United We Vote

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Summary

This paper studies the advantages that a coalition of agents obtains by forming a voting bloc to pool their votes and cast them all together. We identify the necessary and sufficient conditions for an agent to benefit from the formation of the voting bloc, both if the agent is a member of the bloc and if the agent is not part of the bloc. We also determine whether individual agents prefer to participate in or step out of the bloc, and we find the different optimal internal voting rules that aggregate preferences within the coalition.

Keywords: Voting bloc, Coalition formation, Voting rule

JEL Classification: D72, D71

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1 Introduction

Despite the advantages of collaboration, alliances are often broken, groups are dissolved, coalitions split, or they fail to be formed in the first place. Any union of heterogeneous agents may fail to act for the benefit of some of its members. Individual freedom of action is partially curtailed by joining a group and committing to follow its rules. This creates an incentive to abandon the group and proceed alone in a different course of action. There is a trade-off between the potential gains of group action and the sacrifice of individual freedom involved in group formation.

In this paper we examine this trade-off in the context of political competition between agents who can communicate to form a coalition. We model a set of agents who face a vote over a choice of alternatives. We assume that a specific subset of the whole electorate can coalesce to coordinate the voting behavior of its members. Agents coalesce to increase the probability that their preferred outcome is chosen. For example, each country has a vote in the UN General Assembly. The European Union's 25 members could decide to coordinate their foreign policies, agreeing on a common voting position before UN meetings.

A crucial problem is how to choose the common position -each country knows what it wants and it also knows that a coalition of countries will have a better chance of getting what it asks for than a single country. Thus there is an incentive to form such coalition. But if its members have conflicting preferences, what will the coalition stand for? If the coalition intends to act as a "voting bloc" and cast all its votes together, it requires an internal decision-making rule to aggregate the preferences of its members. This internal decision-making rule will map the possibly disparate preferences of the members into a single alternative for which all members of the coalition will vote.

The internal rule that maximizes the aggregate utility for the coalition is simple majority. However, we find that only under certain conditions every member of the coalition benefits from forming a voting bloc if simple majority is chosen as the internal aggregating rule.

If all members do not benefit from simple majority, then the coalition must find other rules to aggregate the preferences of its members. Constitutional design studies the rules that determine how to change voting rules within a society. In our case, we assume that the coalition of agents can form a voting bloc to coordinate their votes only by unanimous agreement of all members. Thus every member of the coalition must be made better off, otherwise the coalition will not be able to function as a voting bloc, because some members will block the project. For instance, any one country of the EU can veto a new EU treaty that intended to unify the foreign policy of its members.

We find that the sufficient conditions for a supermajority internal rule to make every member of the coalition better off are less stringent than those needed for simple majority to do so. We also find that an "opt-out" rule benefits every member in some cases when supermajorities do not. Overall, for a very large set of possible preference profiles there exists some rule that satisfies every member in the coalition.

Imagine a successful coalition that has found one such rule and functions as a voting bloc, casting all its votes together according to the outcome of its internal decision-making rule and making every member better off than if everyone voted individually according to their own preferences. We find that under certain conditions some members will still have an incentive to leave the coalition (if that is possible), thus the coalition will need more than simply to find a rule that benefits everyone to function as a voting bloc: It will need to solve the collective-action problem in which members prefer others but not themselves to participate, although everyone is better off if all of them participate than if no one does.

Several non-cooperative theories of coalition formation with economic applications are surveyed in Carraro (2003). Closer to the motivation of this paper, Buchanan and Tullock (1962) analyze the costs and benefits of forming a coalition and praise the virtues of unanimity as internal voting rule. Barberá and Jackson (2004) let agents choose among several rules and they define "self-stable" voting rules as those that will not be beaten by any other rule if the given voting rule is used to choose among rules. Maggi and Morelli's (2003) study "self-enforcing" rules to determine whether collective action will be taken by a group of agents and they conclude that no other rule but simple majority or unanimity is ever optimal.

A different approach to coalition formation comes from the voting power literature. Gelman (2003) concentrates on the probability of casting a decisive vote in an election and the effect of coalitions over such probability. We focus instead on the probability of getting the desired outcome out of the election and we want to analyze the potential benefits of forming a voting bloc, coalescing with other agents to cast all votes in the same direction.

We are interested in the effect of a single coalition that forms a voting bloc on the degree of satisfaction of its members, how the heterogeneity of such members may affect their gain in utility and which internal voting rules in the coalition may make the voting bloc satisfactory for a broader range of parameters.

These theoretical questions are particularly relevant to the ongoing debates about the need or desire for a common foreign policy in the EU, a purpose that was first vaguely stated in the Maastricht Treaty (1992)[1], but that has been recently the subject of much deeper debates and controversy during the negotiations towards a constitution (started in 2002) and will probably continue to be in the European political agenda for years to come. Therefore, we will frequently refer to the EU as a motivating example along our exposition.

After introducing the model and showing that there is a surplus to be gained by forming a voting bloc in Section 2, in Section 3 we ask whether the formation of the voting bloc will benefit every member of the coalition. In Section 4 we study an "opt-out" rule that allows one agent to stay out of the voting bloc and discuss under what conditions introducing such a rule will benefit all the members of the coalition. In Section 5 we conclude and propose a future agenda of research. Algebraic calculations and proofs are located in the Appendix.

[1] The Treaty on European Union, signed in Maastricht in 1992.

2 The Model. Gains from forming a Voting Bloc

We consider a society formed by $M+N+1$ agents, where M and N are even. These agents (legislators, countries, etc.) face a binary decision: either to keep the status quo, or to vote for an alternative a to replace it. All agents are called to vote either for a (*yes*), or against a (*no*). If the number of favorable votes is equal or higher than a threshold T, then a is implemented.

Each agent strictly prefers either the status quo or the alternative a, and we assume no intensity in preferences. Preferences over lotteries will simply be determined in favor of the lottery that assigns the higher weight to the preferred alternative.

M agents lack coordination powers and will vote individually. The remaining $N+1$ agents can coordinate among themselves and may at wish form a voting bloc. We will call this set of agents the coalition C.

If each of its members agrees, coalition C forms a voting bloc. In this case coalition C will hold an internal meeting to predetermine its voting behavior in the general vote. In the internal meeting, all members of the coalition will vote *yes* or *no* according to their preferences for or against a. Then:

1. If the majority in this internal vote has strictly more than $t(N+1)$ votes, where $t \in [\frac{1}{2}, \frac{N}{N+1}]$, then the majority prevails and all members of coalition C will vote as a bloc in the general election casting $N+1$ votes according to the preferences (either *yes* or *no*) of the majority of the coalition. The outcome of the coalitional internal meeting is binding.

2. If the majority gathers no more than $t(N+1)$ votes in the internal vote, then the coalition fails to act as a bloc in the general election and all members are free to vote according to their individual preferences.

Note that threshold t defines the $t-majority$ rule used by the coalition to decide whether or not it will act as a bloc rolling its internal minorities. A threshold $t \in [\frac{1}{2}, \frac{1}{2}+\frac{1}{N+1})$ corresponds to simple majority, $t \in [\frac{1}{2}+\frac{1}{N+1}, \frac{N}{N+1})$ to a supermajority and $t=\frac{N}{N+1}$ to unanimity.

Forming a voting bloc with unanimity as internal voting rule is in essence identical to not forming a voting bloc, because the coalition will only cast its votes as a bloc if all its members share the same preference, in which case all votes will be cast as they would in the absence of a voting bloc.

If the coalition does not form a voting bloc, then all the members of the coalition will vote according to their individual preferences in the general election.

Coalition members decide to form a voting bloc with a $t-majority$ internal voting rule before the alternative a is specified, so agents do not know if they will prefer alternative a or the status quo. Agents have no power over the specification of alternative a, which is exogenous.

Every agent i has a type p_i, which is the probability that agent i will prefer alternative a to the status quo, once alternative a is revealed.[2] This type can

[2] This probabilistic model of voter uncertainty was first considered by Rae (1969) and developed by Badger (1972) and Curtis (1972).

be interpreted as a propensity for change, or as a displeasure with the status quo in general. Let p_{-i} denote the vector of types of all agents in the society other than i. Types are common knowledge, and so are true preferences once agents learn what the alternative a is.

Each realization of preferences is independent from the others. Once alternative a is revealed, each of possibly many agents with a type p_o has an independent probability p_o of supporting alternative a. Typically several of them will end up supporting a, whereas some others will prefer the status quo.

If the coalition forms a voting bloc, in the internal vote, voting will be sincere and there will be no abstention. With simple majority as the internal decision making rule, if the number of *yes* votes surpasses the number of negative ones, then the whole coalition (now a voting bloc) will cast a total of $(N+1)$ *yes* votes in the general vote which includes all agents in the society. If the number of *no* votes surpasses the number of favorable ones, then the coalition accordingly votes as a bloc casting $(N+1)$ *no* votes in the general vote.

The voting bloc behavior we have described consists of rolling internal minorities to present a common front in the general vote, strengthening the position of the coalition's majority with the minority votes which are "converted" or "swayed" to the majoritarian camp, increasing the chances of eventually getting the outcome the majority wishes (of course, in doing so the probability of getting what the minority wishes decreases).

Proposition 1 *Let type $p_i \in (0,1)$ for each agent i in the society. Then, for any $N \geq 2$ (number of agents in the coalition), $M \geq 2$ (number of agents not in the coalition), and T (threshold to accept alternative a), a coalition of $N+1$ members strictly increases the sum of expected utilities of its members by forming a voting bloc with either simple majority or any supermajority as the internal voting rule. Simple majority rule is the internal voting rule that maximizes sum of expected utilities of the members of the voting bloc.*

The proof is straightforward. We offer a sketch here and details in the Appendix.

Forming a voting bloc only has an effect in utilities if the formation of a bloc and the subsequent rolling of minority votes within the coalition alters the outcome of the general vote. If so, every member of the coalition who is in the coalitional majority benefits from the voting bloc formation, at the cost of every voter in the minority. Since the majority is by definition bigger than the minority, there are more members benefiting than suffering from the bloc, and since the intensity of preferences is set to be equal for every member, in the aggregate forming a voting bloc generates a surplus of utility for the coalition. Any other rule that in some cases fails to roll a minority is giving away this net gain in utility and therefore underperforms in comparison to simple majority in terms of aggregated gains in utility.[3]

[3] The optimality of simple majority as the aggregation rule for a set of agents is proved by Curtis (1972).

It follows from Proposition 1 that if all the members of the coalition share a common type, then forming a voting bloc increases the utility of every member in the coalition. Therefore, a homogeneous coalition of agents who have the same type should always form a voting bloc with simple majority as the internal voting rule to maximize their probability of winning the final vote in a larger electorate. Also from Proposition 1, we derive the following corollary:

Corollary 1 *If all but one of the members of the coalition share a common type, all the homogeneous members benefit from the formation of a voting bloc.*

Suppose all members in C except for i share a common type. Then if member i is in the rolled minority, more of the homogeneous members are in the majority benefiting from the rolling of votes the voting bloc imposes than in the hurt minority, thus in the aggregate the homogeneous members strictly benefit from the bloc. If member i is in the majority, there are at least the same number of homogeneous members in the majority as in the minority. Thus, in the aggregate the bloc is at worst neutral to the homogeneous members. Since both cases are possible, overall there is a surplus for the homogeneous members (maybe not so for the heterogeneous one).

In Sections 3 and 4 we will show under which conditions will every member of an heterogeneous coalition benefit from the formation of a voting bloc. We now ask whether the formation of a voting bloc benefits or harms the interests of the agents who are not part of the coalition. The answer will depend on the voting rule in the general election.

