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Summary

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Keywords: Aggregation, Ambiguity, R&D, Expert Opinions, Convex/Conic Optimization

JEL Classification: C61, D81, Q42

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Ambiguous aggregation of expert opinions: the case of optimal R&D investment *

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Abstract

How should a decision-maker allocate R&D funds when a group of experts provides divergent estimates on a technology's potential effectiveness? To address this question, we propose a simple decision-theoretic framework that takes into account ambiguity over the aggregation of expert opinion and a decision-maker's attitude towards it. In line with the paper's focus on R&D investment, decision variables in our model may affect experts' subjective probability distributions of the future potential of a technology. Using results from convex optimization, we are able to establish a number of analytical results including a closed-form expression of our model's value function, as well as a thorough investigation of its differentiability properties. We apply our framework to original data from a recent expert elicitation survey on solar technology. The analysis suggests that more aggressive investment in solar technology R&D is likely to yield significant dividends even, or rather especially, after taking ambiguous aggregation into account.

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1 Introduction

Innovation is an uncertain process. The history of major R&D programs is paved with failures and dead ends and, eventually, successes. Failures can derive from a funding tap closed too early, or from the plain fact that a technology ultimately proves to be technically or economically infeasible. High opportunity costs and scarce public funding imply that making decisions over competing R&D programs is a delicate and considerably complex task. Importantly, it is a task that needs to take into account uncertainty.

Addressing the uncertainty of R&D programs is complicated by the fact that probabilities of success are very hard to estimate. They tend to be functions of R&D investment itself, and this endogeneity adds formidable challenges to their econometric estimation. Nonetheless, there exists a vast literature, studying patent numbers and/or productivity levels, that provides empirical support to the idea of a positive and strong relationship between R&D funding and innovation (Grossman and Helpman [19]). Of relevance to this paper's empirical exercise, such a connection has also been observed in the specific case of energy R&D investment (Newell et al. [28], Popp [31]).

Although the positive relationship between R&D and technological breakthroughs is well established, characterizing the probability of success given different levels of R&D expenditure is a question that can only partially be addressed by past data. Historical information on costs, patents, and R&D expenditure may be used to get an idea of the general trends, but research programs differ vastly and are, most of the time, not reproducible. Therefore, to account for the uncertainty of specific R&D programs it is often necessary to resort to expert judgments and subjective probabilities.

Structured expert judgment, pioneered in the 1975 Rasmussen Report on nuclear power plant safety, derives probabilistic input for decision problems through experts' quantification of their subjective uncertainties (Morgan and Henrion [27], Cooke [13], O'Hagan et al. [29]). Experts' probability distributions are elicited via transparent protocols and treated as scientific data. The employed elicitation techniques involve recognizing and removing, as much as possible, known psychological biases in judgment (Tversky and Kahneman [35]). They further incorporate consistency checks and structure the variables to be estimated in such a way that experts are called to respond to well-defined and clear questions. Expert elicitation surveys have been used in a wide variety of applications, including energy innovation (Baker et al. [4, 5]), and we refer the reader to [27, 13, 29] for a comprehensive account of this growing literature.

Despite its intuitive appeal, expert elicitation often generates widely-divergent opinions across experts, implying fundamentally different and competing views. To come to grips with this complexity, researchers typically aggregate over expert estimates in some fashion and consider their average. Indeed, there is a rich literature that studies the many different ways such aggregations may be performed. In their general survey papers, Clemen and Winkler [11, 12] broadly distinguish between (i) mathematical approaches and (ii) behavioral approaches. Mathematical aggregation methods are primarily concerned with constructing a single probability distribution on the basis of individual elicited distributions. This is usually pursued either through axiomatic treatments of mathematical formulas of aggregation, or, where possible, through Bayesian statistical methods. By contrast, behavioral approaches involve the direct interaction between experts in order to reach consensus on a single "group" estimate. This interaction can be structured in a number of different ways according to the application at hand.

We take a different approach to the ones outlined above. Motivated by contexts in which Bayesian updating methods are not readily applicable[1], we propose a modeling framework that is inspired by the economic-theory literature on decision making under ambiguity and as such is not concerned with determining a single probability distribution reflecting expert opinion.[2] In constrast to the Bayesian setting, decision-theoretic models of ambiguity are designed to address situations in which a decision maker is unable to posit precise probabilistic structure to physical and economic phenomena. This framework derives from the concept of uncertainty as introduced by Knight [25] to represent a situation in which a decision maker lacks adequate information to assign probabilities to events. Ambiguity is contrasted to risk, which defines settings in which probabilistic structure can be fully captured by a single Bayesian prior.

We now briefly describe the mechanics of our model. In our setting, a decision maker elicits the judgment of a set of experts on the effect of R&D investment on the future cost of a technology. Levels of R&D investment affect the decision maker's problem in two ways: (a) they alter experts' subjective probability distributions on the technology's future cost and (b) they are arguments of a utility function that measures the technology's cost-effectiveness as a function of its future cost and R&D expenditure. As an initial benchmark, our framework posits an equal-weight linear aggregation over experts' diverging probability distributions. Subsequently, it considers enlargements of the set of possible aggregation schemes by parametrizing over their maximum distance, measured via the L_2 norm, with respect to the benchmark equal-weight aggregation. This distance is referred to as *aggregation ambiguity.* Next, our model computes the best-and worst-case expected outcomes of a given level of R&D investment, subject to the feasible set of distributions that is implied by

[1]Say, when a group of experts is interviewed a single time on the long-term potential of an untested technology. Which matches perfectly our paper's empirical application (see Section 4).

[2]See Gilboa and Marinacci [17] for a comprehensive recent survey of this literature.

assigned levels of aggregation ambiguity. Finally, we consider a convex combination of the best and worst-case expected outcomes as a reasonable way to model decision makers' preferences under aggregation ambiguity.

Our model nests in a parametric fashion simple averaging and best/worst-case analysis and allows for an expression of decision-makers' beliefs regarding, and attitude towards, the underlying uncertainty in expert aggregation. Its simple structure allows for precise analytical insights, and we study its properties in depth. Using results from convex and conic optimization, we are able to provide a closed-form expression for our value function and its derivative with respect to aggregation ambiguity. These results enable sensitivity analysis across different levels of aggregation ambiguity and ambiguity attitude. We proceed to investigate the value function's differentiability in R&D investment and, where applicable, provide a closed-form expression for this derivative. This can subsequently be used to to obtain a necessary condition for optimal R&D investment. We conclude the paper's theoretical section by arguing that, while non-differentiability of the value function with respect to investment is in principle possible, it is not likely to be often encountered in practice.

We now discuss our model's relation to the existing literature. Our framework is a variation of the α-maxmin model that has been studied extensively in the decision-theoretic literature beginning with Hurwicz [20] and Arrow and Hurwicz [3]. Later contributions by Gilboa and Schmeidler [18] (whose seminal paper dealt with the pure maxmin model), Ghirardato et al. [16], Chateaunauf et al. [10], and Eichenberger et al. [14] focused on axiomatic treatments of similar models in which a decision maker's actions are modeled by Savage acts [34], i.e. functions from a state space to a space of consequences. Our work departs from these papers in a number of ways. First, the decision variables in our model are not functions. Instead, they are real numbers, representing levels of R&D investment, that enter the value function as arguments of (a) a utility function measuring the technology's cost-effectiveness as well as (b) the set of priors that the decision-maker is taking into account when performing his best- and worst-case analysis. This latter element of action-dependent expert beliefs is non-standard in the decision-theoretic literature. It has been studied in recent axiomatic work by Karni [23], who developed a Bayesian decision theory in which acts influence subjective beliefs. Other researchers have focused on generalizations of the Savage setting along similar lines. Olszewski [30] studied the α-maxmin model in a more abstract environment in which a decision maker is called to choose over sets of lotteries (i.e., priors), while Viero [36] axiomatized the α-maxmin model in a setting in which acts map from states to sets of priors.

The model we propose in this paper combines features of [23, 30, 36] but differs in its explicit treatment of aggregation ambiguity as a parameter input, as well as in its focus on the derivation and differentiability of the value function.[3] Correspondingly, the mathematical machinery we employ is also quite different and, unlike the previously cited economic-theory literature, we do not pursue an axiomatic characterization of preferences in our economic environment. As a result, we do not assert the appeal of our value function on formal principles of rationality. Instead, we see the primary virtues of our approach as being those of intuitiveness and practicality. Admitting a closed-form solution and straightforward interpretation/calibration, our model aims to integrate and operationalize the insights of the deeper contributions of the literature to the decision-making process as it pertains to expert opinions over R&D budgets. Indeed, the model we propose is an outgrowth of the need to develop a tractable theoretical framework to accommodate data from a recent expert elicitation on solar-energy R&D (see paragraph below and Section 4).

We base the empirical analysis of our paper on original data from the ICARUS project (Bosetti et al. [7]), a recent expert elicitation survey on the potential of R&D investment in alternative energy technologies.[4] As an initial step, we use the collected data of the survey to construct experts' subjective probability distributions on the future cost of solar energy conditional on R&D investment. Subsequently, we employ an integrated assessment model (Bosetti et al. [8]) to calculate the benefits of R&D investment (in the form of lower future solar-electricity costs) and use these estimates to inform our assesment of the relevant R&D investment alternatives. The application of our theoretical model to these data suggests that ambiguity plays an important role in assessing the potential of solar technology. The policy implication we are able to cautiously draw is that more aggressive investment in solar technology R&D is likely to yield significant dividends, even (or perhaps especially) after taking ambiguity into account.

Paper outline. The structure of the paper is as follows. Section 2 introduces the decision-theoretic model, while Section 3 analyzes its theoretical properties. Section 4 illustrates the theoretical results with original data from a recent expert elicitation on solar technology. Section 5 provides concluding remarks. All mathematical proofs, tables, figures, as well as non-essential supplementary information are collected in the Appendix.

2 Model Description

Consider a set $\mathcal{N}$ of experts indexed by $n = 1, 2, ..., N$. R&D investment is denoted by a variable $r \in \mathcal{R}$ and the technology's cost by $c \in \mathcal{C}$, where $\mathcal{R}$ and $\mathcal{C}$ are subsets of real numbers. An expert

[3] We elaborate further on the relevant literature and its relation to our work after we have formally defined the model in Section 2.

[4] For more information see www.icarus-project.org.

n's probability distribution of the future cost of technology given investment r is captured by a random variable $C_n(r)$ having a probability distribution function

$$\pi_n(c|r). \tag{1}$$

Note that the decision variables of our model (R&D investment) directly affect experts' subjective probability distributions of the technology's cost. This means that our setting is not amenable to standard decision-theoretic frameworks going back to Savage [34].

Expert beliefs over the economic potential of R&D investment may, and usually do, vary significantly. The question thus naturally arises: How do we make sense of this divergence when studying optimal R&D investment? In the absence of data that could lend greater credibility to one expert over another and form the basis of a Bayesian analysis, one straightforward way would be to simply aggregate over all pdfs π_n as given by Eq. (1), so that we obtain an "aggregate" joint pdf $\bar{\pi}$, where

$$\bar{\pi}(c|r) = \sum_{n=1}^{N} \frac{1}{N} \pi_n(c|r). \tag{2}$$

This approach inherently assumes that each and every expert is equally likely to represent reality, and makes use of simple linear aggregation. While this is standard practice in the applied expert elicitation literature, a great deal of information may be lost in such an averaging-out process, especially when there are huge differences among experts.

