and IV are quite similar. The similarity between the estimates from these specifications is due to the logarithmic transformation being highly linear for large values of T. For instance, the correlation coefficient between ln $(t + 1952)$ and t for $t = 1, 2, 3, \ldots, 25$ is 0.99999. Thus, in cases where estimation of the starting date of a logarithmic time trend does not yield a stable result, a linear trend may provide a good approximation.

Another observation from Table 1 is that estimates in specifications IV and V are quite different. Although specification V has the desirable invariance property described above, the interpretation of the technology variable in this specification is not appealing. This is because the technology variable would be $\exp(\exp(T))$; a very rapidly increasing trend.

It is for these reasons that the linear trend is the preferred specification in this case. In general, the log trend with an estimated start date and the linear trend should be compared. If the estimated start date is "large", or not obtainable, then the linear time trend should be used. Alternatively, with "low" values of the estimated start date, the specification with the estimated start date should be used.

The complete estimates for the model with a linear time trend are given in Table 2. Using these estimates, the Allen-Uzawa elasticities of input demand at mean prices (see Table 4) were recalculated. The estimates are presented with those of MLV in parentheses. As may be observed, the differences between the two sets of estimates are quite substantial. Indeed some of the estimates from specifications III and IV in Table 1 lie outside the confidence intervals estimated using specification I. At the very least, this indicates that care must be taken in interpreting confidence intervals.

Table 2: Restricted Estimates of the Coefficients of the Input Share Demand Functions with Specification IV

Dependent variable S_i	Parameters[a,b]						
	b_{iL}	b_{iN}	b_{iV}	b_{iC}	b_{iM}	b_{it}	b_{i0}
S_L	0.186	-0.024	-0.061	0.083	0.185	-0.008	0.328
	(6.0)	(1.6)	(-6.2)	(3.3)	(4.8)	(-8.2)	(35.7)
S_N	0.024	0.085	0.037	0.070	-0.094	-0.002	0.176
	(-1.6)	(4.0)	(2.6)	(3.8)	(-4.4)	(-2.2)	(18.3)
S_V	-0.061	0.037	0.107	-0.036	0.028	0.003	0.121
	(6.2)	(2.6)	(5.7)	(-3.3)	(2.2)	(3.6)	(10.8)
S_C	0.083	0.070	-0.036	-0.093	-0.023	-0.001	0.170
	(3.3)	(3.8)	(-3.3)	(-2.3)	(0.5)	(-0.7)	(15.8)
S_M	-0.185	0.094	0.028	-0.023	0.274	0.008	0.205
	(-4.8)	(-4.4)	(2.2)	(0.5)	(3.9)	(4.4)	(13.6)

[a,b] See Table 1.

Note: Figures in parentheses are t-statistics.

Table 3 contains some results of particular interest. MLV estimated the elasticity of substitution between labour and capital at 1.132 whereas previous studies have generally yielded estimates of less than unity. The linear specification yields an even higher estimate of 2.972. This result may be explained by noting that unlike previous studies, MLV restricted the definition of capital to plant and machinery. Many other costs normally included in the capital item are segregated into the materials and services category. The very high elasticity of substitution between labour and physical capital obtained with the linear specification is associated with a finding of complementarity between labour and materials and services.

Table 3: Estimated Allen–Uzawa Elasticities of Substitution at Mean Prices with Specification IV

	Labour	Land	Livestock	Capital	Materials and services
Labour		0.483 (-0.103)	-0.715 (-0.588)	2.97 (1.132)	1.97 (2.16)
Land			-0.399 (0.853)	3.22 (2.577)	0.999 (-0.333)
Livestock				-0.475 (0.028)	1.76 (1.670)
Capital					0.47 (1.80)

Note: Figures in parentheses are MLV estimates.

Table 4: Price Elasticities of Input Demand at Mean Prices with Specification IV

	Labour	Land	Livestock	Capital	Materials and services
Labour	0.002 (-0.675)	0.090 (0.018)	0.103 (-0.089)	0.508 (0.191)	0.496 (0.592)
Land	0.119 (0.023)	0.358 (-0.189)	0.058 (-0.129)	0.548 (0.435)	0.252 (0.091)
Livestock	-0.015 (-0.132)	0.074 (-0.154)	-0.113 (-0.177)	0.081 (0.005)	0.445 (0.458)
Capital	0.732 (0.255)	0.594 (0.465)	0.069 (0.004)	1.38 (1.22)	0.117 (0.494)
Materials and services . .	-0.484 (0.486)	-0.185 (-0.060)	0.255 (0.254)	0.080 (0.304)	0.335 (0.938)

Note: Figures in parentheses are MLV estimates.

Thus, the high elasticity of substitution obtained with the linear model can be defended on economic grounds. However, it is of interest to consider whether restrictions which are not supported by the data (such as the requirement for a starting date of 1) should be imposed if they yield more economically plausible results. This is not a cut-and-dried issue. Nevertheless, it is generally desirable to examine alternative possible sources of specification error closely before resorting to remedies of this kind.

4. Concluding Comments

The use of logarithmic time trends especially in conjunction with the translog functional form has become widespread in recent years, e.g. Binswanger (1974), McKay *et al.* (1980 and 1982), Turnovsky, Folie and Ulph (1982), Boyle (1982). However, as has been shown here, estimates so obtained can be extremely sensitive to the essentially arbitrary choice of starting date of the time trend. Consequently, this formulation should be used with care.

If the logarithmic form is used, it would be preferable to estimate the starting date directly, instead of imposing an arbitrary start date of 1. At the very least, a sensitivity analysis of the kind undertaken here is highly desirable. If estimation of the starting date cannot be undertaken or yields unstable estimates, an alternative functional form may be more appropriate.

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