If the rule in the general election is unanimity, then each agent has a veto power over changes to the status quo. If a coalition forms a voting bloc, it removes the veto power from its members, but not from non-members, who therefore benefit from the formation of a voting bloc by the coalition. If the rule in the general election is simple majority, a voting bloc will only change the outcome to make a minority win. That is contrary to the interests of a majority of non-members of the coalition.

Proposition 2 *Let type $p_i \in (0, 1)$ for each agent i in the society and let coalition C form a voting bloc with any internal rule other than unanimity. Then, if the voting rule in the general election is unanimity, every agent not in the coalition strictly benefits from the formation of the voting bloc. If the general voting rule is simple majority, there is a loss in aggregate utility for the agents not in the coalition and the society as a whole.*

We provide the proofs for every proposition in the Appendix.

Even if the formation of a voting bloc is in the aggregate hurting non-members of the coalition, this effect will in general not be uniform. Some agents not in the coalition will win, whereas some lose expected utility if the coalition forms a voting bloc. For instance, suppose that the members of the coalition have types such that almost always the *yes* wins in the coalitional internal vote and a small but significant *no* minority is rolled. Then the bloc behavior by the coalition tilts the general vote in favor of alternative a. Agents with a high type, who are likely to prefer alternative a, will be then happy to see the coalition form a voting bloc. Of course, the behavior of the voting bloc hurts those with a lower type in our example.

To summarize, forming a voting bloc is inconsequential if the coalition uses unanimity as internal voting rule, but with any other rule, forming a bloc gives a surplus in utility to the coalition and simple majority is the internal voting rule that maximizes such surplus. Every agent in the rest of the society benefits from the formation of a bloc by the coalition if the general voting rule is unanimity, but if the general voting rule is simple majority these agents suffer an aggregate loss in utility, although some of them may still benefit from the formation of a voting bloc by the coalition.

In the next section we investigate under what conditions the coalition can reach unanimous agreement among its members to proceed with the formation of a voting bloc and appropriate the surplus in utility that comes with the voting bloc.

3 Achieving consensus to form a Voting Bloc

We wish to find an internal rule for the coalition to aggregate preferences in such a way that maximizes the aggregate utility of its members relative to a default in which the coalition uses unanimity as internal voting rule and all members always vote according to their true preferences in the general election. This internal rule must be such that in expectation every member prefers it to unanimity. Because each member of the coalition can block deviations from unanimity as internal voting rule, the coalition needs full consensus to use any other rule to aggregate its preferences.

Given an internal voting rule v, we say that v is **beneficial for C** if in expectation every member in C is weakly better off using v rather than unanimity as the internal voting rule and some member in C is strictly better off.

In short, a rule is beneficial for the coalition if it benefits all its members to adopt it instead of unanimity.

In this section we assume that the general election rule is simple majority and we investigate which $t - majority$ internal rules are beneficial for C. If there are several beneficial majority rules, we focus on whichever one maximizes the overall surplus for the coalition. We recall from Proposition 1 that the internal voting rule that maximizes the aggregated utility for the coalition is simple majority. However, in an heterogeneous coalition, some members may not benefit from pooling votes in a voting bloc with simple majority.

We label unanimity rule as $\varnothing$ as a reminder that using unanimity as internal voting rule is identical to not forming a voting bloc, or no members joining the voting bloc.

We label as t a $t - majority$ rule in which every member of the coalition participates in the voting bloc, and minorities of size strictly less than $t(N+1)$ are rolled to join the position of the majority of the coalition in the general vote. Simple majority, denoted Sm, refers to the special case in which $t = \frac{1}{2}$.

For any internal voting rule v, we let $EU_i[v]$ denote the expected utility for agent i if the coalition forms a voting bloc with v as internal voting rule.

Before presenting our results, we need to make some assumptions on the types of the agents:

Assumption 1 *The number of favorable votes cast by the M agents not in coalition C follows a symmetric distribution around $\frac{M}{2}$ with some positive probability of casting a quantity of favorable votes different than $\frac{M}{2}$.*

This condition significantly relaxes the standard assumption in the voting power literature that all agents have a common type of 0.5.[4] Instead, it suffices for our model that the M agents can be paired in such way that for each pair (j, j'), $p_j + p_{j'} = 1$, with at least one pair of agents with types strictly between zero and one. We let $f(x)$ denote the probability that the M members cast exactly x favorable votes for alternative a, and we let $F(x) = \sum_{k=0}^{x} f(k)$ be the distribution function of the number of favorable votes cast by the M agents not in the coalition.

We make a milder assumption on the types of the members of coalition C. Namely, we assume that coalition C "leans towards" accepting alternative a. Let $g_{ij}(x)$ denote the probability that x members of $C\backslash\{i, j\}$, the coalition without i or j, prefer alternative a. Then we require the following:

Assumption 2 *For all $k \in [0, \frac{N}{2} - 1]$ and for all $i, j \in C$, $g_{ij}(\frac{N}{2} + k) > g_{ij}(\frac{N}{2} - k - 1)$.*

Note that $g_{lh}(k) = \sum\limits_{\substack{A\subseteq C\backslash\{l,h\} \\ |A|=k}} [\prod\limits_{\substack{i\in A \\ j\in C\backslash(A\cup\{l,h\})}} p_i(1 - p_j)]$.

Assumption 2 states that given any $N-1$ members of the coalition and given any particular majority-minority split of votes in this subset of the coalition, it is more probable that this majority in the subset is for the *yes* side. A sufficient condition for our assumption to hold is that excluding any three members, we can pair the rest in such a way that for each pair (i, i'), $p_i + p_{i'} \geq 1$ and at least one pair is different from $(0.5, 0.5)$.

Let $g_i(x)$ denote the probability that exactly x members of $C\backslash i$, the coalition without i, prefer alternative a. Formally, $g_l(k) = \sum\limits_{\substack{A\subseteq C\backslash l \\ |A|=k}} [\prod\limits_{\substack{i\in A \\ j\in C\backslash(A\cup l)}} p_i(1 - p_j)]$. From Assumption 2 it follows that for all $k \in [1, \frac{N}{2}]$ and for all $i \in C$, $g_i(\frac{N}{2} + k) > g_i(\frac{N}{2} - k)$. We show this in the Appendix.

With these two assumptions on the types of the agents and simple majority as the voting rule in the general election, we find that a member of the coalition will like to form a voting bloc with a $t - majority$ rule as internal voting rule if her type is "high enough": If a given member would benefit from forming a voting bloc with a $t - majority$, then every other member with a higher type would benefit even further.

[4] See, for instance, Gelman (2003).

Lemma 1 *Let $l, h \in C$ such that $p_h \geq p_l$. Then $EU_h[t] - EU_h[\varnothing] \geq EU_l[t] - EU_l[\varnothing]$.*

By Lemma 1 we can focus only on the member with the lowest type to see if she benefits from the formation of a voting bloc with a $t - majority$. If she does, then every member in the coalition benefits from forming a voting bloc with a $t - majority$ rule:

Proposition 3 *Let $l \in C$ be the member with the lowest type. Then a $t - majority$ rule is **beneficial** as an internal voting rule for coalition C if and only if $p_l > p_l^{t,\varnothing}(p_{-l})$.*

In the Appendix we find the exact expression of $p_l^{t,\varnothing}(p_{-l})$ as a function of the types of the agents and the threshold t.

Since the coalition "leans" towards accepting alternative a, the majority within the coalition will be in favor of alternative a more often than not, with the result that the negative votes will be rolled more often than the favorable ones. Therefore it becomes more likely that alternative a wins the general election. Member l only likes such voting behavior if her type is high "enough", where the exact meaning of "enough" is given by the threshold in the Proposition.

The threshold $p_l^{t,\varnothing}(p_{-l})$ converges to 1 as every type p_i, $i \in C \backslash l$ converges to 1. On the other hand as $g_l(\frac{N}{2} + k) - g_l(\frac{N}{2} - k)$ converges to zero (as the distribution of votes by the other members of the coalition converges to a symmetric distribution), the threshold $p_l^{t,\varnothing}(p_{-l})$ converges to $-\infty$. A threshold below zero indicates that member l supports the creation of a voting bloc regardless of her own type.

We illustrate Proposition 3 with the aid of Figure 1, for the specific case of simple majority as the internal voting rule.

To be able to plot the threshold $p_l^{Sm,\varnothing}(p_{-l})$ with respect to only one variable, we assume that the distribution of votes by the agents not in coalition C follows a binomial $Bi(M, \frac{1}{2})$ and that all the members of coalition C except l share a common type r. A single parameter r is sufficient (captures all the relevant information about the types of the rest of members of C) to determine if member l would benefit from the formation of a voting bloc: For any heterogeneous coalition in which all other members except l did not share a common type, that coalition is mapped to one particular value of r such that l evaluates coalition C as if all the other members had a common type r. Therefore, Figure 1 indirectly captures all possible coalitions of size N. In Figure 1 we set $M = 176$ and $N = 24$ to approximate our European Union example. We will use these values in most of our figures.

Our model corresponds to the right half the graph: if the common type r of the N members other than l is bigger than one half, member l will support the formation of a voting bloc with simple majority if the type p_l is above the depicted threshold. The left half of the picture is a symmetric case in which the coalition leans towards rejecting a. Then member l will only support the formation of a bloc if her type is below the threshold.

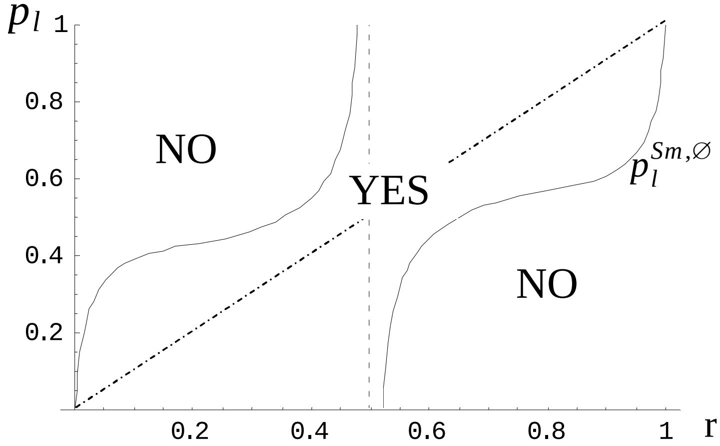

Figure 1: Consensus to form a voting bloc with simple majority

The following proposition tells us that some coalitions that can't form a voting bloc with simple majority, can form a voting bloc with some supermajority rule in such a way that every member's utility increases.

The threshold function $p_l^{t,\varnothing}(p_{-l})$ is decreasing in t. As the parameter t used for the $t-majority$ internal rule increases, the more type profiles for which the $t-majority$ is beneficial for coalition C:

Proposition 4 *Let $t' = t + \frac{1}{N+1}$. Then for any $t \in [\frac{1}{2}, \frac{N-1}{N+1})$, the subset of type profiles for which a $t'-majority$ is **beneficial for C** strictly contains the subset of type profiles for which a $t-majority$ rule is **beneficial for** C.*

The more stringent the supermajority rule the coalition uses, the lower that the type of member l can be and yet allow l to benefit from the formation of a voting bloc with the $t-majority$ internal voting rule.

We depict this result in Figure 2 for $N = 24$, $M = 176$, F is a binomial $Bi(M, \frac{1}{2})$ and every member of C except for l has a type $r > \frac{1}{2}$. Figure 2 presents four possible rules for a coalition the size of the EU: Simple majority, two-thirds majority, four-fifths majority, and nine-tenths majority.