We thus move beyond simple averaging. In our framework each expert n's pdf $\pi_n(c|r)$ is weighted by the decision maker through a second-order probability p_n. The set of admissible second-order distributions $\mathbf{p}$ depends on the amount of ambiguity the decision maker is willing to take into account when aggregating across experts, and in particular on how "far" he is prepared to stray from equal-weight aggregation. Specifically, we consider the set of second-order distributions $\mathrm{P}(b)$ over a set of N experts, parametrized by $b \in \left[0, \frac{N-1}{N}\right]$ where

$$\mathrm{P}(b) = \left\{ \mathbf{p} \in \Re^N : \quad \mathbf{p} \geq \mathbf{0}, \quad \sum_{n=1}^{N} p_n = 1, \quad \sum_{n=1}^{N} \left(p_n - \frac{1}{N} \right)^2 \leq b \right\}. \tag{3}$$

Here, the set $\mathrm{P}(b)$ captures the uncertainty of the decision-maker's aggregation protocol. Thus, we refer to parameter b it as *aggregation ambiguity*. Letting $\boldsymbol{e_N}$ denote a unit vector of dimension N, we see that distributions $\mathbf{p}$ belonging to $\mathrm{P}(b)$ satisfy $||\mathbf{p} - \frac{\boldsymbol{e_N}}{N}||_2 \leq \sqrt{b}$, where $|| \cdot ||_2$ denotes the L_2-norm. Setting $b = 0$ implies complete certainty and adoption of the equal-weight singleton, while $b = \frac{N-1}{N}$ complete ambiguity over the set of all possible second-order distributions.[5]

[5]The latter statement holds in light of the fact that values of $b > \frac{N-1}{N}$ cannot enlarge the feasible set. This is

We briefly provide a potential interpretation of an ambiguity level b in our model. Consider the benchmark equal-weight distribution $\frac{1}{N}\,\boldsymbol{e_N}$. Now take a set of experts $\widehat{\mathcal{N}}$ of cardinality $\widehat{N}$ and begin increasing the collective second-order probability attached to their pdfs. The convex structure of the feasible set $\mathrm{P}(b)$ enables us to provide a tight upper bound on the maximum second-order probability $p(b, \widehat{N}; N)$ that can be placed on this set of experts, as a function of b and $\widehat{N}$:

$$p(b, \widehat{N}; N) = \max_{\mathbf{p}\in \mathrm{P}(b)} \max_{\left\{\widehat{\mathcal{N}}\subseteq\mathcal{N}:\ \left|\widehat{\mathcal{N}}\right|=\widehat{N}\right\}} \sum_{n\in\widehat{\mathcal{N}}} p_n = \min\left\{\frac{\widehat{N}}{N} + \widehat{N}\sqrt{\frac{N-\widehat{N}}{\widehat{N}N}b},\ 1\right\}. \tag{4}$$

To provide a sense of the above formula, let us focus on singleton sets so that $\widehat{N} = 1$. Increasing b from zero to 0.1 mean that the maximum second-order probability that can be assigned to a single agent is 0.48, 0.4, and 0.35 for $N = 5, 10$, and 20 experts, respectively.

Weighting the expert pdfs (1) under all aggregation schemes belonging in $\mathrm{P}(b)$ induces the following set of priors

$$\Pi(b, r) = \left\{\sum_{n=1}^{N} p_n(b)\pi_n(\cdot|r) :\quad \mathbf{p} \in \mathrm{P}(b)\right\} \tag{5}$$

governing the future cost of the technology conditional on R&D investment r. Thus, holding r fixed, an increase in b implies an expansion of the set of priors a decision maker is willing to consider.

Define the real-valued function

$$u\,(c, r) : \mathcal{C} \times \mathcal{R} \mapsto \Re,$$

as representing the utility of R&D investment r, under cost realization c. Now, given investment r, utility u, and the set of second-order distributions $\mathrm{P}(b)$ introduced in (3), we can calculate the best- and worst-case expected outcomes associated with r, given aggregation ambiguity b. This provides a measure of the spread, as measured by utility u, between the worst and best-cases, given a "willingness" to stray from the benchmark equal-weight distribution that is constrained by b. More formally, we consider the functions

$$V_{max}(r|b) = \max_{\pi\in\Pi(b,r)} \int_{\mathcal{C}} u\,(c, r)\,\mathrm{d}\pi(c) \tag{6}$$

$$V_{min}(r|b) = \min_{\pi\in\Pi(b,r)} \int_{\mathcal{C}} u\,(c, r)\,\mathrm{d}\pi(c). \tag{7}$$

Plotting functions (6) and (7) over $b \in [0, (N-1)/N]$ gives decision makers a comprehensive picture of the effectiveness of R&D investment r (as measured by utility u).

because the maximizers of $\sum_{n=1}^{N}\left(p_n - \frac{1}{N}\right)^2$ over the set of probability vectors concentrate all probability mass on one expert, leading to an aggregation ambiguity of $\left(1-\frac{1}{N}\right)^2 + (N-1)\cdot\left(\frac{1}{N}\right)^2 = \frac{N-1}{N}$.

The functions (6)-(7) fix a level of aggregation ambiguity b and subsequently focus on the best and worst cases. As such they capture extreme attitudes towards uncertainty in aggregation. To express more nuanced decision-maker preferences we consider the following value function

$$V(r|b,\alpha) = \alpha \cdot V_{min}(r|b) + (1-\alpha) \cdot V_{max}(r|b) \quad \alpha \in [0,1], \tag{8}$$

representing a convex combination of the worst- and best-cases. The parameter α above captures the decision maker's *ambiguity attitude*. It measures his degree of pessimism given aggregation ambiguity b: the greater (smaller) α is, the more (less) weight is placed on the worst-case scenario. Given values for b and α, Eq. (8) operates as an objective function when searching for optimal investment decisions r.

Relation to the literature. As mentioned in the introduction, the above framework is a variation of the α-maxmin model that has been studied extensively in the decision-theoretic literature beginning with Hurwicz [20] and Arrow and Hurwicz [3]. Later contributions by Gilboa and Schmeidler [18], Ghirardato et al. [16], Chateaunauf et al. [10], and Eichenberger et al. [14] focused on axiomatic treatments of similar models in which a decision maker's actions are modeled by Savage acts [34], i.e. functions from a state space to a space of consequences. In these models (as in most of the relevant literature) acts enter into the value function only as arguments of the utility function u, and have no effect on the set of priors that characterize the model's ambiguity. By contrast, our model does not introduce the notion of a state space, and its decision variables (R&D investment) are real numbers, not functions. Moreover, a decision variable r enters the value function both as an argument of the utility function $u(c,r)$ as well as on the set of priors $\Pi(b,r)$ that the decision-maker will be taking into account when he conducts his best- and worst-case analysis given aggregation ambiguity b. The latter dependency is non-standard. Jaffray [22] had first introduced a similar notion with a decision-theoretic model based on non-additive belief functions, while later Ghirardato [15] analyzed a model in which acts map from states to sets of consequences. More recently, Olszewski [30] studied the α-maxmin model in a related setting in which decision makers are called to choose between sets of lotteries over which the maximum and the minimum payoffs are subsequently computed. Moreover, Viero [36] axiomatized the α-maxmin model in a setting in which acts map from states to sets of lotteries, and thus can be viewed as a generalization of the model of Olszewski. Ahn [1] adopted a similar environment to Olszewski, but the decision maker in his model has preferences that incorporate aversion to ambiguity in a manner similar to the smooth ambiguity model of Klibanoff et al. [24]. All of the above papers focus on settings that are considerably more abstract to ours, and do not easily lend themselves to the kind of optimization-centered analysis we pursue. Of greater resemblance to this work, Karni [23] proposed a Bayesian model with action-dependent subjective probabilities similar to Eq. (1). However, Karni was primarily interested in providing an axiomatic characterization of his model and did not consider extending it to a multiple-prior setting. An additional difference of our framework with respect to all of the above is its specific consideration of aggregation ambiguity through sets $\mathrm{P}(b)$ and $\Pi(b, r)$. Searching for the maximum and minimum payoff of an investment subject to an aggregation ambiguity b is reminiscent, at least in spirit, to the quantile-maximization model of Rostek [33].

3 Theoretical Analysis

In this section we focus on the optimization problems (6) and (7) and analyze the behavior of value function $V(r|b, \alpha)$, as we vary ambiguity levels b and R&D investment r. Using results from convex optimization we are able to derive closed-form expression for this function and establish its differentiability in b (almost everywhere). To the best of our knowledge these results are novel, as are the proof techniques we employ. Differentiablity with respect to r is more subtle and we use the results of Milgrom and Segal [26] to provide ranges of b and α for which it holds. Where applicable, we ease notation in the following manner:[6]

$$
\begin{aligned}
u_n(r) &\equiv \int_{\mathcal{C}} u(c, r) \mathrm{d}\pi_n(c|r), \\
V_{max}(r, b) &\equiv V_{max}(r|b) = \max_{\pi \in \Pi(b,r)} \int_{\mathcal{C}} u\left(c, r\right) \mathrm{d}\pi(c) = \max_{\mathbf{p} \in \mathrm{P}(b)} \sum_{n=1}^{N} p_n u_n(r) && (9) \\
V_{min}(r, b) &\equiv V_{min}(r|b) = \min_{\pi \in \Pi(b,r)} \int_{\mathcal{C}} u\left(c, r\right) \mathrm{d}\pi(c) = \min_{\mathbf{p} \in \mathrm{P}(b)} \sum_{n=1}^{N} p_n u_n(r). && (10)
\end{aligned}
$$

Eqs. (9) and (10) are valid by the linearity of the expectation operator. Optimization problems (9) and (10) are convex programs with a simple structure and thus amenable to rich analysis. We begin by proving that their optimal cost functions are continuous, monotonic, and concave/convex in b.

Proposition 1 *Fix $r \in \mathcal{R}$. The function $V_{max}(r, b)$ ($V_{min}(r, b)$) defined in Eq. (9) (Eq. 10) is increasing (decreasing) and concave (convex) in b. Both functions are continuous in b.*

Before we state our next result we need to introduce additional notation. First, let $\mathcal{N}_k(r)$ denote the set of experts sharing the k'th order statistic of $\{u_1(r), u_2(r), ..., u_N(r)\}$. There are a total of $N(r)$ such sets where, depending on the problem instance, $N(r)$ can be any number in $\{1, 2, ..., N\}$,

[6] While b is a parameter, we will abuse notation and, where convenient, consider it a variable.

and we define $N_k(r) = |\mathcal{N}_k(r)|$. Furthermore, let $\mathcal{N}_k^+(r) = \bigcup_{i=k}^N \mathcal{N}_i(r)$, $\mathcal{N}_k^-(r) = \bigcup_{i=1}^k \mathcal{N}_i(r)$ and $N_k^+(r) = \left|\mathcal{N}_k^+(r)\right|$, $N_k^-(r) = \left|\mathcal{N}_k^-(r)\right|$. Our model structure enables us to easily show the following Lemma.

Lemma 1 *Fix $r \in \mathcal{R}$ and consider the optimization problems (9) and (10). Define ambiguity levels $b^*_{max}(r) \equiv \frac{1}{N_{N(r)}(r)} - \frac{1}{N}$ and $b^*_{min}(r) \equiv \frac{1}{N_1(r)} - \frac{1}{N}$. $V_{max}(r, b)$ is strictly increasing in $b \in [0, b^*_{max}(r)]$ and equal to $\max_{n\in\mathcal{N}} u_n(r)$ in $b \in [b^*_{max}(r), \frac{N-1}{N}]$. $V_{min}(r, b)$ is strictly decreasing in $b \in [0, b^*_{min}(r)]$ and equal to $\min_{n\in\mathcal{N}} u_n(r)$ in $b \in [b^*_{min}(r), \frac{N-1}{N}]$.*

Lemma 1 suggests that $b^*_{max}(r)$ and $b^*_{min}(r)$ are important thresholds. They represent the level of aggregation ambiguity above which the set $\mathrm{P}(b)$ allows for the maximum (minimum) expert estimate to be attained as an objective function value of (9) ((10)). Our next result establishes that for levels of ambiguity smaller than these extreme values, the optimal solutions of problems (9) and (10) will be unique and bind the quadratic ambiguity constraint associated with set $\mathrm{P}(b)$.