We see how the range of parameters for which a voting bloc would benefit every member increases as the supermajority rule becomes more stringent. However, since simple majority maximizes the overall surplus for the coalition, setting higher thresholds for approval of a common position diminishes the value of the voting bloc, although it may help to bring an outlier on board.

Aiming to maximize the utility of the coalition subject to not hurting any member, the optimizing solution is the lowest possible supermajority that would benefit (or at least leave indifferent) the member with the lowest type. A consequence of Proposition 4 is that for any $t' \in [\frac{1}{2}, \frac{N-1}{N+1})$ there exists a profile of

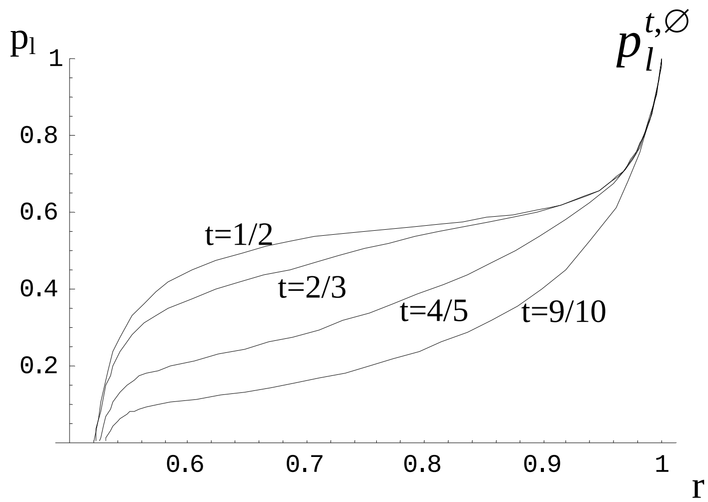

Figure 2: Supermajority rules

types such that t' maximizes the surplus for the coalition among the class of beneficial $t-majority$ rules.

In the remainder of this section we investigate how changes in the size of the coalition or the heterogeneity of types of its members affect which rules the coalition will be able to use to the benefit of all its members.

We find that if the size of the coalition is too large, then no coalition in which all members but l share a common type $r > p_l$ can form a voting bloc with simple majority.

Proposition 5 *Let M be fixed. Let $p_l < r$ for $l \in C$ and $p_i = r$ for all $i \in C \backslash \{l\}$. There exists some $\overline{N}$ such that if $N > \overline{N}$, simple majority is not* ***beneficial for*** **C**.

As the coalition becomes very large relative to M, the internal majority coincides with the external majority unless the coalition is almost evenly split. The coalition is more likely to vote for a than against a. Since member l is the member with the lowest type, conditional on the coalition being evenly split, member l is more likely to be against a, thus on the losing side. Therefore, if the coalition becomes so large that rolling its votes only affects the outcome when the coalition is almost evenly split, the member with the lowest type rejects the formation of a bloc with simple majority. In the limit, only a fully homogeneous coalition where every member has the same type could form a voting bloc with simple majority.

Figure 3 shows the convergence of the threshold $p_l^{Sm,\varnothing}(r, N)$ to r as N gets large, given that F is a binomial $Bi(M, \frac{1}{2})$, $M = 60$ and $p_i = r \; \forall i \in C \backslash l$. The

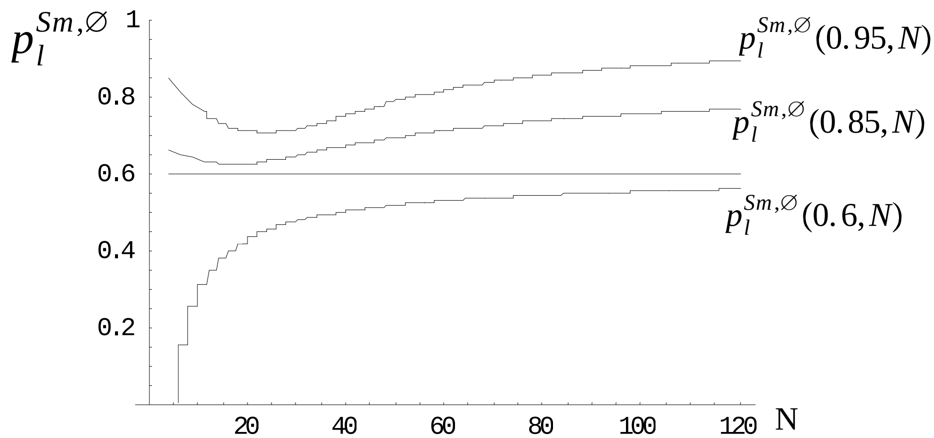

Figure 3: Convergence of $p_l^{Sm,\varnothing}(r, N)$ to r in a dichotomous coalition.

three plots (from bottom to top) correspond to a common value $r = 0.6$ (we show the convergence asymptote as well), a common value 0.85 and a common value 0.95 for the N members minus l in the coalition.

Beyond size, we ask how heterogeneity affects the chances of a coalition forming a voting bloc. We know that simple majority is beneficial for any homogeneous coalition, whereas heterogeneous coalitions may run into obstacles. Nevertheless, we show by means of an example that the possibility of forming a voting bloc with simple majority is not monotonic with heterogeneity.

We compare three coalitions with the same mean type and the same lowest type. We measure heterogeneity by the standard deviation of types. We find that the most homogeneous and the least homogeneous of the three coalitions cannot form a voting bloc with simple majority, whereas the intermediate one can.

Example 1 *Let all agents not in C have a type $p_m = 0.5$ and let there be 10 of them.*

Let C_1 be a coalition of agents with types {0.445, 0.75, 0.75, 0.75, 0.75}. The mean type is 0.689. The standard deviation 0.1368. If $C = C_1$, then $p_l^{Sm,\varnothing}(p_{-l}) = 0.4943$ and l rejects the formation of a voting bloc with simple majority as internal voting rule.

Let C_2 be another coalition of agents with types {0.445, 0.5, 0.5, 1, 1}, mean type 0.689, standard deviation 0.2847. If $C = C_2$, then $p_l^{Sm,\varnothing}(p_{-l}) = 0.441$ so forming a voting bloc with simple majority benefits every member of the coalition.

Let C_3 be yet another coalition of agents with types {0.445, 0.45, 0.55, 1, 1}, mean type 0.689, standard deviation 0.2869. If $C = C_3$, then $p_l^{Sm,\varnothing}(p_{-l})$ $= 0.446$ so once again member l vetoes the formation of a voting bloc with simple *majority.*

For coalitions C_1 and C_3 in Example 1, using a two-thirds majority or a three-quarters majority (or any other value of t that requires a majority of 4 to 1 to roll the minority) every member benefits from forming a voting bloc. In C_1, using $t = 3/4$, $p_l^{3/4,\varnothing}(p_{-l}) = 0.39$ so member l favors the formation of a voting bloc that rolls only minorities of size one. Similar results hold for coalition C_3.

Considering coalition C_1 in Example 1, we quantify the impact that the formation of a voting bloc would have over the outcome in the general election. We show the results in Table 1. The numbers represent the probability that the event indicated in each row occurs, given the internal rule the coalition uses. In the second column, the coalition uses unanimity or forms no bloc, in the third column it forms a bloc with simple majority and in the fourth column it uses a 3/4 majority.

TABLE 1	No bloc	1/2 maj	3/4 maj
a approved	69.52%	79.58%	73.45%
a approved given l likes a	79.82%	90.00%	84.72%
a approved given l dislikes a	61.26%	71.22%	64.42%
l satisfied with outcome	57.02%	56.02%	57.44%
$j \in C \backslash l$ satisfied with outcome	67.26%	74.70%	70.57%
$m \notin C$ satisfied with outcome	59.44%	53.51%	57.61%

Since the coalition leans towards a, forming a voting bloc makes approval of a more likely. All the members except for l benefit and all non-members are hurt forming a voting bloc. Member l is hurt if simple majority is used, so in order to benefit all its members, the coalition has to select a supermajority that makes l better off, but attenuates the advantage for all the other members. We see that forming a voting bloc with simple majority would have a substantial impact: the probability of approving a increases ten percentage points and the probability of getting the desired outcome out of the election would increase seven percentage points for all members of coalition C_1 but l. Using a 3/4 majority reduces this benefit of a voting bloc to roughly a half, but it makes all members of C_1 more likely to see their preference prevail in the general election.

In this section we have described the necessary and sufficient condition for a coalition to be able to form a voting bloc with a majority rule. We show that although simple majority always maximizes the aggregate surplus, there are type profiles for which simple majority is not beneficial for the coalition but some supermajority rules are and the coalition can choose one of them to gain some of the surplus of a voting bloc benefiting all its members.

4 An Opt-Out rule

In this section we explore a more nuanced rule, which consists of forming a voting bloc with all but one of the members of the coalition. The excluded member does not participate in the internal vote of the voting bloc, but votes directly and according to her true preferences in the general election.

This scheme differs from expelling one member from the coalition in a crucial detail: The exclusion is voluntary, the member who does not participate in the voting bloc agrees to the formation of the voting bloc without her, and hence she opts to be out, or "opts-out". The member who opts-out has to benefit from the formation of the voting bloc by the other members, otherwise she would rather veto the whole project and keep unanimity in place as the voting rule to aggregate votes in the coalition.[5]

We denote by *Out* the "Opt-Out for l" rule in which the member l with the lowest type does not participate in the voting bloc which is formed by every other member of the coalition and simple majority is chosen as internal voting rule.[6]

Throughout this section we assume that the general election rule is simple majority.

Our first result on opt-out rules considers under the conditions under which the member of the coalition who stays out of the voting bloc benefits from its formation.

Proposition 6 *The formation of a voting bloc with simple majority rule by every member of coalition C except l, benefits member l if and only if $p_l > p_l^{Out,\varnothing}(p_{-l})$.*

We provide the expression of $p_l^{Out,\varnothing}(p_{-l})$ in the Appendix.

The threshold function $p_l^{Out,\varnothing}(p_{-l})$ is always positive given our assumptions on types; it is not increasing with respect to the type of all other members of the coalition and it does not always converge to one as the types of the members of the coalition do. This last feature guarantees that in some cases in which member l rejects forming a voting bloc with simple majority she benefits from the formation of a bloc with an "Opt-Out for l" rule. If all the other members also benefit from the "Opt-Out for l" rule, then this rule is *beneficial for C* and it offers a solution for a coalition which couldn't form a bloc with $t-majority$ rules. The next Proposition states this result.

Proposition 7 *If $4 \leq N \leq M$, there exist type profiles for which an "Opt-Out for l" internal voting rule is **beneficial for** $\mathbf{C}$ and no $t-majority$ rule is. If $N = 2$ or $N > M$, there exists no type profile for which "Opt-Out for l" is **beneficial for** $\mathbf{C}$ and simple majority is not.*

If $N = 2$, allowing one member to step-out reduces the bloc to size two, which is identical to not forming a bloc at all, or forming it with unanimity. If

[5] Famous opt-outs in the European Union include the UK and Denmark with regards to the European Monetary Union: Their approval to the Maastricht Treaty was necessary for the monetary union to bring about the euro, and they supported the implementation of the treaty, whilst staying out of the project. If they had deemed it harmful to their interests, they could have refused to sign it.

[6] We could extend the results in this section to consider opting-out rules in which l stayed out and the participating members chose a supermajority as internal voting rule, but for simplicity we focus on the rule that will maximize the surplus for the members who participate in the bloc given that l will not join them.

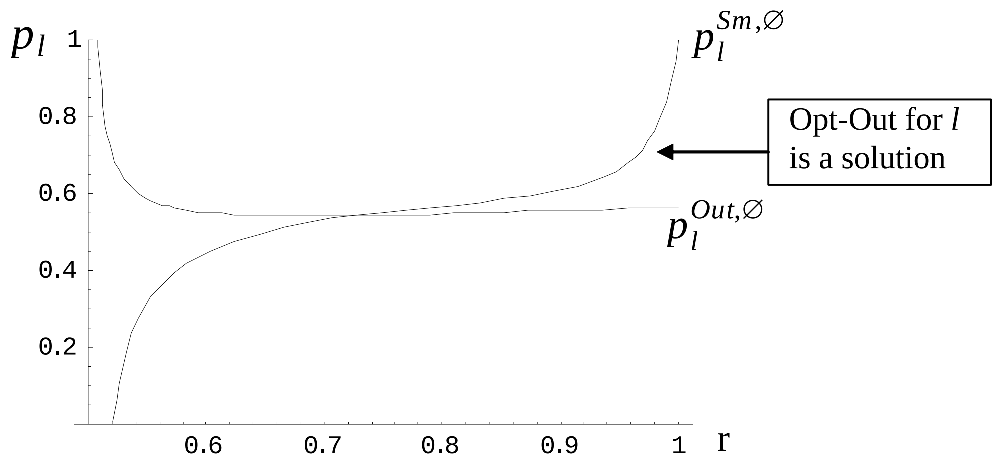

Figure 4: An Opt-Out rule as a solution

$N > M$, then the coalition acts as a dictator even if one member opts-out, thus the member who opts-out cannot be better off out of the voting bloc than in the voting bloc.

We use Figure 4 to gain some insight about the threshold $p_l^{Out,\varnothing}(p_{-l})$ and its comparison with $p_l^{Sm,\varnothing}(p_{-l})$. As in all figures, we assume that the distribution of votes by the agents not in coalition C is a binomial distribution $Bi(M, \frac{1}{2})$ and that all the members of coalition C except l share a common type r. We set $M = 176$ and $N = 24$.

Looking at Figure 4, we notice that if types are in the area below the threshold $p_l^{Sm,\varnothing}(p_{-l})$ and above $p_l^{Out,\varnothing}(p_{-l})$, l would veto forming a voting bloc with simple majority if it included all the members. By allowing l to stay out, the coalition can form a voting bloc with simple majority with every other member and in expectation raise the utility of every member including l.

This result casts a favorable light over "opt-out" rules. On the other hand, "opt-out" rules have two setbacks: If the coalition is heterogeneous (and not just dichotomous with all members but l sharing a common type) and l opts out, it is possible that the member with the next lowest type opposes the formation of the reduced bloc. Allowing this member to opt-out as well may simply pass the problem to the next member until the bloc fully unravels and every member but the last two opt-out, which negates the purpose of a voting bloc.

Even if this unravelling does not take place, there is a second latent complication to opt-out rules: if the coalition allows for the member with the lowest type to opt-out, then other members may also request to opt-out, even if they benefit from the formation of a voting bloc, simply because they would benefit even more by opting-out. If the coalition lets every member join in or stay out of the voting bloc, then it faces a "free-rider" problem, where some members who would benefit from joining the voting bloc, may prefer to opt-out and passively take advantage of the pooling of votes by other coalition partners. From our concept of free-riding we exclude situations in which a member opts-out of a voting bloc and benefits from the pooling of votes by the other members if the member who opts-out would be hurt by a voting bloc that included him.

Definition 1 *Member l "free-rides" if she would have benefitted from forming and participating in a voting bloc, but benefits even more as a result of opting-out.*

With no opt-out rules, there is no chance to free-ride, since the coalition faces an "all-or-none" binary decision: either every member joins the voting bloc, or the bloc is not formed. If instead members can individually choose whether to join in or to stay out, some may not choose to join in. In the following Proposition we explore whether a member would prefer to participate in or to stay out of a voting bloc formed by every other member of the coalition.

Proposition 8 *Member l prefers to participate in a voting bloc formed by the coalition with simple majority as internal decision rule better than to opt-out and not participate in the pooling of votes by the rest of the members of the coalition if and only if $p_l > p_l^{Sm,Out}(p_{-l})$.*

We provide the exact expression of $p_l^{Sm,Out}(p_{-l})$ in the Appendix.

From Proposition 3 we obtained the condition for l to benefit from forming and participating in a voting bloc with simple majority. Proposition 8 now states when will member l prefer to opt-out from such a bloc. Combining Propositions 3 and 8 we obtain Proposition 9, which demonstrates the downside of opt-out rules: They create a free-riding problem when member l would benefit from participating in the bloc but prefers to opt-out.

Proposition 9 *An "Opt-Out for l" rule creates a free-rider problem if and only if $p_l^{Sm,\varnothing}(p_{-l}) < p_l < p_l^{Sm,Out}(p_{-l})$. If $4 \leq N \leq M$, there exist type profiles for which this condition is met. If $N = 2$ or $N > M$, this condition cannot hold and free-riding cannot occur.*

If $p_l < p_l^{Sm,Out}(p_{-l})$, member l prefers not to participate in the voting bloc and she free-rides if participating would be better for l than not forming a voting bloc.

When member l compares the utility of being in the bloc to the utility of being out of the bloc, she compares the probability of affecting the general outcome such that it coincides with her wishes when she votes directly in the general election, to the same probability when she is voting through the voting bloc.

This analysis of the utility of being in or out of a voting bloc bears some resemblance, but is not equivalent, to the comparison of probabilities of being decisive (of being pivotal in the final outcome) that occupy the voting power

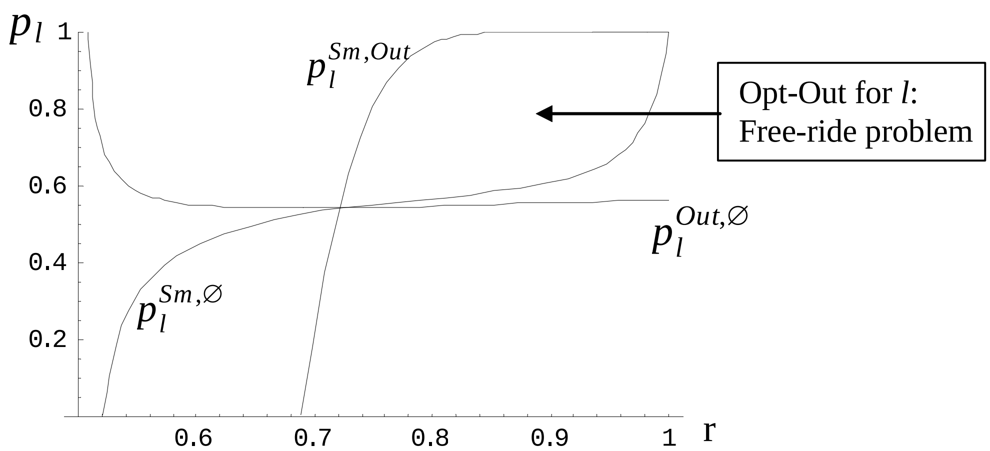

Figure 5: Opt-out and free-ride

literature. In a nutshell, in the voting power literature the agents seek to maximize their probability of being able to alter the outcome, whereas in our model they only care to alter the outcome towards their preference:

Suppose an agent can change the outcome against his preference by joining a voting bloc and casting a vote against his preference, whereas if he stays out of the bloc his vote is irrelevant. Then in our model this agent is indifferent between being out of the bloc and being irrelevant but getting the desired outcome, or being in the bloc and being crucial to obtain the desired outcome.[7]

Let us visualize when a member will prefer to opt-out and free-ride on her coalition partners with the aid of Figure 5, where again $N = 24$, $M = 176$, the 24 members other than l in coalition C share a common type r and the number of favorable votes by agents not in the coalition follows a binomial $Bi(M, \frac{1}{2})$.

If type p_l is above $p_l^{Sm,\varnothing}(p_{-l})$ but below $p_l^{Sm,Out}(p_{-l})$, member l would have supported a voting bloc with simple majority and no opt-outs better than no bloc, but she prefers to opt-out if she can. If she opts-out, the overall utility for the coalition is reduced.

If any member can opt-out, then coalitions of size less than M face an even worse problem: for some configurations of types all members would prefer to opt-out. If $N > M$, then the coalition forming a voting bloc is a dictator and thus votes of agents not in the voting bloc do not count and no member can gain from opting-out.

As a summary, allowing a member to opt-out can be a good solution in a coalition with great homogeneity of types and one outlier, but in many other

[7] For a rigorous study of the relation and differences between voting power and probability of success or satisfaction (the approach we take), we recommend Laruelle and Valenciano (2005).

instances it can generate free-rider problems, which are aggravated if the possibility to opt-out is extended to every member.

5 Conclusions and Extensions

A coalition of agents who are a part of a larger electorate or assembly facing a vote may choose to form a voting bloc. The coalition will then pool its votes and cast them all together according to the outcome of an internal vote. We have shown that forming a voting bloc generates a surplus in the aggregate utility of the members of the coalition and we have checked that simple majority is the internal rule for the voting bloc that maximizes such surplus.

However, if there is heterogeneity among the members of the coalition, the surplus will not be evenly shared. In the absence of transfers, the formation of a voting bloc may be detrimental to some members of the coalition. Ordering members from "least likely" to "most likely" to support changes to the status quo, we find a single cutting point or threshold separating those members of the coalition who support the formation of a voting bloc, and those who reject it. This implies that either every agent in the coalition supports the voting bloc, or at most, agents at one tail of the distribution of types reject it. It will never be the case that extremists from both tails reject the formation of a voting bloc.

We find it interesting to compare this result with a different model of federalism, by Crémer and Palfrey (1996) and (1999). Crémer and Palfrey argue that moderate voters, with preferences closer to the median of the Union, will advocate federalism and unified policies. Complementing their work, our paper provides a rationale for at least one set of extreme voters to wish for a common foreign policy: Suppose the binary choice is between the status quo and "change" and forming a voting bloc the aggregate vote for the whole coalition will most likely be "change". Then voters within each country who want "change" see a better chance of getting it through a unified federal government.

Under the motivation that each member of the coalition can veto the formation of any kind of voting bloc, we have analyzed possible solutions to achieve unanimous support by all members of the coalition to proceed with some form of voting bloc when some member of the coalition opposes the solution which maximizes the total surplus of utility (a voting bloc formed by the whole coalition with simple majority as internal decision rule).

Using qualified majority rules (supermajorities) as the internal voting rules reduces the overall surplus in aggregate utility but it makes it easier to achieve unanimous support for the formation of a voting bloc: the higher the threshold of the qualified majority rule, the less likely that any agent would be hurt by the formation of a bloc. This result contrasts with the findings of Maggi and Morelli's model (2003), in which only simple majority or unanimity are ever found to be optimal, but their focus is on homogeneous agents.

In the last section we have considered another solution: for some range of parameters, allowing an extreme agent who opposes the formation of a voting bloc to opt-out and not participate in the bloc is sufficient to achieve unanimous support (including support by the member who chooses to opt-out) for the formation of a voting bloc by the rest of the coalition. Though it may look strange to require an agent that does not participate in the bloc to acquiesce to its formation, we think the salient example of the EU (where each country has a veto power over changes on fundamental treaties and thus can stop an initiative regardless of whether it includes or excludes the vetoing country) provides enough motivation for the relevance of this result.

Our theoretical results have practical implications, suggesting that a collection of countries with some similarity in their policy preferences would do better by forging a common foreign policy that was not based in unanimity. In particular, each of the 25 members of the EU would be more likely to see its preference prevail at a UN Assembly meeting (or at any international forum that grants one vote per country) if the Union first pre-determined how it will cast all its 25 votes according to an internal voting rule that rolled minorities within the EU.