Proposition 2 *Fix $r \in \mathcal{R}$. Suppose $b \in [0, b^*_{max}(r)]$ and consider the maximization problem (9). There exists a unique optimal solution $\mathbf{p^{max}}(r, b)$ and it must satisfy the quadratic constraint of set (3) with equality. For $b \in (b^*_{max}(r), \frac{N-1}{N}]$ all probability vectors $\mathbf{p^{max}}(r, b)$ satisfying $p^{max}_n(r, b) = 0$ for $n \notin \mathcal{N}_{N(r)}(r)$ and $\sum_{n\in\mathcal{N}_{N(r)}} \left(p^{max}_n(r, b) - \frac{1}{N}\right)^2 \leq b - \frac{N^-_{N(r)-1}(r)}{N^2}$ will be optimal solutions of (9). Analogous results apply to the minimization problem (10).*

We are now ready to prove the paper's first main result. Theorem 1 establishes that functions $V_{max}(r, b)$ and $V_{min}(r, b)$ are differentiable with respect to b everywhere on $\left(0, \frac{N-1}{N}\right)$ except at the points $b^*_{max}(r)$ and $b^*_{min}(r)$ respectively. Moreover, it formalizes a straightforward monotonicity property of the optimal solutions of (9) and (10) that is essential to the derivation of the value function pursued in Theorem 2. In proving Theorem 1 we make extensive use of results from conic optimization, in particular the duality theory of second-order cone programming (see Alizadeh and Goldfarb [2]).

Theorem 1 *Fix $r \in \mathcal{R}$.*

(a) The function $V_{max}(r, b)$ ($V_{min}(r, b)$) is differentiable with respect to b everywhere on $b \in \left(0, \frac{N-1}{N}\right)$ except $b^*_{max}(r)$ ($b^*_{min}(r)$).

(b) Let $\mathbf{p^{max}}(r, b)$ ($\mathbf{p^{min}}(r, b)$) denote an optimal solution of $V_{max}(r, b)$ ($V_{min}(r, b)$). The following levels of aggregation ambiguity

$$b^{max}_k(r) = \left\{\tilde{b} : \left\{p^{max}_n(r, b) = 0 \;\; \forall n \in \mathcal{N}^-_k(r)\right\} \;\Leftrightarrow\; b \geq \tilde{b}\right\}, \quad (11)$$

$$b^{min}_k(r) = \left\{\tilde{b} : \left\{p^{min}_n(r, b) = 0, \;\; \forall n \in \mathcal{N}^+_{N(r)-k+1}(r)\right\} \;\Leftrightarrow\; b \geq \tilde{b}\right\}, \quad (12)$$

where $k \in \{1, 2, ..., N(r)-1\}$, are well-defined and strictly increasing in k.

Part (b) of Theorem 1 implies that $b_k^{max}(r)$ and $b_k^{min}(r)$ can be interpreted in the following way. In the case of problem (9), $b_k^{max}(r)$ denotes the minimum level of ambiguity such that for all $b \geq b_k^{max}(r)$ no probability mass is ever allocated to experts having an $u_n(r)$ that is less than or equal to the k'th order statistic of $\{u_1(r), u_2(r), ..., u_N(r)\}$. Conversely, in the case of problem (10), $b_k^{min}(r)$ denotes the minimum level of ambiguity such that for all $b \geq b_k^{min}(r)$ no probability mass is allocated to experts having an $u_n(r)$ that is greater than or equal to the $(N(r)-k+1)$'th order statistic of $\{u_1(r), u_2(r), ..., u_N(r)\}$. While the existence and monotonicity of these ambiguity thresholds is intuitively clear, their proofs are relatively involved.

Having established the differentiability with respect to b of $V_{max}(r, b)$ and $V_{min}(r, b)$, we go on to provide a set of differential equations that they must satisfy. These differential equations will prove valuable in the subsequent derivation of V_{max} and V_{min}.

Proposition 3 *Fix $r \in \mathcal{R}$ and let $\mathbf{p^{max}}(r, b)$ denote the unique optimal solution of maximization problem (9) as a function of $b \in [0, b^*_{max}(r)]$. Suppose expert n_k satisfies $n_k \in \mathcal{N}_k(r)$. Consider $b_k^{max}(r)$ defined in Eq. (11). $V_{max}(r, b)$ satisfies the following differential equation:*

$$2\frac{\partial}{\partial b}V_{max}(r, b)\left(p_{n_k}^{max}(r, b) - \frac{1}{N} - b\right) = u_{n_k}(r) - V_{max}(r, b), \quad b \in (0, b_k^{max}(r)). \tag{13}$$

Analogous results apply for the minimization problem (10) and $V_{min}(r, b)$.

Before presenting the paper's second main result, let $u_{(k)}(r)$ denote the k'th order statistic of $\{u_1(r), u_2(r), ..., u_N(r)\}$, where $k = 1, 2, ..., N(r)$. Now, define the following quantities (where $\overline{u}_{N(r)+1}(r)^+ = \overline{u}_0^-(r) \equiv 0$)

$$\overline{u}_k(r)^+ = \frac{\sum_{n \in \mathcal{N}_k^+(r)} u_n(r)}{N_k^+(r)}, \quad \widehat{u}_k(r)^+ = u_{(k)}(r) - \overline{u}_{k+1}(r)^+, \quad k = 1, 2, ..., N(r) \tag{14}$$

$$\overline{u}_k(r)^- = \frac{\sum_{n \in \mathcal{N}_k^-(r)} u_n(r)}{N_k^-(r)}, \quad \widehat{u}_k(r)^- = u_{(k)}(r) - \overline{u}_{k-1}(r)^-, \quad k = 1, 2, ..., N(r). \tag{15}$$

The term $\overline{u}_k(r)^+$ ($\overline{u}_k(r)^-$) is simply an average of the values of the set $\{u_1(r), u_2(r), ..., u_N(r)\}$ that are greater (smaller) than or equal to its k'th order statistic. The term $\widehat{u}_k(r)^+$ ($\widehat{u}_k(r)^-$) captures the distance between the k'th order statistic and the average of the $u_n(r)$'s that are strictly greater (smaller) than it.

Now, we define the following quantities that play an important role for functions $V_{max}(r, b)$ and $V_{min}(r, b)$ ($\mathbf{1}\{\cdot\}$ denotes an indicator function).

$$C_k^+(r) = \sqrt{\frac{1 + (N-1)\mathbf{1}\{k=1\}}{N_k^+(r)}\left[\frac{N^+_{N(r)-1}(r)N_{N(r)}(r)}{N_{N(r)-1}(r)}\left(u_{(N(r))}(r) - \overline{u}_{N(r)-1}(r)^+\right)^2 + \sum_{l=k+1}^{N(r)-1}\frac{N_l^+(r)N_{l-1}(r)}{N_{l-1}^+(r)}\left(\widehat{u}_{l-1}(r)^+\right)^2\right]}$$

$$b_k^+(r) = \frac{1}{N_{k+1}^+(r)}\left[\left(\frac{C_{k+1}^+(r)}{\widehat{u}_k(r)^+}\right)^2 + \frac{N_k^-(r)}{N}\right], \qquad k = 1, 2, ..., N(r)-2 \tag{16}$$

$$C_{N(r)-1}^+(r) = \sqrt{\frac{N_{N(r)}(r)}{N_{N(r)-1}(r)} \cdot (1 + (N-1)\mathbf{1}\{N(r)=2\}) \cdot \left(u_{(N(r))}(r) - \overline{u}_{N(r)-1}(r)^+\right)}, \quad b_{N(r)-1}^+(r) = \frac{1}{N_{N(r)}(r)} - \frac{1}{N}$$

$$C_k^-(r) = -\sqrt{\frac{1 + (N-1)\mathbf{1}\{k=1\}}{N_{N(r)-k+1}^-(r)}\left[\frac{N_2^-(r)N_1(r)}{N_2(r)}\left(u_{(1)}(r) - \overline{u}_2(r)^-\right)^2 + \sum_{l=2}^{N(r)-k}\frac{N_l^-(r)N_{l+1}(r)}{N_{l+1}^-(r)}\left(\widehat{u}_{l+1}(r)^-\right)^2\right]}$$

$$b_k^-(r) = \frac{1}{N_{N(r)-k}^-(r)}\left[\left(\frac{C_{k+1}^-(r)}{\widehat{u}_{N(r)-k+1}(r)^-}\right)^2 + \frac{N_{N(r)-k}^+(r)}{N}\right], \qquad k = 1, 2, ..., N(r)-2 \tag{17}$$

$$C_{N(r)-1}^-(r) = \sqrt{\frac{N_1(r)}{N_2(r)} \cdot (1 + (N-1)\mathbf{1}\{N(r)=2\}) \cdot \left(u_{(1)}(r) - \overline{u}_2(r)^-\right)}, \quad b_{N(r)-1}^-(r) = \frac{1}{N_1(r)} - \frac{1}{N}.$$

(Clearly, the first two expressions of Eqs. (16) and (17) are applicable only if $N(r) \geq 3$.)

We are now ready to state our second main result and provide a closed-form expression for $V_{max}(r,b)$ and $V_{min}(r,b)$, and therefore the value function (8).

Theorem 2 *Fix $r \in \mathcal{R}$. Consider the optimization problems (9) and (10) and the vectors $(\boldsymbol{C}^+(r), \boldsymbol{b}^+(r))$ and $(\boldsymbol{C}^-(r), \boldsymbol{b}^-(r))$ defined in Eqs. (16) and (17). The vectors $\boldsymbol{b}^+(r)$ and $\boldsymbol{b}^-(r)$ satisfy*

$$b_k^+(r) = b_k^{max}(r) \quad k \in \{1, 2, ..., N(r)-1\}$$
$$b_k^-(r) = b_k^{min}(r) \quad k \in \{1, 2, ..., N(r)-1\},$$

where $b_k^{max}(r)$ and $b_k^{min}(r)$ are defined in Eqs. (11)-(12). The functions $V_{max}(r,b)$ and $V_{min}(r,b)$ are equal to

$$V_{max}(r,b) = \begin{cases} \frac{\sum_{n=1}^N u_n(r)}{N} + C_1^+(r)\sqrt{b} & b \in \left[0, b_1^+(r)\right) \\ \overline{u}_k(r)^+ + C_k^+(r)\sqrt{N_k^+(r)b - \frac{N_{k-1}^-(r)}{N}} & b \in \left[b_{k-1}^+(r), b_k^+(r)\right), \ k = 2, 3, ..., N(r)-1 \\ \max_{n\in\mathcal{N}} u_n(r) & b \in \left[b_{N(r)-1}^+(r), \frac{N-1}{N}\right] \end{cases}$$

$$V_{min}(r,b) = \begin{cases} \frac{\sum_{n=1}^N u_n(r)}{N} + C_1^-(r)\sqrt{b} & b \in \left[0, b_1^-(r)\right) \\ \overline{u}_{N(r)-k+1}(r)^- + C_k^-(r)\sqrt{N_{N(r)-k+1}^-(r)b - \frac{N_{N(r)-k+2}^+(r)}{N}} & b \in \left[b_{k-1}^-(r), b_k^-(r)\right), \ k = 2, ..., N(r)-1 \\ \min_{n\in\mathcal{N}} u_n(r) & b \in \left[b_{N(r)-1}^-(r), \frac{N-1}{N}\right]. \end{cases}$$

Theorem 2 shows that, keeping r fixed, $V_{max}(r,b)$ and $V_{min}(r,b)$ will be concatenations of approroiately-specified square-root-like functions.[7] These concatenations occur at levels of ambigu-

[7] These results are consistent with the more general analysis of Section 4.2 in Iyengar [21]. However, Iyengar uses different arguments and does not prove differentiability in b, nor does he derive and interpret formulas for V_{max} and V_{min} and their optimal solutions $\mathbf{p^{max}}$, $\mathbf{p^{min}}$ (we provide the latter in Corollary 1). Instead, his analysis is concerned with determining complexity bounds for the computation of the optimal cost of (10).

ity $\boldsymbol{b^{max}}, \boldsymbol{b^{min}}$ which are interpreted by Eqs. (11)-(12), and can be computed explicitly through Eqs. (16) and (17). The curvature of these functions is driven by the dispersion of experts' expected estimates, as captured by the quantities $(\widehat{u}_k(r)^+, \widehat{u}_k(r)^-)$ of Eqs. (14) and (15). Plotting functions $V_{max}(r,b)$ and $V_{min}(r,b)$ over the entire range of $b \in \left[0, \frac{N-1}{N}\right]$ provides a concise visualization of expert beliefs on the effect of an investment r.

We verify and illustrate Theorem 2 for the simple case in which $N(r) = 2$. We focus on V_{max} as the argument for V_{min} is analogous. That $V_{max}(r,b) = \max_{n\in\mathcal{N}} u_n(r)$ for $b \geq b^*_{max}(r)$ follows by Lemma 1 so we proceed by considering $b \in [0, b^*_{max}(r))$. In this case, by first principles it is easy to see that the optimal solution of (9) will increase the probability share of all experts $n \in \mathcal{N}_2(r)$ by an equal amount ϵ, which in turn will be offset by a uniform decrease in the probability shares of experts $n \in \mathcal{N}_1(r)$. Since by Proposition 2 the quadratic ambiguity constraint will bind, ϵ must satisfy

$$N_2(r)\epsilon^2 + N_1(r)\left(\frac{\epsilon N_2(r)}{N_1(r)}\right)^2 = b \quad\Rightarrow\quad \epsilon = \sqrt{\frac{N_1(r)b}{NN_2(r)}}.$$

Thus, we may deduce that

$$\begin{aligned} V_{max}(r,b) &= N_2(r)\left(\frac{1}{N} + \sqrt{\frac{N_1(r)b}{NN_2(r)}}\right)\max_n u_n(r) + N_1(r)\left(\frac{1}{N} - \frac{N_2(r)}{N_1(r)}\sqrt{\frac{N_1(r)b}{NN_2(r)}}\right)\min_n u_n(r) \\ &= \frac{\sum_{n\in\mathcal{N}} u_n(r)}{N} + \sqrt{\frac{N_1(r)N_2(r)}{N}}\left(\max_{n\in\mathcal{N}} u_n(r) - \min_{n\in\mathcal{N}} u_n(r)\right)\sqrt{b}, \quad b \in [0, b^*_{max}(r)). \end{aligned} \tag{18}$$

Note that V_{max}'s kink at $b = b^*_{max}$ is verified through first principles.

On the other hand, applying Eqs. (16) to the case $N(r) = 2$, we obtain

$$C_1^+(r) = \sqrt{\frac{N_2(r)N}{N_1(r)}}\left(u_{(2)}(r) - \overline{u}_1(r)^+\right) = \sqrt{\frac{N_1(r)N_2(r)}{N}}\left(\max_{n\in\mathcal{N}} u_n(r) - \min_{n\in\mathcal{N}} u_n(r)\right)$$

so that Theorem 2 implies

$$V_{max}(r,b) = \frac{\sum_{n\in\mathcal{N}} u_n(r)}{N} + \sqrt{\frac{N_1(r)N_2(r)}{N}}\left(\max_{n\in\mathcal{N}} u_n(r) - \min_{n\in\mathcal{N}} u_n(r)\right)\sqrt{b}, \quad b \in [0, b^*_{max}(r)),$$

which is consistent with Eq. (18).

We can now integrate the various results we have established to characterize the optimal solutions of (9) and (10).

Corollary 1 *Consider $r \in \mathcal{R}$ and $b \in \left[0, \frac{N-1}{N}\right]$. Propositions 2 and 3 and Theorems 1 and 2 allow us to explicitly characterize the sets of optimal solutions $\mathbf{P^{max}}(r,b)$ and $\mathbf{P^{min}}(r,b)$ of optimization problems (9) and (10) respectively. Refer to the Appendix for the an explicit expression of these sets.*

Corollary 1 provides succinct expressions for the optimal expert probabilities given investment r and aggregation ambiguity b.

We now shift the focus of our analysis to investigate the differentiability of value function $V(r|b,\alpha)$, given by Eq. (8), with respect to r.[8] Here, the picture is considerably more subtle. We use the results of Milgrom and Segal [26] to state the following Theorem.

Theorem 3 *Fix* $b \in \left[0, \frac{N-1}{N}\right]$ *and* $\alpha \in [0,1]$ *and consider the value function* $V(r|b,\alpha)$ *given by Eq. (8). Assume* $\mathcal{R} = [r_m, r_M] \subset \Re$ *and that the functions* $u_n(r)$ *are continuously differentiable on* $\mathcal{R}$ *for all* $n \in \mathcal{N}$. *Let* $\mathbf{P^{max}}(r,b)$ *and* $\mathbf{P^{min}}(r,b)$ *denote the sets of optimal solutions of problems (9) and (10) respectively, as given by Corollary 1. The function* $V(\cdot|b,\alpha) : \mathcal{R} \to \Re$ *is differentiable at* $r_0 \in (r_m, r_M)$ *if and only if the sets*

$$\left\{\alpha \sum_{n=1}^{N} p_n^{min} \frac{\mathrm{d}}{\mathrm{d}r} u_n(r_0) \middle| \mathbf{p^{min}} \in \mathbf{P^{min}}(r_0,b)\right\} \quad \text{and} \quad \left\{(1-\alpha) \sum_{n=1}^{N} p_n^{max} \frac{\mathrm{d}}{\mathrm{d}r} u_n(r_0) \middle| \mathbf{p^{max}} \in \mathbf{P^{max}}(r_0,b)\right\}$$

are singletons. In that case,

$$\frac{\mathrm{d}V}{\mathrm{d}r}(r_0|b,\alpha) = \sum_{n=1}^{N} \left(\alpha p_n^{min}(r_0,b) + (1-\alpha) p_n^{max}(r_0,b)\right) \frac{\mathrm{d}}{\mathrm{d}r} u_n(r_0), \tag{19}$$

for all pairs of optimal $\mathbf{p^{min}}(r_0,b) \in \mathbf{P^{min}}(r_0,b)$ *and* $\mathbf{p^{max}}(r_0,b) \in \mathbf{P^{max}}(r_0,b)$.

Theorem 3 in combination with Proposition 2 allows us to establish the differentiability of the value function $V(r|b,\alpha)$ at a point $r = r_0$ for a range of b and α.

Corollary 2 *Suppose the assumptions of Theorem 3 hold. The function* $V(\cdot|b,\alpha) : \mathcal{R} \to \Re$ *is differentiable at* $r_0 \in (r_m, r_M)$ *for all* $b \in [0, \min\{b^*_{max}(r_0|\alpha), b^*_{min}(r_0|\alpha)\}]$ *where* $b^*_{max}(r|\alpha) = b^*_{max}(r)$ *if* $\alpha < 1$ *and* $\frac{N-1}{N}$ *otherwise and* $b^*_{min}(r|\alpha) = b^*_{min}(r)$ *if* $\alpha > 0$ *and* $\frac{N-1}{N}$ *otherwise. The derivative is given by Eq. (19) where* $\mathbf{p^{max}}(r_0,b)$ *and* $\mathbf{p^{min}}(r_0,b)$ *are uniquely defined by Corollary 1.*

Conversely, Theorem 3 also suggests that it is likely for the function $V(r|b,\alpha)$ to be non-differentiable at a value r_0 for a nontrivial range of b and α. This non-differentiability is due to the fact that for $b > b^*_{max}(r_0)$ ($b^*_{min}(r_0)$) optimization problem $\max_{p\in P(b)} \sum_{n=1}^{N} p_n u_n(r_0)$ ($\min_{p\in P(b)} \sum_{n=1}^{N} p_n u_n(r_0)$) admits multiple optimal solutions. For this reason, a derivative at $r = r_0$ will generally fail to exist. Proposition 4 formalizes this observation.

Proposition 4 *Suppose the conditions of Theorem 3 hold and consider* $r_0 \in (r_m, r_M)$.

[8] We now return to considering b as a parameter of the value function.

(a) *Suppose there exist two experts* n_1 *and* n_2 *satisfying* $\frac{\mathrm{d}u_{n_1}}{\mathrm{d}r}(r_0) \neq \frac{\mathrm{d}u_{n_2}}{\mathrm{d}r}(r_0)$ *and* $n_1, n_2 \in \mathcal{N}_1(r_0)$ $(\mathcal{N}_{N(r_0)}(r_0))$. *Then the function* $V_{min}(r|b)$ $(V_{max}(r|b))$ *is not differentiable at* $r = r_0$ *for all* $b > b^*_{min}(r_0)$ $(b > b^*_{max}(r_0))$.

(b) *Suppose* $\alpha \in (0,1)$. *If there exist experts* n_1 *and* n_2 *satisfying* $\frac{\mathrm{d}u_{n_1}}{\mathrm{d}r}(r_0) \neq \frac{\mathrm{d}u_{n_2}}{\mathrm{d}r}(r_0)$ *such that* $n_1, n_2 \in \mathcal{N}_1(r_0)$ *then the function* $V(r|b,\alpha)$ *is not differentiable at* $r = r_0$ *for all* $b > b^*_{min}(r_0)$. *If there exist experts* n_3 *and* n_4 *satisfying* $\frac{\mathrm{d}u_{n_3}}{\mathrm{d}r}(r_0) \neq \frac{\mathrm{d}u_{n_4}}{\mathrm{d}r}(r_0)$ *such that* $n_3, n_4 \in \mathcal{N}_{N(r_0)}(r_0)$ *then the function* $V(r|b,\alpha)$ *is not differentiable at* $r = r_0$ *for all* $b > b^*_{max}(r_0)$.

In instances described by Proposition 4, it is clear that one cannot use first-order conditions to establish the potential optimality of an R&D investment r_0. This non-differentiability is an unsatisfying, though not entirely unexpected, consequence of the maxmin nature of our model. It stems from the fact that beyond a certain level of aggregation ambiguity there may exist a multiplicity of aggregation schemes that yield the absolute maximum and minimum payoffs. In the remainder of this section, we suggest that such non-smoothness issues may be, to a significant degree, mitigated.

Remark 1. Given Corollary 2 and Proposition 4, we would like to narrow the range of b and α over which $V(r|b,\alpha)$ would fail to be differentiable. For this purpose, we provide a plausible lower bound on problematic ranges of b with the following informal argument. Consider carefully the continuously differentiable functions $u_n(r) = \int_{c\in\mathcal{C}} u(c,r)\mathrm{d}\pi_n(c|r)$. Since subjective probability distributions will generally differ across experts, then, assuming the domain $\mathcal{C}$ is moderately large, it is unlikely that at any point r_0 we will have more than 2 experts sharing the same value, including the maximum and minimum values of $\{u_1(r_0), u_2(r_0), ..., u_N(r_0)\}$. Therefore, it is likely that $N_1(r) \leq 2$ and $N_{N(r)}(r) \leq 2$ for all $r \in \mathcal{R}$. This observation leads to the following bound

$$\min_{r\in\mathcal{R}} \min\{b^*_{max}(r), b^*_{min}(r)\} \geq \frac{1}{2} - \frac{1}{N} = \frac{N-2}{2N},$$

so that Corollary 2 implies that $V(\cdot|b,\alpha)$ will be, at the very least, everywhere differentiable for any choice of $\alpha \in [0,1]$ and $b \leq \frac{N-2}{2N}$. If we assume that decision makers are constrained in their maximal choice of b (say, because equal-weight aggregation is deemed "fair" and/or to ensure that no single group of experts is given too much weight) then potentially problematic ranges of $b \geq \frac{N-2}{N}$ are less likely to be considered and the negative result of Proposition 4 loses its bite.

Related to the above, Corollary 2 implies the diffentiability of $V(\cdot|b,\alpha)$ at all $r_0 \in (r_m, r_M)$, for all b and α, for an important special case: that in which there exists a pair of experts that are consistently the most optimistic and pessimistic across all levels of R&D. If surveyed experts have different backgrounds, such consistently optimistic and pessimistic biases may well occur.