We could extend this model to incorporate decision costs of forming a voting bloc, making it harder to achieve unanimous support for the voting bloc. Alternatively, we could envision a more favorable setting for voting blocs considering economies of scale. The joint expression of will by a united coalition may wield more power in the general electorate than the individual sum of the votes of the coalition members. That is difficult to justify in terms of votes, but is much more reasonable if lobbying, exerting political pressure or otherwise influencing others are parts of the actions that come with voting in one or the other direction.

Another extension within the framework of the model would be to consider correlation in the realization of the preferences of the members of the coalition, possibly through a correlation matrix. A way to partially incorporate correlation without adding too substantial complications would be to define a "leader of the coalition" and then let the type of every other agent be a two-dimensional vector with the probability of supporting the alternative if the leader does and if the leader does not support the alternative.

A more ambitious extension that we think deserves future research consists on allowing several coalitions, not just one, to form voting blocs. Ideally, any subset of agents would be allowed to form a voting bloc and we would look for stable partitions of the space of agents into voting blocs. We find strong justification for our assumption of a single coalition by considering the formation of a voting bloc in the case of international relations, where the 25 countries in the EU participate in a project that is to a large extent unique. However, another very natural scenario in which voting blocs may occur, in fact an even more appropriate one, is any legislature in which political parties may be formed. If we want to explain party formation, we need to allow for different parties to exist. Starting with a set of individual legislators, parties would be each one of the voting blocs that are formed.

Some models studying the incentives to party formation are grounded on a distributive politics setting, where parties help agents to get a share of the pie, as in Baron (1989). Jackson and Moselle (2002) attempt to model coalition and party formation with both distributive and ideological dimensions. Our approach would try to explain party formation solely on the grounds of enhanced probabilities of getting the desired ideological outcome.

We leave these and other developments for future research.

6 Appendix

6.1 Proposition 1

Proof. Let s denote the size of the minority in the coalitional internal vote, S the size of the majority in such vote, thus $s+S=N+1$, let $EU_i^T(t)$ denote the expected utility for agent $i \in C$ given that the whole coalition C forms a voting bloc with t as the internal voting rule and given that T is the number of votes needed in the general election for alternative a to be implemented. Similarly let $EU_i^T(\varnothing)$ denote the expected utility for agent $i \in C$ if the coalition does not form a voting bloc and T is the threshold in the general election. Then: $\sum_{i \in C} EU_i(t) - \sum_{i \in C} EU_i^T(\varnothing) = \sum_{k=1}^{N/2} (N+1-2k) * \Pr[s=k] * \Pr[\textit{minority is rolled}] * \Pr[\textit{rolling k votes alters outcome}]$.

Given any $t \in [\frac{1}{2}, \frac{N}{N+1})$ and any profile of types such that $p_i \in (0,1)$ for all i in the society, if $s=1$ then $\Pr[\textit{minority is rolled}]$ is equal to one and all the other terms in the expression are strictly positive. For any $s \in [1, \frac{N}{2}]$ all terms are weakly positive, thus the aggregated expected surplus in utility for the coalition generated by the voting bloc is strictly positive.

Let Sm denote simple majority as internal voting rule for coalition C, and let vr denote any other internal voting rule for coalition C. Under Sm, all the votes of the coalition are always cast in favor of the position chosen by the majority of the coalition, thus no other rule can give more votes to the position favored by a majority of the coalition. Therefore, if for some voting behavior the outcome in the general election depends on whether the coalition uses vr or Sm, it must be that the outcome under Sm is the one favored by the majority, and under vr the one favored by the minority. Let $v \in \{0,1\}^{M+N+1}$ be a realization of the preferences of all the agents in the society, where 0 represents a *no* preference and 1 a *yes* preference. Let $\Pr(v)$ be the probability that the realization v occurs, according to the vector of types of all agents. Let $q(vr|v)$ denote the probability that given preferences v, the outcome in the general election depends on whether C uses rule vr or Sm as internal voting rule. Then:

$$\sum_{i \in C} EU_i(Sm) - \sum_{i \in C} EU_i^T(vr) = \sum_{v \in \{0,1\}^{M+N+1}} (S-s) \Pr(v) q(vr|v).$$

This term is always non-negative and it is strictly positive for any rule vr that with positive probability will bring about a different outcome than Sm. ■

6.2 Proposition 2

Proof. Let $T = M + N + 1$. Then if some $j \notin C$ opposes alternative a, alternative a will be rejected regardless of the votes of every other agent. In this case the formation of a voting bloc by C does not affect the utility of any agent.

If all M agents not in C support alternative a and C forms no bloc, then a is approved if and only if all members of C support a. If all M agents not in C support alternative a and C forms a voting bloc with a $t - majority$ internal voting rule, then a is implemented if strictly more than $t(N+1)$ members in C support a. Given that $p_i \in (0,1)\ \forall i \in C$ and given that the $t - majority$ internal voting rule for C is either simple majority or any supermajority, the probability that more than $t(N+1)$ members in C support a is strictly greater than the probability that every member in C supports a. Thus, the expected utility for any agent m not in C with $p_m > 0$ is strictly greater if C forms a voting bloc.

Let instead $T = \frac{M+N}{2} + 1$. Then the sum of expected utilities for the whole society is maximized without a voting bloc, because the maximal social welfare is achieved by always implementing the wishes of the majority, as proved by Curtis (1972). A voting bloc reduces the total social welfare by making minorities win. Since the formation of a voting bloc increases the sum of expected utilities of the members of the coalition, it must be that it decreases the sum of expected utilities of the agents not in the coalition. ■

6.3 Claim from Assumption 2

Assumption 2: For all $k \in [0, \frac{N}{2} - 1]$ and for all $i, j \in C$, $g_{ij}(\frac{N}{2} + k) > g_{ij}(\frac{N}{2} - k - 1)$.

We want to show: For all $k \in [1, \frac{N}{2}]$ and for all $i \in C$, $g_i(\frac{N}{2} + k) > g_i(\frac{N}{2} - k)$.

Proof. $g_{ij}(x)$ is the distribution of a sum of $N-1$ independent Bernoulli trials, each trial taking the type of a member of $C \backslash \{i,j\}$ as probability of success. The sum of independent Bernoulli trials is a unimodal distribution, as shown by Darroch (1964). Therefore, $g_{ij}(\frac{N}{2} + k - 1) > g_{ij}(\frac{N}{2} - k)$ implies $g_{ij}(\frac{N}{2} + k - 1) > g_{ij}(\frac{N}{2} - k - 1)$.

For any i, j, $g_i(\frac{N}{2} + k) - g_i(\frac{N}{2} - k)$ is equal to:

$$\begin{aligned}
& p_j g_{ij}(\frac{N}{2} + k - 1) + (1 - p_j) g_{ij}(\frac{N}{2} + k) - p_j g_{ij}(\frac{N}{2} - k - 1) - (1 - p_j) g_{ij}(\frac{N}{2} - k) \\
> \ & (2p_j - 1) g_{ij}(\frac{N}{2} + k - 1) - (2p_j - 1) g_{ij}(\frac{N}{2} - k - 1) \\
> \ & (2p_j - 1)[g_{ij}(\frac{N}{2} + k - 1) - g_{ij}(\frac{N}{2} - k - 1)].
\end{aligned}$$

Since this is true for any i, j, and since by Assumption 2 at least two members have a type over a half, then for any i this last expression is true for at least one j with $p_j > \frac{1}{2}$. Then $g_i(\frac{N}{2} + k) - g_i(\frac{N}{2} - k) > 0$. ■

6.4 Lemma 1

Let A and B be two internal voting rules for coalition C. Let us define two functions, which depend on the rules A and B, and the vector of types p_{-i}:

$\alpha_i(B, A, p_{-i})$ = Probability that, given that member i prefers *yes,* the outcome in the general election is *yes* if the coalition uses rule B and *no* if it uses rule A.

$\beta_i(B, A, p_{-i})$ = Probability that, given that member i prefers *no,* the outcome in the general election is *no* if the coalition uses rule B and *yes* if it uses rule A.

We will use these functions in most of the proofs in this Appendix. In Lemma 1, we want to show that $EU_h[t] - EU_h[\varnothing] - (EU_l[t] - EU_l[\varnothing]) \geq 0$.

Proof. $EU_l[t] - EU_l[\varnothing] =$

$$p_l * \alpha_l(t, \varnothing, p_{-l}) + (1 - p_l) * \beta_l(t, \varnothing, p_{-l}) - p_l * \alpha_l(\varnothing, t, p_{-l}) - (1 - p_l) * \beta_l(\varnothing, t, p_{-l})$$

and similarly $EU_h[t] - EU_h[\varnothing] =$

$$p_h * \alpha_h(t, \varnothing, p_{-h}) + (1 - p_h) * \beta_h(t, \varnothing, p_{-h}) - p_h * \alpha_h(\varnothing, t, p_{-h}) - (1 - p_h) * \beta_h(\varnothing, t, p_{-h}).$$

In Step 1 we show that

$$p_h * \alpha_h(t, \varnothing, p_{-h}) + (1 - p_h) * \beta_h(t, \varnothing, p_{-h}) - p_l * \alpha_l(t, \varnothing, p_{-l}) - (1 - p_l) * \beta_l(t, \varnothing, p_{-l})$$

is positive. In Step 2, we show that

$$-p_h * \alpha_h(\varnothing, t, p_{-h}) - (1 - p_h) * \beta_h(\varnothing, t, p_{-h}) + p_l * \alpha_l(\varnothing, t, p_{-l}) + (1 - p_l) * \beta_l(\varnothing, t, p_{-l})$$

is also positive, thus adding all the terms, $EU_h[t] - EU_h[\varnothing] - (EU_l[t] - EU_l[\varnothing])$ is also positive.

Step 1:

Let $\lceil x \rceil$ denote the smallest integer equal or larger than x and similarly let $\lfloor x \rfloor$ denote the largest integer smaller or equal to x. With this convention, for $i = \{l, h\}$,

$$\alpha_i(t, \varnothing, p_{-i}) = \sum_{k=\lceil tN \rceil}^{N-1} g_i(k)[F(\frac{M+N}{2} - k - 1) - F(\frac{M-N}{2} - 1)].$$

Then, writing $g_l(k)$ as $g_l(k) = p_h g_{lh}(k-1) + (1 - p_h) g_{lh}(k)$, we obtain:

$$p_l * \alpha_l(t, \varnothing, p_{-l}) = \sum_{k=\lceil tN \rceil}^{N-1} p_l[p_h * g_{lh}(k-1) + (1 - p_h) g_{lh}(k)][F(\frac{M+N}{2} - k - 1) - F(\frac{M-N}{2} - 1)]$$

and

$$p_h * \alpha_h(t, \varnothing, p_{-h}) = \sum_{k=\lceil tN \rceil}^{N-1} p_h[p_l * g_{lh}(k-1) + (1 - p_l) g_{lh}(k)][F(\frac{M+N}{2} - k - 1) - F(\frac{M-N}{2} - 1)],$$

so:

$$p_h * \alpha_h(t, \varnothing, p_{-h}) - p_l * \alpha_l(t, \varnothing, p_{-l}) = \sum_{k=\lceil tN \rceil}^{N-1} [(p_h - p_l) g_{lh}(k)][F(\frac{M+N}{2} - k - 1) - F(\frac{M-N}{2} - 1)].$$

which relabeling the counter in the summation becomes:

$$\sum_{k=\lceil tN \rceil - \frac{N}{2}}^{\frac{N}{2}-1} [(p_h - p_l) g_{lh}(\frac{N}{2} + k)][F(\frac{M}{2} - k - 1) - F(\frac{M-N}{2} - 1)].$$