Corollary 3 *Suppose the conditions of Theorem 3 hold and there exist two experts n_1 and n_2 such that $u_{n_1}(r) > u_n(r)$ for all $n \neq n_1$ and $u_{n_2}(r) < u_n(r)$ for all $n \neq n_2$, for all $r \in \mathcal{R}$. In other words, experts n_1 and n_2 are consistently the most optimistic and pessimistic across all levels of R&D. Then Theorem 3 implies that $V(r|b,\alpha)$ is differentiable at all $r_0 \in (r_m, r_M)$ for all choices of b and α, with its derivative given by Eq. (19).*

Hence, for problem instances satisfying the conditions of Corollary 3, first-order conditions given by Eq. (19) may be always invoked to solve for the maximizer of the value function $V(r|b,\alpha)$, regardless of the values of b and α.

Remark 2. Finally, if our problem instance is such that Remark 1 and Corollary 3 are not applicable, we may always address the potential non-differentiability of the value function (8) by slightly perturbing the $u_n(r)$ functions so that for all $r \in \mathcal{R}$ we have $N_1(r) = N_{N(r)}(r) = 1$. Since the $u_n(r)$ functions are assumed to be continuously differentiable, such small perturbations would be, at least in our view, defensible.

4 Empirical Application to Solar-Technology R&D

We base the empirical application of our paper to original data collected by the ICARUS survey, an expert elicitation on the potential of solar technologies. During the course of 2010-2011, the ICARUS survey collected expert judgments on future costs and technological barriers of different Photovoltaic (PV) and Concentrated Solar Power (CSP) technologies.[9] Sixteen leading European experts from academia, the private sector, and international institutions took part in the survey. The elicitation collected probabilistic information on (1) the year-2030 expected cost of the technologies; (2) the role of public European Union R&D investments in affecting those costs; and (3) the potential for the deployment of these technologies (both in OECD and non-OECD countries). We refer readers interested in the general findings of the survey to Bosetti et al. [7] and we focus here on the data on future costs as they form the basis of our analysis.

Current 5-year EU R&D investment in solar technology is estimated at 165 million US dollars. The ICARUS study elicited the probabilistic estimates of the 16 experts on the 2030 solar electricity

[9]The survey is part of a 3-year ERC-funded project on innovation in carbon-free technologies (ICARUS - Innovation for Climate chAnge mitigation: a study of energy R&D, its Uncertain effectiveness and Spillovers www.icarus-project.org).

cost (2005 c$/kWh) under three future *Scenarios*: (1) keeping current levels of R&D constant until 2030, (2) increasing them by 50%, and (3) increasing them by 100%. Coherent responses were obtained from 14 out of the 16 experts so the analysis that follows focuses solely on them. We used linear interpolation of the survey's collected data (generally 3-6 points of each expert's cumulative distribution function(cdf) conditional on R&D investment) to compute a pdf for each expert $n \in \{1, 2, ..., 14\}$, given the three relevant levels of R&D investment denoted by $r \in \{r_1, r_2, r_3\}$ (here r_i refers to Scenario i).[10] These pdfs represent experts' subjective probability distributions of the cost of technology as denoted in Eq. (1). Figure 1 plots the corresponding cdfs as well as the cdf that the aggregate pdf (2) leads to, under all three Scenarios.

[FIGURE 1 here]

As one can see in Figure 1 there is considerable disagreement between experts over the potential of solar technology. This disagreement is particularly acute under Scenario 1, and diminishes as R&D levels increase. Nonetheless, the breakthrough nature of innovation and the need to cross certain firm cost thresholds, means that ambiguity in expert estimates remains an important concern, even under Scenario 3. This will become apparent in the analysis to follow.

We measure the utility of an investment via its net payoff. Denoting the benefit associated to a technology cost c by the function $B(c)$ and the opportunity cost of an investment r by $O(r)$, this is given by the following utility function:

$$u(c, r) = B(c) - O(r). \tag{20}$$

The next section describes how we provide numerical values for $B(\cdot)$ and $O(\cdot)$.

Quantifying benefits and opportunity costs of solar technology R&D. Expected benefits of solar technology R&D investments are quantified via a general equilibrium intertemporal model that can account for a range of macro-economic feedbacks and interactions. These include the effects of energy and climate change policies, the competition for innovation resources with other power technologies, the effect of growth, as well as a number of other factors.[11] To capture the long-term nature of such investments, the integrated assessment model is run over the time horizon 2105-2100 in 5-year time periods for the whole range of exogenously-imposed possible 2030-costs of solar power

[10]Please refer to section A2 of the Appendix for more information on how expert pdfs were constructed from the survey data.

[11]The analysis is carried out using the World Induced Technical Change Hybrid (WITCH) model (Bosetti et al. [8]), an energy-economy-climate model that has been used extensively for economic analysis of climate change policies. See www.witchmodel.org for a list of applications and papers.

that we are considering. Subsequently, simulation results are compared to the benchmark case in which the cost of solar power is so high that the technology is not competitive with alternative production modes. For each possible 2030 solar-power cost, the benefit to the European Union is quantified by the discounted EU-consumption improvement over the entire time-horizon 2005-2100 with respect to the case where solar technology is not competitive. Table 1 summarizes the results.

[TABLE 1 here]

Three important assumptions are at the basis of the numbers reported in Table 1. First, as the survey concentrated on public EU R&D investment and the effects of increasing it, we disregard spillovers and technological transfers to the rest of the world and consider only the consumption improvement for Europe. Second, we evaluate the benefit of alternative 2030 costs of solar power assuming that no carbon policy is in place and that no special constraints on other technologies are imposed (e.g., a partial ban on nuclear technology). Third, we discount cash flows using a 3% discount rate. Although our choice is well in the range of discount rates adopted for large scale public projects, it is important to note that the cost threshold for positive payoffs is robust for a wide range of more myopic discount rate values. Our assumptions all err on the side of being conservative about the potential payoffs of solar-technology R&D.

We now explain how we calculate the costs of solar R&D investment. Given an R&D investment r, we assume that actual R&D spending is fixed at r during the period 2005-2030, in line with the survey questions. After 2030 we assume that spending drops to half its initial value, i.e. $r/2$, and remains at that level until 2100. This drop occurs because we assume that post-2030 funds represent the government's commitment to maintain the technological gains achieved by 2030. We now derive the discounted opportunity cost of such expenditure streams of solar-technology R&D spending. In doing so we follow Popp [32] and assume that, at every time period, this opportunity cost is equal to 4 times the original investment. Thus, in our model the opportunity cost of a level of R&D investment r is given by the net present value of the stream $O(t)$ where $O(t) = 4 \cdot r$ for $t = 1, 2, ..., 6$ and $2 \cdot r$ for $t = 7, 8, ..., 20$ (once again we use a 3% discount rate). Table 2 summarizes these results for the three R&D Scenarios that the ICARUS survey focused on.

[TABLE 2 here]

Application of the decision-theoretic framework. We now extend our analysis to explicitly account for aggregation ambiguity and adopt the decision-theoretic model introduced in Section 2.[12] Our objective is to compare the three R&D Scenarios, and we do not consider optimizing over

[12] All simulations are performed in Mathematica.

a continuous R&D domain $\mathcal{R}$. We make this choice primarily because we wish to keep the applied section brief and pursue more in-depth empirical analysis in future work.[13]

Figure 2 plots $V_{max}(r, b^2)$ and $V_{min}(r, b^2)$ over $b \in \left[0, \sqrt{\frac{13}{14}}\right] \approx [0, .96]$ for the three Scenarios. The parametrization b^2 is adopted since it allows us to (a) dampen the curvature of the original functions as given by Theorem 2 and (b) interpret the parameter b as a bound on the Euclidean distance of admissible aggregation schemes with respect to the benchmark equal-weight aggregation.

[FIGURE 2 here]

Focusing first on Scenario 1, we note that pure aggregation of expert opinion (corresponding to $b = 0$) yields a payoff of approximately $\$1.36 \times 10^9$. We observe that the worst-case payoff drops to about $\$-3.4 \times 10^9$ at $b \approx .25$ at which point it largely stops being sensitive to changes in b, slowly asymptoting to its minimum value of $\$-3.67 \times 10^9$; in contrast, the best-case one increases steadily to a maximum value of $\$22.7 \times 10^9$ at the maximum level of $b = .96$. Under Scenario 2, the payoff under zero ambiguity is equal to $\$7.8 \times 10^9$. Subequently, we see that the worst-case payoff drops to 0 at $b = 0.15$, at which point it keeps decreasing at a smaller rate until it practically reaches its minimum value of $\$-5.5 \times 10^9$ at $b \approx .55$. Conversely, the best-case payoff rises steadily to about $\$32 \times 10^9$ for $b \approx .55$ at which point it continues to rise at a much smaller rate until it reaches a maximum value of $\$33.3 \times 10^9$ at $b = .96$. Thus for Scenario 2, aggregation uncertainty becomes largely unimportant once b reaches the threshold of 0.55. Under Scenario 3 the unambiguous payoff is around $\$20 \times 10^9$, significantly higher than both other Scenarios. The worst-case payoff drops relatively smoothly to a minimum value of $\$-7.35 \times 10^9$ for $b = .96$, while the best-case one rises at a comparatively higher rate to $\$70.9 \times 10^9$.

It is clear that aggregation ambiguity is important under Scenario 3, for both the worst- and best-case payoffs, significantly more so than under Scenarios 1 and 2. This fact is interesting in light of Figure 1, which shows that experts' pdfs are much more dispersed under Scenarios 1 and 2 than they are under 3. The reason behind this seemingly unexpected result is straightforward. As Table 1 suggests, expected payoffs of R&D investment are very sensitive at low cost values, i.e., less than 8c\$/kWh. The more aggressive investment of Scenario 3 has a greater effect on these lower cost values, and therefore its best- and worst-case payoffs are in turn more sensitive to changes in b.

We now consider the effect of ambiguity attitude on the decision maker's problem. Figure 3 plots the value function $V(r|b^2, \alpha)$ given by Eq. (8) for all three investment Scenarios, over

[13] Indeed, constructing plausible approximations of experts' $u_n(r)$ functions over an interesting range of r will likely require further engagement with the experts.

all levels of aggregation ambiguity and a decision-maker's attitude toward it. This allows policy makers to visualize the effects of the three R&D investment decisions over the entire range of possible ambiguity levels and ambiguity attitudes. As we expect from Figure 2, Scenario 3 fares much better than both 1 and 2 over a very wide range of b and α, and is much more sensitive to changes in both.

[FIGURE 3 here]

Figure 4 goes a step further and compares the three R&D Scenarios for all possible combinations of b and α. Following the color scheme of Figure 2, a region's color corresponds to the Scenario that performs the best within it, while the bold numbers within regions denote the relative order of the three Scenarios within this range of (b, α) (e.g., an expression "321" means Scenario 2 dominates 1, and Scenario 3 dominates both 2 and 1).

[FIGURE 4 here]

Figure 4 makes clear that Scenario 3 dominates 1 and 2 for an extremely wide range of combinations of b and α. Conversely, Scenario 1 is the best option for a combination of very high b and α. Somewhat surprisingly, we see that Scenario 2 is dominated by either 1 or 3 for all possible combinations of b and α and thus will never be chosen by a decision maker whose preferences are captured by Eq. (8). Thus, on the basis of the presented data, it is clear that policy makers should opt for the most aggressive R&D investment, unless they are both (a) open to ignoring a very large set of surveyed experts (b) extremely concerned about the possibility of worst-case failure. Moreover, assuming all three options are readily implementable, they can safely disregard the middle-range R&D investment implied by Scenario 2.

5 Conclusion

Determining the optimal portfolio of government R&D is an important task, especially at times of public funding scarcity. As R&D programs imply uncertain returns, it is important to assess these investments using probabilistic tools. Expert elicitation surveys can play an important role in this process if used to capture in a transparent and objective way subjective probabilities that can be used as scientific data.

Yet, gathered information can vary substantially across experts. In particular, if the elicitation is designed correctly is should exactly aim at covering all prevailing "visions" about that specific technology. The different backgrounds and perspectives that experts bring to the elicitation process imply that collected subjective probability distributions will, more often than not, span a wide and potentially confusing spectrum.