Now, noting that for $i = \{l, h\}$,

$$\beta_i(t, \varnothing, p_{-i}) = \sum_{k=1}^{\lfloor (1-t)N \rfloor} g_i(k)[F(\frac{M+N}{2}) - F(\frac{M+N}{2} - k)].$$

and omitting a very similar step we directly obtain that

$$(1 - p_h)\beta_h(t, \varnothing, p_{-h}) - (1 - p_l)\beta_l(t, \varnothing, p_{-l}) =$$
$$= - \sum_{k=1}^{\lfloor (1-t)N \rfloor} [(p_h - p_l) g_{lh}(k-1)][F(\frac{M+N}{2}) - F(\frac{M+N}{2} - k)]$$

which, since $\lceil tN \rceil + \lfloor (1-t)N \rfloor = N$, relabeling the counter in the summation becomes:

$$(1 - p_h)\beta_h(t, \varnothing, p_{-h}) - (1 - p_l)\beta_l(t, \varnothing, p_{-l}) =$$
$$= - \sum_{k=\lceil tN \rceil - \frac{N}{2}}^{\frac{N}{2}-1} [(p_h - p_l) g_{lh}(\frac{N}{2} - k - 1)][F(\frac{M+N}{2}) - F(\frac{M}{2} + k)].$$

By Assumption 1,

$$F(\frac{M+N}{2}) - F(\frac{M}{2} + k) = F(\frac{M}{2} - k - 1) - F(\frac{M-N}{2} - 1),$$

it follows

$$(1 - p_h)\beta_h(t, \varnothing, p_{-h}) + p_h * \alpha_h(t, \varnothing, p_{-h}) - (1 - p_l)\beta_l(t, \varnothing, p_{-l}) - p_l * \alpha_l(t, \varnothing, p_{-l}) =$$
$$= \sum_{k=\lceil tN \rceil - \frac{N}{2}}^{\frac{N}{2}-1} [(p_h - p_l)[F(\frac{M+N}{2}) - F(\frac{M}{2} + k)][g_{lh}(\frac{N}{2} + k) - g_{lh}(\frac{N}{2} - k - 1)]$$

which is positive by Assumption 2.

Step 2:

Noting that for $i = \{l, h\}$,

$$\alpha_i(\varnothing, t, p_{-i}) = \sum_{k=0}^{\lfloor (1-t)N \rfloor - 1} g_i(k)[F(\frac{M+N}{2}) - F(\frac{M+N}{2} - k - 1)] \text{ and}$$

$$\beta_i(\varnothing, t, p_{-i}) = \sum_{k=\lceil tN \rceil + 1}^{N} g_i(k)[F(\frac{M+N}{2} - k) - F(\frac{M-N}{2} - 1)],$$

and repeating the same steps as in Step 1, we get:

$$p_h * \alpha_h(\varnothing, t, p_{-h}) - p_l * \alpha_l(\varnothing, t, p_{-l}) = \sum_{k=0}^{\lfloor (1-t)N \rfloor - 1} [(p_h - p_l) g_{lh}(k)][F(\frac{M+N}{2}) - F(\frac{M+N}{2} - k - 1)]$$

which relabeling the counter in the summation becomes:

$$p_h * \alpha_h(\varnothing, t, p_{-h}) - p_l * \alpha_l(\varnothing, t, p_{-l}) = \sum_{k=\lceil tN \rceil - \frac{N}{2}}^{\frac{N}{2} - 1} [(p_h - p_l) g_{lh}(\frac{N}{2} - k - 1)][F(\frac{M+N}{2}) - F(\frac{M}{2} + k)]$$

and

$$(1 - p_h)\beta_h(\varnothing, t, p_{-h}) - (1 - p_l)\beta_l(\varnothing, t, p_{-l}) =$$
$$= - \sum_{k=\lceil tN \rceil + 1}^{N} [(p_h - p_l) g_{lh}(k - 1)][F(\frac{M+N}{2} - k) - F(\frac{M-N}{2} - 1)]$$

which, relabeling once again, becomes

$$(1 - p_h)\beta_h(\varnothing, t, p_{-h}) - (1 - p_l)\beta_l(\varnothing, t, p_{-l}) =$$
$$= - \sum_{k=\lceil tN \rceil - \frac{N}{2}}^{\frac{N}{2} - 1} [(p_h - p_l) g_{lh}(\frac{N}{2} + k)][F(\frac{M}{2} - k - 1) - F(\frac{M-N}{2} - 1)].$$

Therefore,

$$-(1 - p_h)\beta_h(\varnothing, t, p_{-h}) - p_h * a_h(\varnothing, t, p_{-h}) + (1 - p_l)\beta_l(\varnothing, t, p_{-l}) + p_l * \alpha_l(\varnothing, t, p_{-l}) =$$
$$\sum_{k=\lceil tN \rceil - \frac{N}{2}}^{\frac{N}{2} - 1} [(p_h - p_l)[F(\frac{M}{2} - k - 1) - F(\frac{M-N}{2} - 1)][g_{lh}(\frac{N}{2} + k) - g_{lh}(\frac{N}{2} - k - 1)]$$

which is also positive by Assumption 2.

It follows that $EU_h[t] - EU_h[\varnothing] - (EU_l[t] - EU_l[\varnothing]) \geq 0$. ■

6.5 Lemma 2

To shorten the proofs of Propositions 3, 6 and 8 we introduce a second lemma:

Lemma 2 *Given two internal voting rules A and B for coalition C, a member $i \in C$ is indifferent between A and B if:*
$p_i = \frac{\beta_i(A,B,p_{-i})-\beta_i(B,A,p_{-i})}{\alpha_i(B,A,p_{-i})-\alpha_i(A,B,p_{-i})+\beta_i(A,B,p_{-i})-\beta_i(B,A,p_{-i})}$.

Proof. $EU_i(B)-EU_i(A) = p_i[\alpha_i(B,A,p_{-i})-\alpha_i(A,B,p_{-i})]+(1-p_i)[\beta_i(B,A,p_{-i})-\beta_i(A,B,p_{-i})]$, where the functions α_i and β_i are as defined in the proof of Lemma 1.

Equating to zero and solving for p_i we get:
$p_i = \frac{\beta_i(A,B,p_{-i})-\beta_i(B,A,p_{-i})}{\alpha_i(B,A,p_{-i})-\alpha_i(A,B,p_{-i})+\beta_i(A,B,p_{-i})-\beta_i(B,A,p_{-i})}$. ■

6.6 Proposition 3

Let simple majority be the general election rule. Then the formation of a voting bloc by the coalition with t as the internal decision rule benefits member $l \in C$ if and only if $p_l > p_l^{t,\varnothing}(p_{-l})$.

Proof. By Lemma 2, member l is indifferent between a voting bloc with a $t-majority$ or a voting bloc with unanimity (identical to no voting bloc) if:
$p_l^{t,\varnothing}(p_{-l}) = \frac{\beta_l(\varnothing,t,p_{-l})-\beta_l(t,\varnothing,p_{-l})}{\alpha_l(t,\varnothing,p_{-l})-a_l(\varnothing,t,p_{-l})+\beta_l(\varnothing,t,p_{-l})-\beta_l(t,\varnothing,p_{-l})}$,
where:

$$\begin{aligned}
\alpha_l(t,\varnothing,p_{-l}) &= \sum_{k=\lceil tN\rceil}^{N-1} g_l(k)[F(\frac{M+N}{2}-k-1)-F(\frac{M-N}{2}-1)];\\
\beta_l(t,\varnothing,p_{-l}) &= \sum_{k=1}^{\lfloor(1-t)N\rfloor} g_l(k)[F(\frac{M+N}{2})-F(\frac{M+N}{2}-k)];\\
a_l(\varnothing,t,p_{-l}) &= \sum_{k=0}^{\lfloor(1-t)N\rfloor-1} g_l(k)[F(\frac{M+N}{2})-F(\frac{M+N}{2}-k-1)]\\
\text{and } \beta_l(\varnothing,t,p_{-l}) &= \sum_{k=\lceil tN\rceil+1}^{N} g_l(k)[F(\frac{M+N}{2}-k)-F(\frac{M-N}{2}-1)].
\end{aligned}$$

The derivative with respect to p_l of the surplus for member l generated by the voting bloc with internal voting rule t is equal to the denominator of $p_l^{t,\varnothing}(p_{-l})$, which relabeling the counter in the four summations in the denominator, is equal to:

$$\sum_{k=\lceil tN\rceil-\frac{N}{2}}^{\frac{N}{2}-1} g_l(\frac{N}{2}+k)[F(\frac{M}{2}-k-1)-F(\frac{M-N}{2}-1)]$$
$$-\sum_{k=\lceil tN\rceil-\frac{N}{2}}^{N/2} g_l(\frac{N}{2}-k)[F(\frac{M+N}{2})-F(\frac{M}{2}+k)]$$
$$+\sum_{k=\lceil tN\rceil-\frac{N}{2}+1}^{N/2} g_l(\frac{N}{2}+k)[F(\frac{M}{2}-k)-F(\frac{M-N}{2}-1)]$$
$$-\sum_{k=\lceil tN\rceil-\frac{N}{2}+1}^{N/2} g_l(\frac{N}{2}-k)[F(\frac{M+N}{2})-F(\frac{M}{2}+k-1)]$$

$$= \sum_{k=\lceil tN\rceil-\frac{N}{2}+1}^{N/2} g_l(\frac{N}{2}+k)[F(\frac{M}{2}-k)-F(\frac{M-N}{2}-1)+F(\frac{M}{2}-k-1)-F(\frac{M-N}{2}-1)]$$
$$+g_l(\lceil tN\rceil)[F(\frac{M+N}{2}-1-\lceil tN\rceil)-F(\frac{M-N}{2}-1)$$
$$-\sum_{k=\lceil tN\rceil-\frac{N}{2}+1}^{N/2} g_l(\frac{N}{2}-k)[F(\frac{M+N}{2})-F(\frac{M}{2}+k-1)+F(\frac{M+N}{2})-F(\frac{M}{2}+k)]$$
$$-g_l(N-\lceil tN\rceil)[F(\frac{M+N}{2})-F(\frac{M-N}{2}+\lceil tN\rceil)].$$

then note that, by assumption,

$$F(\frac{M}{2}-k-1)=1-F(\frac{M}{2}+k); F(\frac{M}{2}-k)=1-F(\frac{M}{2}+k-1)$$

and $2F(\frac{M-N}{2}-1)=2-2F(\frac{M+N}{2})$.

Substitute accordingly to get:

$$\sum_{k=\lceil tN\rceil-\frac{N}{2}+1}^{N/2} [g_l(\frac{N}{2}+k)-g_l(\frac{N}{2}-k)][2F(\frac{M+N}{2})-F(\frac{M}{2}+k)-F(\frac{M}{2}+k-1)]$$
$$+g_l(\lceil tN\rceil)[\sum_{k=0}^{N-1-\lceil tN\rceil} g_l(\frac{M-N}{2}+k)]-g_l(\lfloor(1-t)N\rfloor)\sum_{k=\lceil tN\rceil+1}^{N} f(\frac{M-N}{2}+k)=$$

$$= \sum_{k=\lceil tN\rceil-\frac{N}{2}+1}^{N/2} [g_l(\frac{N}{2}+k)-g_l(\frac{N}{2}-k)][2F(\frac{M+N}{2})-F(\frac{M}{2}+k)-F(\frac{M}{2}+k-1)]$$
$$+[g_l(\lceil tN\rceil)-g_l(\lfloor(1-t)N\rfloor)]\sum_{k=0}^{N-1-\lceil tN\rceil} f(\frac{M-N}{2}+k)$$

Since $\lceil tN \rceil + \lfloor (1-t)N \rfloor = N$ and $t \geq \frac{1}{2}$, if follows $g_l(\lceil tN \rceil) > g_l(\lfloor (1-t)N \rfloor)$ and thus the denominator is positive.

Therefore,

$$EU_l[t] - EU_l[\varnothing] > 0 \Longleftrightarrow p_l > p_l^{t,\varnothing}.$$

Then, by Lemma 1, $EU_l[t] - EU_l[\varnothing] > 0 \Longrightarrow EU_i[t] - EU_i[\varnothing] > 0$ for all $i \in C$.

As a corollary note, if the internal voting rule is simple majority, then $\lceil tN \rceil = (1-t)N = \frac{N}{2}$, $[g_l(\lceil tN \rceil) - g_l(\lfloor (1-t)N \rfloor)] = 0$ and the threshold $p_l^{t,\varnothing}(p_{-l})$ simplifies to:

$$p_l^{Sm,\varnothing}(p_{-l}) = \frac{\sum_{k=\frac{N}{2}+1}^{N} g_l(k)[F(\frac{M+N}{2}-k) - F(\frac{M-N}{2}-1)] - \sum_{k=1}^{N/2} g_l(k)[F(\frac{M+N}{2}) - F(\frac{M+N}{2}-k)]}{\sum_{k=1}^{N/2}[g_l(\frac{N}{2}+k) - g_l(\frac{N}{2}-k)] * [2F(\frac{M+N}{2}) - F(\frac{M}{2}+k) - F(\frac{M}{2}+k-1)]}.$$

■

6.7 Proposition 4

Let $V^t \subset V$ be the subset of type profiles such that a $t-majority$ rule is beneficial for C and let $t' = t + \frac{1}{N+1}$.

WTS: For any $t \in [\frac{1}{2}, \frac{N-1}{N+1})$, V^t is strictly contained in $V^{t'}$.

Proof. $V^t = \{\overrightarrow{p} \in \Re^{N+M} : A1, A2 \text{ hold and } p_l > p_l^{t,\varnothing}(p_{-l})\}$.

$V^{t'} = \{\overrightarrow{p} \in \Re^{N+M} : A1, A2 \text{ hold and } p_l > p_l^{t+\frac{1}{N+1},\varnothing}(p_{-l})\}$.

It suffices to show that $p_l^{t,\varnothing}(p_{-l}) > p_l^{t+\frac{1}{N+1},\varnothing}(p_{-l})$ for any p_{-l}.

Suppose $p_l = p_l^{t,\varnothing}(p_{-l})$. Then $EU_l[t] - EU_l[\varnothing] = 0$

Let $S_{l|k}$ denote the probability that member l is in the majority of the coalition, given that the minority is of size k and let $s_{l|k} = 1 - S_{l|k}$ denote the probability that member l is in the minority of the coalition, given that the minority is of size k. Then $EU_l[t] - EU_l[\varnothing] =$

$$\sum_{k=1}^{\lfloor (1-t)N \rfloor} (S_{l|k} - s_{l|k}) prob[Minority\ size = k] prob[rolling\ k\ votes\ alters\ outcome].$$

Note that $S_{l|k} - s_{l|k}$ is decreasing in k. The bigger the minority, the more likely l is in it. Then, $EU_l[t] - EU_l[\varnothing] = 0$ implies that for $k = \lfloor (1-t)N \rfloor > 1$, $(S_{l|k} - s_{l|k}) < 0$. Then:

$$\sum_{k=1}^{\lfloor (1-t)N \rfloor - 1} (S_{l|k} - s_{l|k}) prob[Minority\ size = k] prob[rolling\ k\ votes\ alters\ outcome] > 0.$$

But

$$\sum_{k=1}^{\lfloor (1-t)N \rfloor - 1} (S_{l|k} - s_{l|k}) prob[Minority\ size = k] prob[rolling\ k\ votes\ alters\ outcome]$$

is equal to $EU_l[t'] - EU_l[\varnothing]$, so $p_l > p_l^{t',\varnothing}(p_{-l})$. ■

6.8 Proposition 5

Proof. Let M be fixed. Let all the N members in $C\backslash\{l\}$ have a common type r. Then:

$$EU_l[Sm] - EU_l[\varnothing] = p\sum_{k=0}^{\frac{M}{2}-1} g_l(\frac{N}{2}+k)F(\frac{M}{2}-k-1) + (1-p)\sum_{k=0}^{\frac{M}{2}-1} g_l(\frac{N}{2}-k)[1-F(\frac{M}{2}+k)]$$
$$-p\sum_{k=0}^{\frac{M}{2}-1} g_l(\frac{N}{2}-k-1)[1-F(\frac{M}{2}+k)] - (1-p)\sum_{k=0}^{\frac{M}{2}-1} g_l(\frac{N}{2}+k+1)[F(\frac{M}{2}-k-1)]$$

Since $F(\frac{M}{2}-k-1) = [1-F(\frac{M}{2}+k)]$, this is equal to:

$$\sum_{k=0}^{\frac{M}{2}-1} \frac{N!}{(\frac{N}{2}-k)!(\frac{N}{2}+k)!}[p * r^{\frac{N}{2}+k}(1-r)^{\frac{N}{2}-k} + (1-p)r^{\frac{N}{2}-k}(1-r)^{\frac{N}{2}+k}]F(\frac{M}{2}-k-1)$$
$$-\sum_{k=0}^{\frac{M}{2}-1} \frac{N!}{(\frac{N}{2}-k-1)!(\frac{N}{2}+k+1)!}[p * r^{\frac{N}{2}-k-1}(1-r)^{\frac{N}{2}+k+1}$$
$$+(1-p)r^{\frac{N}{2}+k+1}(1-r)^{\frac{N}{2}-k-1}[F(\frac{M}{2}-k-1)].$$

Equating to zero and simplifying:

$$\sum_{k=0}^{\frac{M}{2}-1} p\{\frac{1}{N-2k}[r^{2k+1}(1-r) - r(1-r)^{2k+1}] - \frac{1}{N+2k+2}[(1-r)^{2k+2} - r^{2k+2}]\}$$
$$= \quad \sum_{k=0}^{\frac{M}{2}-1} (\frac{r^{2k+2}}{N+2k+2} - \frac{(1-r)^{2k+1}}{N-2k})$$

Now we break this equation into $\frac{M}{2}$ different equations, imposing that for each $k \in \{0, \frac{M}{2}-1\}$,

$$p\{\frac{1}{N-2k}[r^{2k+1}(1-r) - r(1-r)^{2k+1}] - \frac{1}{N+2k+2}[(1-r)^{2k+2} - r^{2k+2}]\}$$
$$= \quad (\frac{r^{2k+2}}{N+2k+2} - \frac{(1-r)^{2k+1}}{N-2k})$$

A solution to this system of equations (with just one unknown) also solves the original equation. For each individual equation:

$$p = \frac{(N-2k)r^{2k+2} - (N+2k+2)r(1-r)^{2k+1}}{r(1-r)[r^{2k} - (1-r)^{2k}](N+2k+2) - (N-2k)[(1-r)^{2k+2} - r^{2k+2}]}$$

which, as $N \longrightarrow \infty$, converges to

$$\frac{r^{2k+2} - r(1-r)^{2k+1}}{r^{2k+1}(1-r) - r(1-r)^{2k+1} - (1-r)^{2k+2} + r^{2k+2}} = r$$

So $\lim_{N-->\infty} p_l^{Sm,\varnothing}(p_{-l}) = r$. Since by assumption $p_l < r$, this implies that for N large enough, $p_l < p_l^{Sm,\varnothing}(p_{-l})$ and then member l would be hurt if C forms a voting bloc with simple majority. ■

6.9 Proposition 6

Proof. Let *Out* denote the internal voting rule for the coalition under which N members form a voting bloc with simple majority and member l stays out of the bloc and does not pool her vote with the rest of the coalition. By Lemma 2 member l is indifferent between rules *Out* and $\varnothing$ if:
$p_l^{Out,\varnothing}(p_{-l}) = \frac{\beta_l(\varnothing,Out,p_{-l})-\beta_l(Out,\varnothing,p_{-l})}{\alpha_l(Out,\varnothing,p_{-l})-\alpha_l(\varnothing,Out,p_{-l})+\beta_l(\varnothing,Out,p_{-l})-\beta_l(Out,\varnothing,p_{-l})}$, where:

$$\begin{aligned}
\alpha_l(Out,\varnothing,p_{-l}) &= \sum_{k=\frac{N}{2}+1}^{N-1} g_l(k)[F(\frac{M+N}{2}-k-1) - F(\frac{M-N}{2}-1)]; \\
\beta_l(Out,\varnothing,p_{-l}) &= \sum_{k=1}^{\frac{N}{2}-1} g_l(k)[F(\frac{M+N}{2}) - F(\frac{M+N}{2}-k)]; \\
\alpha_l(\varnothing,Out,p_{-l}) &= \sum_{k=1}^{\frac{N}{2}-1} g_l(k)[F(\frac{M+N}{2}-1) - F(\frac{M+N}{2}-k-1)] \\
\text{and } \beta_l(\varnothing,Out,p_{-l}) &= \sum_{k=\frac{N}{2}+1}^{N-1} g_l(k)[F(\frac{M+N}{2}-k) - F(\frac{M-N}{2})].
\end{aligned}$$

Relabeling the counters in the summations, we can write the denominator $\alpha_l(Out,\varnothing,p_{-l}) - \alpha_l(\varnothing,Out,p_{-l}) + \beta_l(\varnothing,Out,p_{-l}) - \beta_l(Out,\varnothing,p_{-l})$ as:

$$\begin{aligned}
&\sum_{k=1}^{\frac{N}{2}-1} g_l(\frac{N}{2}+k)[F(\frac{M}{2}-k-1) - F(\frac{M-N}{2}-1)] \\
&-\sum_{k=1}^{\frac{N}{2}-1} g_l(\frac{N}{2}-k)[F(\frac{M+N}{2}-1) - F(\frac{M}{2}+k-1)] \\
&+\sum_{k=1}^{\frac{N}{2}-1} g_l(\frac{N}{2}+k)[F(\frac{M}{2}-k) - F(\frac{M-N}{2})] - \sum_{k=1}^{\frac{N}{2}-1} g_l(\frac{N}{2}-k)[F(\frac{M+N}{2}) - F(\frac{M}{2}+k)].
\end{aligned}$$

Since

$$[F(\frac{M}{2}-k)-F(\frac{M-N}{2})+F(\frac{M}{2}-k-1)-F(\frac{M-N}{2}-1)]=$$
$$[F(\frac{M+N}{2})-F(\frac{M}{2}+k)+F(\frac{M+N}{2}-1)-F(\frac{M}{2}+k-1)],$$

the denominator simplifies to

$$\sum_{k=1}^{\frac{N}{2}-1}[g_l(\frac{N}{2}+k)-g_l(\frac{N}{2}-k)][F(\frac{M}{2}-k)-F(\frac{M-N}{2})+F(\frac{M}{2}-k-1)-F(\frac{M-N}{2}-1)]$$

and $p_l^{Out,\varnothing}(p_{-l})=\frac{\sum_{k=1}^{\frac{N}{2}-1}\{g_l(\frac{N}{2}+k)[F(\frac{M}{2}-k)-F(\frac{M-N}{2})]-g_l(\frac{N}{2}-k)[F(\frac{M+N}{2})-F(\frac{M}{2}+k)]\}}{\sum_{k=1}^{\frac{N}{2}-1}[g_l(\frac{N}{2}+k)-g_l(\frac{N}{2}-k)][F(\frac{M}{2}-k)-F(\frac{M-N}{2})+F(\frac{M}{2}-k-1)-F(\frac{M-N}{2}-1)]}$.