Condensing all of the problem's uncertainty into one single average probability distribution, especially in cases where Bayesian methods cannot be readily applied, may conceal important imformation and yield policy recommendations that are not robust. To deal with this issue, we proposed and analyzed a novel decision-theoretic framework inspired by the well-studied α-maxmin model. In line with the paper's focus on R&D investment, decision variables in our model affect experts' subjective probability distributions of the future potential of a technology. We applied our framework to original data from a recent expert elicitation survey on solar technology. The analysis suggested that more aggressive investment in solar technology R&D is likely to yield substantial benefits even after ambiguity over expert opinion has been taken into account.

We conclude by noting that while this paper has been motivated by the issue of R&D allocation, the model and analysis presented herein are general and can be applied to other contexts of decision making under ambiguity.

Appendix

A1: Proofs

To ease notation, in our proofs we suppress dependence on R&D investment r except where necessary.

Proposition 1. We prove the Proposition for $V_{max}(b)$ (the argument for $V_{min}(b)$ is analogous). That $V_{max}(b)$ is increasing in b follows by definition. Consider the optimization problems given by the right-hand-side of Eq. (9) for $b_1 \in [0, \frac{N-1}{N}]$ and $b_2 \geq b_1$ and denote their optimal solutions by $\mathbf{p^{max}}(b_1)$ and $\mathbf{p^{max}}(b_2)$ respectively. By feasibility we may note the following:

$$\sum_{n=1}^{N} \left(p_n^{max}(b_1) - \frac{1}{N}\right)^2 \leq b_1, \quad \sum_{n=1}^{N} \left(p_n^{max}(b_2) - \frac{1}{N}\right)^2 \leq b_2. \tag{21}$$

Consider a convex combination of b_1 and b_2 given by $b(\lambda) = \lambda b_1 + (1-\lambda)b_2$ for some $\lambda \in [0,1]$ and the optimization problem

$$V_{max}(b(\lambda)) = \max_{\mathbf{p} \in \mathrm{P}(b(\lambda))} \sum_{n=1}^{N} p_n m_n. \tag{22}$$

To prove concavity of V_{max} in b it suffices to show that

$$V_{max}(b(\lambda)) \geq \lambda V_{max}(b_1) + (1-\lambda)V_{max}(b_2).$$

To this end, consider the probability vector given by

$$\mathbf{p}(\lambda) = \lambda \mathbf{p^{max}}(b_1) + (1-\lambda)\mathbf{p^{max}}(b_2).$$

By feasibility of $\mathbf{p^{max}}(b_1)$ and $\mathbf{p^{max}}(b_2)$ we immediately deduce that $\mathbf{p}(\lambda) \geq \mathbf{0}$ and that $\sum_{n=1}^{N} p_n(\lambda) = 1$. Now we may write

$$\begin{aligned}
\sum_{n=1}^{N} \left(p_n(\lambda) - \frac{1}{N}\right)^2 &= \sum_{n=1}^{N} \left(\lambda\left(p_n^{max}(b_1) - \frac{1}{N}\right) + (1-\lambda)\left(p_n^{max}(b_2) - \frac{1}{N}\right)\right)^2 \\
&\overset{\text{triangle ineq.}}{\leq} \left[\lambda\left(\sum_{n=1}^{N}\left(p_n^{max}(b_1) - \frac{1}{N}\right)^2\right)^{\frac{1}{2}} + (1-\lambda)\left(\sum_{n=1}^{N}\left(p_n^{max}(b_2) - \frac{1}{N}\right)^2\right)^{\frac{1}{2}}\right]^2 \\
&\overset{(21)}{\leq} \left[\lambda\sqrt{b_1} + (1-\lambda)\sqrt{b_2}\right]^2 \leq \left[\sqrt{\lambda b_1 + (1-\lambda)b_2}\right]^2 = b(\lambda).
\end{aligned} \tag{23}$$

By Eq. (23) and the observations immediately preceding it we can conclude that $\mathbf{p}(\lambda)$ is feasible for optimization problem (22). Thus we may write

$$\begin{aligned}
V_{max}(b(\lambda)) \geq \sum_{n=1}^{N} p(\lambda)_n u_n &= \lambda \sum_{n=1}^{N} p_n^{max}(b_1)u_n + (1-\lambda)\sum_{n=1}^{N} p_n^{max}(b_2)u_n \\
&= \lambda V_{max}(b_1) + (1-\lambda)V_{max}(b_2),
\end{aligned}$$

where the last equality follows from the assumed optimality of $\mathbf{p^{max}}(b_1)$ and $\mathbf{p^{max}}(b_2)$. We now proceed to show continuity. By concavity $V_{max}(b)$ will be continuous on the open interval $(0, \frac{N-1}{N})$ so we need only consider the endpoints 0 and $\frac{N-1}{N}$. Since $V_{max}(b)$ is increasing in b we must have $\lim_{b \to (\frac{N-1}{N})^-} V_{max}(b) \leq V_{max}(\frac{N-1}{N})$. However, if $\lim_{b \to (\frac{N-1}{N})^-} V_{max}(b) < V_{max}(\frac{N-1}{N})$ then we reach a contradiction if we apply concavity to $(N-1)/N$ and other values of b.

To prove continuity at $b = 0$ consider an $\epsilon > 0$. Now let $\delta > 0$ and write

$$\begin{aligned}
|V_{max}(\delta) - V_{max}(0)| &= V_{max}(\delta) - V_{max}(0) = \sum_{n=1}^{N} \left(p_n^{max}(\delta) - \frac{1}{N} \right) u_n \\
&\leq \max_{n \in \mathcal{N}} |u_n| \sum_{n=1}^{N} \left| p_n^{max}(\delta) - \frac{1}{N} \right| \\
&\overset{\text{Hölder's ineq.}}{\leq} \max_{n \in \mathcal{N}} |u_n| \left[\sum_{n=1}^{N} \left(p_n^{max}(\delta) - \frac{1}{N} \right)^2 \right]^{\frac{1}{2}} \leq \max_{n \in \mathcal{N}} |u_n| \sqrt{\delta}.
\end{aligned}$$

Thus, any choice of $0 < \delta < \frac{\epsilon^2}{(\max_{n \in \mathcal{N}} |u_n|)^2}$ will ensure that $|V_{max}(\delta) - V_{max}(0)| < \epsilon$, completing the proof. ■

Lemma 1. The function $V_{max}(b)$ is bounded above by u_n for any $n \in \mathcal{N}_{N(r)}$. This upper bound is attained by a probability vector $\mathbf{p}$ if and only if it satisfies

$$\sum_{n \in \widehat{\mathcal{N}}} p_n = 1, \quad \text{for some } \widehat{\mathcal{N}} \subseteq \mathcal{N}_{N(r)}$$

Consider a subset $\widehat{\mathcal{N}} \subseteq \mathcal{N}_{N(r)}$, with cardinality $\widehat{N}$. Eq. (4) implies that the value of b at which it first becomes possible to assign probability 1 to subset $\widehat{\mathcal{N}}$ is given by

$$b(\widehat{N}; N) = \frac{1}{\widehat{N}} - \frac{1}{N}.$$

The minimizer of $b(\widehat{N}; N)$ over $\widehat{\mathcal{N}} \subseteq \mathcal{N}_{N(r)}$ is the entire set $\mathcal{N}_{N(r)}$, yielding the desired result.

Now consider $b < b^*_{max}$ and the optimal solution $\mathbf{p^{max}}(b)$. As $b < b^*_{max}$ there must exist a $j \neq \mathcal{N}_{N(r)}$ such that $p_j^{max}(b) > 0$. Now consider increasing b by an amount ϵ. For $\delta > 0$ small enough the solution $\tilde{\mathbf{p}}$ which is identical to $\mathbf{p^{max}}(b)$ except that $\tilde{p}_j = p_j^{max}(b) - \delta$ and $\tilde{p}_k = p_k^{max}(b) + \delta$ for some $k \in \mathcal{N}_{N(r)}$ will be feasible and result in a strictly greater objective value, so that $V_{max}(b + \epsilon) > V_{max}(b)$. Equivalent reasoning applies to the V_{min} case. ■

Proposition 2. Suppose first that $b = b^*_{max}$. It is clear here that the unique optimal solution is given by $\mathbf{p^{max}}$ such that $p_n^{max} = 1/N_{N(r)}$ for all $n \in \mathcal{N}_{N(r)}$ and $p_n^{max} = 0$ otherwise. The quadratic ambiguity constraint binds by the definition of b^*_{max}.

Consider now the case $b < b^*_{max}$ and suppose there exists an optimal solution $\mathbf{p^{max}}(b)$ such that the quadratic ambiguity constraint is slack. As $b < b^*_{max}$ there must exist an $j \neq \mathcal{N}_{N(r)}$ such that $p_j^{max}(b) > 0$. For $\epsilon > 0$ small enough the solution $\tilde{\mathbf{p}}^{\mathbf{max}}$ in which $\tilde{p}_j = p_j^{max}(b) - \epsilon$ and $\tilde{p}_k = p_k^{max}(b) + \epsilon$ for some $k \in \mathcal{N}_{N(r)}$ will be feasible and result in a strictly greater objective value, contradicting $\mathbf{p^{max}}(b)$'s optimality. Thus, all optimal solutions must satisfy the quadratic ambiguity constraint with equality.

We now prove uniqueness. Suppose there exist two optimal solutions $\mathbf{p^{max,1}}$ and $\mathbf{p^{max,2}}$. By the preceding argument they must bind the quadratic ambiguity constraint. Consider the set of probability vectors given by their convex combinations

$$\mathbf{p}(\lambda) = \lambda \mathbf{p^{max,1}} + (1-\lambda)\mathbf{p^{max,2}}, \ \ \lambda \in [0,1].$$

For $\lambda \in (0,1)$, $\mathbf{p}(\lambda)$ will satisfy the ambiguity constraint with strict inequality, since:

$$\begin{aligned}
\sum_{n=1}^{N}\left(p_n(\lambda) - \frac{1}{N}\right)^2 &= \sum_{n=1}^{N}\left(\lambda\left(p_n^{max,1} - \frac{1}{N}\right) + (1-\lambda)\left(p_n^{max,2} - \frac{1}{N}\right)\right)^2 \\
&\overset{\text{strict convexity}}{<} \sum_{n=1}^{N}\left[\lambda\left(p_n^{max,1} - \frac{1}{N}\right)^2 + (1-\lambda)\left(p_n^{max,2} - \frac{1}{N}\right)^2\right] \\
&= \lambda\sum_{n=1}^{N}\left(p_n^{max,1} - \frac{1}{N}\right)^2 + (1-\lambda)\sum_{n=1}^{N}\left(p_n^{max,2} - \frac{1}{N}\right)^2 \\
&= \lambda b + (1-\lambda)b = b.
\end{aligned}$$

Thus all solutions $\mathbf{p}(\lambda)$ are feasible. That they are optimal follows trivially by the assumed optimality of $\mathbf{p^{max,1}}, \mathbf{p^{max,2}}$ and the linear objective function of (9). But this is a contradiction as all optimal solutions must satisfy the quadratic ambiguity constraint with equality. The second claim of the Proposition regarding $b > b_{max}$ is trivial. ■

Theorem 1. We prove the result for V_{max}; the argument for V_{min} is analogous. To do so we need to invoke results from conic duality. We begin with part (a). Given $\mathbf{x} = (x_0, \bar{\mathbf{x}}) \in \Re^{n+1}$ we introduce the following notation to denote inclusion in a second-order cone of dimension $n+1$

$$(x_0, \bar{\mathbf{x}}) \in \mathcal{L}^2_{n+1} \ \ \Leftrightarrow x_0 \geq ||\bar{\mathbf{x}}||_2.$$

We follow Alizadeh and Goldfarb [2] to write (9) as a primal conic program $\mathcal{P}(b)$ and introduce its dual $\mathcal{D}(b)$ (for clarity, next to the primal constraints we indicate the corresponding dual variables):

$$
\begin{array}{llll}
\max\limits_{\mathbf{p},\mathbf{q},q_0,\theta} & \sum_{n=1}^{N} u_n p_n & \min\limits_{\mathbf{y},y_0,\boldsymbol{\gamma},\beta_0,\mathbf{z_p},\mathbf{z_q},z_{q0},\mathbf{z}_\theta} & y_0 + \sqrt{b+\frac{1}{N}}\beta_0 \\
\text{s.t.} & -p_n + q_n = 0,\ \ \forall n \in \mathcal{N},\ \ (y_n) & \text{s.t.} & -y_0 - y_n + z_{pn} = -u_n,\ \ \forall n \in \mathcal{N} \\
 & \sum_{n=1}^{N} -p_n = -1,\ \ (y_0) & & y_n + z_{qn} = 0,\ \ \forall n \in \mathcal{N} \\
\mathcal{P}(b) & \theta_n = 0,\ \ \forall n \in \mathcal{N},\ \ (\gamma_n) & & \gamma_n + z_{\theta n} = 0,\ \ \forall n \in \mathcal{N} \\
 & q_0 = \sqrt{b+\frac{1}{N}},\ \ (\beta_0) & \mathcal{D}(b) & -\beta_0 + z_{q0} = 0 \\
 & (p_n,\theta_n) \in \mathcal{L}_2^2,\ \ \forall n \in \mathcal{N} & & (z_{pn}, z_{\theta n}) \in \mathcal{L}_2^2,\ \ \forall n \in \mathcal{N} \\
 & (q_0,\mathbf{q}) \in \mathcal{L}_{n+1}^2, & & (z_{q0}, -\mathbf{z_q}) \in \mathcal{L}_{n+1}^2.
\end{array}
$$

Since both the primal and the dual have feasible strictly interior solutions, strong duality holds (see Theorem 13 of [2]). Without loss of generality, we can immediately simplify $\mathcal{D}(b)$ by setting $\mathbf{z}_\theta = \boldsymbol{\gamma} = \mathbf{0}$ and $\mathbf{z_p} \geq \mathbf{0}$. Correspondingly, we can eliminate the variable $\mathbf{z_q}$ by replacing it with $-\mathbf{y}$. Finally, it is evident that at optimality the quadratic constraint of the dual will be binding so that $z_{q0}^* = \beta_0^* = \sqrt{\sum_{n=1}^N (-y_n)^2} = \sqrt{\sum_{n=1}^N y_n^2}$. Collecting all of these observations we may re-write the dual in the following much simpler way:

$$
\begin{aligned}
\mathcal{D}_1(b) = \min_{\mathbf{y},y_0} \quad & y_0 + \sqrt{b+\frac{1}{N}}\sqrt{\sum_{n=1}^{N} y_n^2} \\
\text{s.t.} \quad & -u_n + y_0 + y_n \geq 0,\ \ n = 1,2,...,N.
\end{aligned} \tag{24}
$$

Examining (24) we deduce that at optimality $y_n^* = \max(0, u_n - y_0)$. Thus we may simplify the dual even further to an unconstrained optimization problem with just one variable:

$$
\mathcal{D}_2(b) = \min_{y_0} y_0 + \sqrt{b+\frac{1}{N}}\sqrt{\sum_{n=1}^{N} \max(0, u_n - y_0)^2}. \tag{25}
$$

By strong duality the dual optimal objective will be bounded between $\frac{1}{N}\sum_{n=1}^N u_n$ and $u_{(N(r))}$. We immediately see that solutions satisfying $y_0 > u_{(N(r))}$ result in strictly greater objective function values than $y_0 = u_{(N(r))}$, so that we can safely disregard them. Conversely, solutions satisfying $y_0 < 0$ yield

$$
\begin{aligned}
y_0 + \sqrt{b+\frac{1}{N}}\sqrt{\sum_{n=1}^{N} \max(0, u_n - y_0)^2} &= y_0 + \sqrt{b+\frac{1}{N}}\sqrt{\sum_{n=1}^{N} (u_n - y_0)^2} \\
&> y_0 + \sqrt{Nb+1}(|u_{(1)}| - y_0) = \sqrt{Nb+1}|u_{(1)}| + y_0(1 - \sqrt{Nb+1}).
\end{aligned}
$$

Thus, values of $y_0 < \frac{|u_{(N(r)|}}{1-\sqrt{Nb+1}}$ result in a strictly greater objective function value than $y_0 = u_{(N(r))}$ and hence can also be disregarded. With these observations we may rewrite the dual (25) in the following way:

$$\mathcal{D}_3(b) = \min_{y_0 \in \left[\frac{|u_{(N(r))}|}{1-\sqrt{Nb+1}}, u_{(N(r)}\right]} y_0 + \sqrt{b + \frac{1}{N}} \sqrt{\sum_{n=1}^{N} \max(0, u_n - y_0)^2} \tag{26}$$

The domain of $\mathcal{D}_3(b)$ is thus compact, for any $b > 0$. For values of $b \in [0, b^*_{max})$ we know that the optimal solution of the primal will be strictly less than $u_{(N(r))}$. Thus, strong duality implies that for all $b \in (0, b^*_{max})$, any optimal solution $y^*(b)$ must satisfy $y^*(b) < u_{(N(r)-1)}$. However, notice that the objective function of $\mathcal{D}_3$ is strictly convex for $y_0 < u_{(N(r)-1)}$. Thus, we may deduce that when $b \in (0, b^*_{max})$ $\mathcal{D}_3(b)$ admits a unique optimal solution $y^*_0(b)$.

The above observation implies that we can apply Danskin's theorem (see Proposition B.25 in Bertsekas [6]) to conclude that the optimal dual objective value, and therefore by strong duality $V_{max}(b)$ as well, is differentiable at all $b \in (0, b^*_{max})$ and that

$$\frac{\mathrm{d}V_{max}}{\mathrm{d}b}(b) = \frac{\sqrt{\sum_{n=1}^{N} \max(0, u_n - y^*_0(b))^2}}{2\sqrt{b + \frac{1}{N}}}, \quad b \in (0, b^*_{max}). \tag{27}$$

Before we proceed with investigating the endpoints $b = 0$ and b_{max}, we show that $y^*_0(b)$ is strictly increasing in $b \in (0, b^*_{max})$. Consider $b_1 < b_2$ with both belonging in $(0, b^*_{max})$ and their optimal solutions $y^*_0(b_1)$ and $y^*_0(b_2)$. By uniqueness of $y^*_0(b)$ in this range of b we have

$$y^*_0(b_1) + \sqrt{b_1 + \frac{1}{N}} \sqrt{\sum_{n=1}^{N} \max(0, u_n - y^*_0(b_1))^2} < y^*_0(b_2) + \sqrt{b_1 + \frac{1}{N}} \sqrt{\sum_{n=1}^{N} \max(0, u_n - y^*_0(b_2))^2}$$

$$y^*_0(b_2) + \sqrt{b_2 + \frac{1}{N}} \sqrt{\sum_{n=1}^{N} \max(0, u_n - y^*_0(b_2))^2} < y^*_0(b_1) + \sqrt{b_2 + \frac{1}{N}} \sqrt{\sum_{n=1}^{N} \max(0, u_n - y^*_0(b_1))^2}.$$

Summing the above inequalities and rearranging terms yields

$$\left(\sqrt{b_2 + \frac{1}{N}} - \sqrt{b_1 + \frac{1}{N}}\right) \left(\sqrt{\sum_{n=1}^{N} \max(0, u_n - y^*_0(b_1))^2} - \sqrt{\sum_{n=1}^{N} \max(0, u_n - y^*_0(b_2))^2}\right) > 0$$

$$\Rightarrow \sqrt{\sum_{n=1}^{N} \max(0, u_n - y^*_0(b_1))^2} - \sqrt{\sum_{n=1}^{N} \max(0, u_n - y^*_0(b_2))^2} > 0 \Rightarrow y^*_0(b_2) > y^*_0(b_1).$$

We discuss now the differentiability of V_{max} at $b \in \{0, b^*_{max}\}$. At $b = 0$ the domain of (26) is no longer bounded below and therefore we can no longer invoke Danskin's theorem. Consequently, we reason in a different way. By continuity (recall Proposition 1) we must have

$$\lim_{b\to 0^+} V_{max}(b) = \sum_{n=1}^{N} \frac{u_n}{N} \Leftrightarrow \lim_{b\to 0^+} y^*(b) + \sqrt{\frac{1}{N}}\sqrt{\sum_{n=1}^{N} \max(0, u_n - \lim_{b\to 0^+} y^*(b))^2} = \sum_{n=1}^{N} \frac{u_n}{N}$$

The strict monotonicity of $y^*(b)$ and Hölder's inequality imply that $\lim_{b\to 0^+} y^*(b) < 0$. Subsequently, simple algebra obtains:

$$\lim_{b\to 0^+} y^*(b) - \frac{\sum_{n=1}^{N} u_n}{N} = -\sqrt{\frac{1}{N}\sum_{n=1}^{N}(u_n - \lim_{b\to 0^+} y^*(b))^2}$$

If we take squares now on both sides and re-apply Hölder's inequality, we see that

$$\lim_{b\to 0^+} y^*(b) = -\infty \quad \Rightarrow \quad \lim_{b\to 0^+} \frac{\mathrm{d}V_{max}}{\mathrm{d}b}(b) = +\infty.$$

Now we consider $b = b^*_{max}$. Note that the optimal solution $y_0^*(b^*_{max})$ is not unique; instead it can take any value in the interval $[u_{(N(r)-1)}, u_{(N(r))}]$. Hence Danskin's theorem implies that the subdifferential of $V_{max}(b)$ at b^*_{max} will consist of all convex combinations of $\frac{\sqrt{N_{(N(r))}}(u_{(N(r))} - u_{(N(r)-1)})}{2\sqrt{b^*_{max} + \frac{1}{N}}}$ and 0.

We now prove part (b). Let us go back to the original primal-dual pair $(\mathcal{P}(b), \mathcal{D}(b))$ and consider a pair of optimal solutions of the primal and dual problems. By Proposition 2 the primal optimal solution $(\mathbf{p}^*(b), \mathbf{q}^*(b), \boldsymbol{\theta}^*(b), q_0^*(b))$ is unique, while our reasoning in part (a) established the uniqueness of the optimal dual variables $(\beta_0^*(b), \mathbf{y}^*(b), y_0^*(b), \mathbf{z}^*_{\mathbf{p}}(b), \mathbf{z}^*_{\mathbf{q}}(b))$. Applying Theorem 16 and part (ii) of the complementarity conditions of Lemma 15 of Alizadeh and Goldfarb [2], we arrive at the following conditions:

$$q_0^*(b)z^*_{qn}(b) + \beta_0^*(b)q_n^*(b) = 0 \Leftrightarrow -\sqrt{b+\frac{1}{N}}y_n^*(b) + \sqrt{\sum_{n=1}^{N} y_n^*(b)^2}p_n^*(b) = 0, \quad n = 1, 2, ..., N$$

$$\Leftrightarrow -\sqrt{b+\frac{1}{N}}\ \max(0, u_n - y_0^*(b)) + \sqrt{\sum_{n=1}^{N}\max(0, u_n - y_0^*(b))^2}\ p_n^*(b) = 0, \quad n = 1, 2, ..., N \tag{28}$$

When $b < b^*_{max}$, strong duality implies $y_0^*(b) < u_{(N(r)-1)}$ which in turn ensures $\sum_{n=1}^{N}\max(0, u_n - y_0^*(b))^2 > 0$. As mentioned earlier, when $b = b^*_{max}$ $y_0^*(b)$ can take any value in $[u_{(N(r)-1)}, u_{(N(r))}]$ so we choose one that again yields $\sum_{n=1}^{N}\max(0, u_n - y_0^*(b^*_{max}))^2 > 0$. Hence, the complementarity conditions (28) yield

$$p_n^*(b) = 0 \Leftrightarrow u_n - y_0^*(b) \le 0, \quad n = 1, 2, ..., N. \tag{29}$$

Since $y_0^*(b)$ is strictly increasing in b in $(0, b^*_{max})$ and $\lim_{b\to 0^+} y_0^*(b) = -\infty$ and $\lim_{b\to b^*_{max}-} y^*(b) = u_{(N(r)-1)}$, Eq. (29) implies the existence of a set $\{b_1, b_2, ..., b_{N(r)-1}\}$ such that

$$0 < b_1 < b_2 < .... < b_{N(r)-1} = b^*_{max}$$
$$\left\{\left\{p_n^*(b) = 0 \;\; \forall n \in \mathcal{N}_k^-\right\} \;\Leftrightarrow\; b \geq b_k\right\}, \;\; \forall k = 1, 2, ..., N(r) - 1.$$