The difference in utility for agent l between the formation of a bloc without l and no bloc at all is

$$EU_l[Out]-EU_l[\varnothing]=$$
$$=p_l[\alpha_l(Out,\varnothing,p_{-l})-\alpha_l(\varnothing,Out,p_{-l})]+(1-p_l)[\beta_l(Out,\varnothing,p_{-l})-\beta_l(\varnothing,Out,p_{-l})].$$

The derivative of $EU_l[Out]-EU_l[\varnothing]$ with respect to p_l coincides with the denominator of the threshold $p_l^{Out,\varnothing}(p_{-l})$, which is positive. Therefore, for p_l above the threshold $p_l^{Out,\varnothing}(p_{-l})$, member l prefers the formation of a voting bloc in which l does not participate better than not forming any bloc at all; whereas for p_l below $p_l^{Out,\varnothing}(p_{-l})$, member l prefers to form no bloc than to form a bloc in which l does not participate. ■

6.10 Proposition 7

Let V be the set of type profiles $(p_1,p_2,...,p_{N+M+1})$ satisfying Assumptions 1 and 2. Let $V^t\subset V$ be the subset of type profiles such that a $t-majority$ rule is beneficial for C, let $(V^t)^C$ be its complement such that $V^t\cup(V^t)^C\equiv V$ and let $V^{Out}\subset V$ be the subset of type profiles such that an "Opt-Out for l" rule is beneficial for C. Then, for any M and for any $4\leq N\leq M$, $(V^t)^C\cap V^{Out}$ is not empty.

Proof. Suppose $4\leq N\leq M$, $p_l=(1-\delta)$ and $p_j=(1-\varepsilon)$ $\forall j\in C\backslash\{l\}$. Then:

$$
\begin{aligned}
EU_l[Out] - EU_l[\varnothing] &= (1-\delta) \sum_{k=\lceil t(N-1)\rceil+1}^{N-1} g_l(k)[F(\frac{M+N}{2} - k - 1) - F(\frac{M-N}{2} - 1)] \\
&+\delta \sum_{k=1}^{\lfloor (1-t)(N-1)\rfloor} g_l(k)[F(\frac{M+N}{2}) - F(\frac{M+N}{2} - k)] \\
&-(1-\delta) \sum_{k=1}^{\lfloor (1-t)(N-1)\rfloor} g_l(k)[F(\frac{M+N}{2} - 1) - F(\frac{M+N}{2} - k - 1)] \\
&-\delta \sum_{k=\lceil t(N-1)\rceil+1}^{N-1} g_l(k)[F(\frac{M+N}{2} - k) - F(\frac{M-N}{2})].
\end{aligned}
$$

As ε converges to zero, $\frac{g_l(k)}{g_l(N-1)}$ converges to zero for any $k < N-1$ and $EU_l[Out] - EU_l[\varnothing]$ converges to:

$(1-\delta)g_l(N-1)[F(\frac{M-N}{2})-F(\frac{M-N}{2}-1)]-\delta g_l(N-1)[F(\frac{M-N}{2}+1)-F(\frac{M-N}{2})]$ which is positive for a sufficiently low δ, provided that $f(\frac{M-N}{2}) > 0$.

Then, there exist a $\delta > 0$ and $\overline{\varepsilon} > 0$ such that for all $\varepsilon < \overline{\varepsilon}$, $EU_l[Out] - EU_l[\varnothing] > 0$.

Therefore, if $4 \leq N \leq M,$ and the types of all the members but l converge to 1, member l with type $p_l = (1-\delta)$ will benefit from a voting bloc with an "Opt-Out for l" rule. Given that all the other members of C share a common type, they would all benefit from forming a voting bloc without l. Since $p_l^{t,\varnothing}(p_{-l})$ converges to 1, member l would not benefit from a $t-majority$ internal voting rule in a voting bloc that includes every member. It follows that if $\varepsilon < \overline{\varepsilon}$, any profile of types in which $p_l = (1-\delta)$ and $p_j = (1-\varepsilon)$ $\forall j \in C\backslash\{l\}$ is in V^{Out} but not in V^t, thus $V^{Out} \nsubseteq V^t$.

If $N > M$, then $EU_l[Sm] - EU_l[Out] = 2g_l(\frac{N}{2})[F(\frac{M+N}{2}) - F(\frac{M}{2})] > 0$, thus thus $V^{Out} \subset V^{Sm}$. If $N = 2$, then Out coincides with $\varnothing$ and V^{Out} is empty. ■

6.11 Proposition 8

Proof. By Lemma 2, member l will be indifferent between participating in the voting bloc or opting out if

$p_l^{Sm,Out}(p_{-l}) = \frac{\beta_l(Out,Sm,p_{-l})-\beta_l(Sm,Out,p_{-l})}{\alpha_l(Sm,Out,p_{-l})-\alpha_l(Out,Sm,p_{-l})+\beta_l(Out,Sm,p_{-l})-\beta_l(Sm,Out,p_{-l})}$, where:

$\alpha_l(Sm,Out,p_{-l}) = g_l(\frac{N}{2})[F(\frac{M}{2}-1) - F(\frac{M-N}{2}-1)]$; $\beta_l(Sm,Out,p_{-l}) = g_l(\frac{N}{2})[F(\frac{M+N}{2})-F(\frac{M}{2})]$; $\alpha_l(Out,Sm,p_{-l}) = \sum_{k=0}^{\frac{N}{2}-1} g_l(k)f(\frac{M+N}{2})$; $\beta_l(Out,Sm,p_{-l}) = \sum_{k=\frac{N}{2}+1}^{N} g(k)f(\frac{M-N}{2})$.

Since $F(\frac{M}{2}-1)-F(\frac{M-N}{2}-1) = F(\frac{M+N}{2})-F(\frac{M}{2})$ and $f(\frac{M+N}{2}) = f(\frac{M-N}{2})$ it follows that $\alpha_l(Sm,Out,p_{-l}) = \beta_l(Sm,Out,p_{-l})$ and we can simplify the denominator to $\alpha_l(Out, Sm, p_{-l}) - \beta_l(Out, Sm, p_{-l}) =$

$$[\sum_{k=\frac{N}{2}+1}^{N} g(k) - \sum_{k=0}^{\frac{N}{2}-1} g_l(k)]f(\frac{M+N}{2}) = \sum_{k=1}^{N/2}[g_l(\frac{N}{2}+k) - g_l(\frac{N}{2}-k)]f(\frac{M+N}{2})$$

and $p_l^{Sm,Out}(p_{-l}) = \frac{\sum_{k=\frac{N}{2}+1}^{N} g(k)f(\frac{M-N}{2}) - g_l(\frac{N}{2})[F(\frac{M+N}{2}) - F(\frac{M}{2})]}{\sum_{k=1}^{N/2}[g_l(\frac{N}{2}+k) - g_l(\frac{N}{2}-k)]f(\frac{M+N}{2})}$.

The advantage for member l of staying in, $EU_l[Sm] - EU_l[Out] =$

$$= p_l[\alpha_l(Sm, Out, p_{-l}) - a_l(Out, Sm, p_{-l})] + (1-p_l)[\beta_l(sm, Out, p_{-l}) - \beta_l(Out, Sm, p_{-l})].$$

Its derivative with respect to p_l coincides with the denominator of $p_l^{Sm,Out}(p_{-l})$. Since $g_l(\frac{N}{2}+k) > g_l(\frac{N}{2}-k)$ for all $k \in [1, \frac{N}{2}]$, the denominator and thus the derivative are positive. Therefore, member l prefers to stay in if type p_l is above the threshold $p_l^{Sm,Out}(p_{-l})$ and member l prefers to opt-out than to stay in if $p_l < p_l^{Sm,Out}(p_{-l})$. ∎

6.12 Proposition 9

Proof. The first statement comes straightforward from Propositions 3 and 8. For the second one, suppose $p_i = p \; \forall i \in C$. Then all members benefit from the formation of a voting bloc with simple majority: $EU_i[Sm] > EU_i[\varnothing] \; \forall i \in C$. The extra gains of stepping out for member l when simple majority is the general voting rule are:

$$EU_l[Out] - EU_l[Sm] = p\sum_{k=0}^{\frac{N}{2}-1} g_l(k)f(\frac{M+N}{2}) + (1-p)\sum_{k=\frac{N}{2}+1}^{N} g_l(k)f(\frac{M-N}{2})$$
$$-pg_l(\frac{N}{2})[F(\frac{M}{2}-1) - F(\frac{M-N}{2}-1)] - (1-p)g_l(\frac{N}{2})[F(\frac{M+N}{2}) - F(\frac{M}{2})]$$

For $p \in (1/2, 1)$ and any N, M :

$$\begin{aligned} EU_l[Out] - EU_l[Sm] &> (1-p)\sum_{k=\frac{N}{2}+1}^{N} g_l(k)f(\frac{M-N}{2}) - g_l(\frac{N}{2})[F(\frac{M+N}{2}) - F(\frac{M}{2})] \\ &> (1-p)f(\frac{M-N}{2})g_l(N) - \frac{1}{2}g_l(\frac{N}{2}), \end{aligned}$$

which letting $\alpha = f(\frac{M-N}{2})$ and $\beta = \frac{N!}{\frac{N}{2}!\frac{N}{2}!}$ is equal to:

$\alpha(1-p)p^N - \frac{1}{2}\beta p^{N/2}(1-p)^{N/2} = (1-p)p^{N/2}[\alpha p^{N/2} - \frac{1}{2}\beta(1-p)^{\frac{N-2}{2}}]$, which is positive if and only if

$$2\alpha p^{N/2} \geq \beta(1-p)^{\frac{N-2}{2}} \iff \frac{p^{N/2}}{(1-p)^{\frac{N-2}{2}}} \geq \frac{\beta}{2\alpha} \iff \frac{p}{(1-p)^{\frac{N-2}{N}}} \geq \left(\frac{\beta}{2\alpha}\right)^{2/N}.$$

Letting $\gamma > 0$ be any number such that $p \geq \gamma$, this last inequality will be satisfied if

$$\frac{\gamma}{(1-p)^{\frac{N-2}{N}}} \geq \left(\frac{\beta}{2\alpha}\right)^{2/N} \Longleftrightarrow (1-p)^{\frac{N-2}{N}} \leq \gamma\left(\frac{2\alpha}{\beta}\right)^{2/N} \Longleftrightarrow (1-p) \leq \gamma\left(\frac{2\alpha}{\beta}\right)^{\frac{2}{N-2}}$$
$$\Longleftrightarrow \quad p \geq 1-\gamma\left(\frac{2\alpha}{\beta}\right)^{\frac{2}{N-2}},$$

which is less than one for $N \in [4, M]$, provided that $f(\frac{M-N}{2}) > 0$.

Note that if $N = 2$, then $\left(\frac{2\alpha}{\beta}\right)^{\frac{2}{N-2}} = 0$; whereas if $N > M$, $\alpha = 0$.

If $N = 2$, then $EU_l[Out] = EU_l[\varnothing]$, thus $EU_l[Sm] > EU_l[\varnothing]$ implies $EU_l[Sm] > EU_l[Out]$.

If $N > M$, then $f(\frac{M+N}{2}) = f(\frac{M-N}{2}) = 0$, thus

$$EU_l[Out] - EU_l[Sm] = -pg_l(\frac{N}{2})F(\frac{M}{2}-1) - (1-p)g_l(\frac{N}{2})[1-F(\frac{M}{2})] < 0.$$

■

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