■

Proposition 3. We prove the result for V_{max}; the argument for V_{min} is analogous. Focusing on optimization problem (9), we introduce Lagrangian multipliers and write the Karush-Kuhn-Tucker (KKT) conditions:

$$u_n - 2\lambda\left(p_n - \frac{1}{N}\right) + \mu + \nu_n = 0, \;\; n \in \{1, 2, ..., N\} \tag{30}$$

$$\lambda\left(\sum_{n=1}^N \left(p_n - \frac{1}{N}\right)^2 - b\right) = 0, \;\; \lambda \geq 0 \tag{31}$$

$$\sum_{n=1}^N \left(p_n - \frac{1}{N}\right)^2 \leq b, \;\; \sum_{n=1}^N p_n = 1, \;\; \mathbf{p} \geq \mathbf{0} \tag{32}$$

$$\nu_n p_n = 0, \;\; \nu_n \geq 0, \;\; n \in \{1, 2, ..., N\}. \tag{33}$$

Since our problem is concave with affine equality constraints and satisfies Slater's condition (see section 5.2.3 in [9]), strong duality holds and the KKT conditions (30)-(33) will be necessary and sufficient for both primal and dual optimality. In other words, the duality gap is zero and the vector $(\mathbf{p}^*, \boldsymbol{\nu}^*, \lambda^*, \mu^*)$ satisfies (30)-(33) if and only $\mathbf{p}^*$ and $\lambda^*, \boldsymbol{\nu}^*, \mu^*$ are primal and dual optimal respectively (see section 5.5.3 in [9]).

From Proposition 2 we know that there exists a unique primal optimal solution $\mathbf{p}^*$. By strong duality, the Lagrangean dual problem admits an optimal solution, and we refer to it by $\lambda^*, \boldsymbol{\nu}^*, \mu^*$.[14] Since $V_{max}(b)$ is differentiable (Theorem 1) and strong duality holds we follow Section 5.6.3 in Boyd and Vandenberghe [9] to deduce the following simple relation:

$$\frac{\mathrm{d}}{\mathrm{d}b} V_{max}(b) = \lambda^{max}(b), \;\; b \in (0, b^*_{max}). \tag{34}$$

Eq. (34) means that we can now focus on calculating the Lagrange multiplier $\lambda^{max}(b)$. Before we do so we note the following useful identity

$$\sum_{n=1}^N \left(p_n^{max}(b) - \frac{1}{N}\right)^2 \;\;=\;\; \sum_{n=1}^N p_n^{max}(b)\left(p_n^{max}(b) - \frac{1}{N}\right) - \frac{1}{N}\sum_{n=1}^N p_n^{max}(b) + \sum_{n=1}^N \frac{1}{N^2}$$

[14] Note that at this point one can manipulate the KKT conditions (30)-(33) to show that the Lagrangean dual's optimal solution is also unique.

$$= \sum_{n=1}^{N} p_n^{max}(b)\left(p_n^{max}(b) - \frac{1}{N}\right). \tag{35}$$

Multiplying both sides of Eq. (30) by $p_n^{max}(b)$ and then summing over all $n = 1, 2, .., N$ obtains

$$\sum_{n=1}^{N} u_n p_n^{max}(b) - 2\lambda^{max}(b)\sum_{n=1}^{N} p_n^{max}(b)\left(p_n^{max}(b) - \frac{1}{N}\right) + \mu^{max}(b)\sum_{n=1}^{N} p_n^{max}(b) = 0$$

$$\overset{(35)}{\Rightarrow} \quad \sum_{n=1}^{N} u_n p_n^{max}(b) - 2\lambda^{max}(b)\sum_{n=1}^{N}\left(p_n^{max}(b) - \frac{1}{N}\right)^2 + \mu^{max}(b) = 0$$

$$\overset{\text{Prop. 2}}{\Rightarrow} \quad \mu^{max}(b) = 2\lambda^{max}(b)\cdot b - \sum_{n=1}^{N} u_n p_n^{max}(b). \tag{36}$$

Now we consider Eq. (30) for expert $n_k \in \mathcal{N}_k$. By part (b) of Theorem 1 we must have $p_{n_k}^{max}(b) > 0$ if and only if $b \in [0, b_k^{max})$. Substituting the value of $\mu^{max}(b)$ obtained in Eq. (36), and applying the complementary slackness condition (33) we obtain

$$u_{n_k} - 2\lambda^{max}(b)\left(p_{n_k}^{max}(b) - \frac{1}{N}\right) = \sum_{n=1}^{N} u_n p_n^{max}(b) - 2\lambda^{max}(b)\cdot b$$

$$\overset{(34)}{\Rightarrow} \quad 2\frac{\mathrm{d}}{\mathrm{d}b}V_{max}(b)\left(p_{n_k}^{max}(b) - \frac{1}{N} - b\right) = u_{n_k} - V_{max}(b), \quad b \in (0, b_k^{max}). \tag{37}$$

■

Theorem 2. We focus on V_{max}; the argument for V_{min} is symmetric. Recall the definition of b_k^{max} of Eq. (11). Consider first $b \in (0, b_1^{max})$ so that $p_n^{max}(b) > 0$ for all $b \in (0, b_1^{max})$ and $n \in \mathcal{N}$. Recalling Proposition 3 and adding Eqs. (37) for all $n \in \mathcal{N}$ yields the following differential equation

$$-2Nb\frac{\mathrm{d}V_{max}(b)}{\mathrm{d}b} = -NV_{max}(b) + \sum_{n\in\mathcal{N}} u_n, \quad b \in (0, b_1^{max}). \tag{38}$$

Solving differential equation (38) leads to the following expression:

$$V_{max}(b) = C_1^{max}\sqrt{b} + \frac{\sum_{n\in\mathcal{N}} u_n}{N}, \quad b \in [0, b_1^{max}), \tag{39}$$

where C_1^{max} is a constant to be determined. Consider now $b \in [b_{k-1}^{max}, b_k^{max})$ for $k \in \{2, 3, ..., N(r) - 1\}$. In this range of b we will have $p_n^{max}(b) > 0$ if and only $n \in \mathcal{N}_k^+$. Adding Eqs. (37) for all such $n \in \mathcal{N}_k^+$ yields the following differential equation

$$2\left(\frac{N_{k-1}^-}{N} - N_k^+ b\right)\frac{\mathrm{d}V_{max}}{\mathrm{d}b} = \sum_{n\in\mathcal{N}_k^+} u_n - N_k^+ V_{max}(b), \quad b \in [b_{k-1}^{max}, b_k^{max}) \tag{40}$$

Solving differential equation (40) gives the following:

$$V_{max}(b) = \overline{u}_k^+ + C_k\sqrt{N_k^+ b - \frac{N_{k-1}^-}{N}}, \ \ b \in \left[b_{k-1}^{max}, b_k^{max}\right), \tag{41}$$

for $k \in \{2, 3, ..., N(r) - 1\}$, where C_k^{max} is a constant to determined. Finally since $b_{N(r)-1}^{max} = b_{max}^*$ we use Lemma 1 to conclude

$$V_{max}(b) = \max_{n\in\mathcal{N}} u_n, \ \ b \in \left[b_{N(r)-1}^{max}, \frac{N-1}{N}\right]. \tag{42}$$

Putting together Eqs. (39), (41), and (42) we see that V_{max} will equal

$$V_{max}(b) = \begin{cases} \frac{\sum_{n\in\mathcal{N}} u_n}{N} + C_1^{max}\sqrt{b} & b \in [0, b_1^{max}) \\ \overline{u}_k^+ + C_k^{max}\sqrt{N_k^+ b - \frac{N_{k-1}^-}{N}} & b \in \left[b_{k-1}^{max}, b_k^{max}\right), \ k = 2, 3, ..., N(r) - 1 \\ \max_{n\in\mathcal{N}} u_n & b \in \left[b_{N(r)-1}^{max}, \frac{N-1}{N}\right] \end{cases} \tag{43}$$

for appropriately chosen constants $(C_1^{max}, C_2^{max}, ..., C_{N(r)-1}^{max})$ and $\left(b_1^{max}, b_2^{max}, ..., b_{N(r)-1}^{max}\right)$. By Proposition 1 and Theorem 1, V_{max} is continuous everywhere and differentiable everywhere at $(0, \frac{N-1}{N})$ except b_{max}^*. Thus, the vectors $(C_1^{max}, C_2^{max}, ..., C_{N(r)-1}^{max})$ and $\left(b_1^{max}, b_2^{max}, ..., b_{N(r)-1}^{max}\right)$ must fulfill these criteria of continuity and differentiability and are thus uniquely determined by the following system of nonlinear equations (44)-(51):

Case 1: $N(r) = 2$**.**

$$\frac{\sum_{n\in\mathcal{N}} u_n}{N} + C_1^{max}\sqrt{b_1^{max}} = \max_{n\in\mathcal{N}} u_n \tag{44}$$

$$b_1^{max} = \frac{1}{N_2^{max}} - \frac{1}{N}. \tag{45}$$

Case 2: $N(r) \geq 3$.

$$\frac{\sum_{n\in\mathcal{N}} u_n}{N} + C_1^{max}\sqrt{b_1^{max}} = \overline{u}_2^+ + C_2^{max}\sqrt{N_2^+ b_1^{max} - \frac{N_1^-}{N}} \tag{46}$$

$$\frac{C_1^{max}}{\sqrt{b_1^{max}}} = \frac{C_2^{max} N_2^+}{\sqrt{N_2^+ b_1^{max} - \frac{N_1^-}{N}}} \tag{47}$$

$$\overline{u}_k^+ + C_k^{max}\sqrt{N_k^+ b_k^{max} - \frac{N_{k-1}^-}{N}} = \overline{u}_{k+1}^+ + C_{k+1}^{max}\sqrt{N_{k+1}^+ b_k^{max} - \frac{N_k^-}{N}}, \quad k = 2, 3, ..., N(r) - 2 \tag{48}$$

$$\frac{C_k^{max} N_k^+}{\sqrt{N_k^+ b_k^{max} - \frac{N_{k-1}^-}{N}}} = \frac{C_{k+1}^{max} N_{k+1}^+}{\sqrt{N_{k+1}^+ b_k^{max} - \frac{N_k^-}{N}}}, \quad k = 2, 3, ..., N(r) - 2 \tag{49}$$

$$\overline{u}^+_{N(r)-1} + C^{max}_{N(r)-1}\sqrt{N^+_{N(r)-1}b^{max}_{N(r)-1} - \frac{N^-_{N(r)-2}}{N}} = \max_{n\in\mathcal{N}} u_n \tag{50}$$

$$b^{max}_{N(r)-1} = \frac{1}{N^{max}_{N(r)}} - \frac{1}{N} \tag{51}$$

It now remains to show that the solution of System (44)-(51) satisfies $(\boldsymbol{C^{max}}, \boldsymbol{b^{max}}) = (\boldsymbol{C^+}, \boldsymbol{b^+})$, where the latter are given by Eqs. (16). We begin with Case 1 and $N(r) = 2$. That $b^{max}_1 = b^+_1$ is trivially true. Then, Eq. (44) immediately yields

$$C^{max}_1 = \sqrt{\frac{N_{N(r)}}{N_{N(r)-1}} \cdot N} \cdot \left(u_{(N(r))} - \overline{u}^+_{N(r)-1}\right) = C^+_1.$$

We now focus on Case 2 and $N(r) \geq 3$. Once again, $b^{max}_{N(r)-1} = b^+_{N(r)-1}$ is trivially true, whence Eq. (50) implies

$$C^{max}_{N(r)-1} = \sqrt{\frac{N_{N(r)}}{N_{N(r)-1}}} \cdot \left(u_{(N(r))} - \overline{u}^+_{N(r)-1}\right)^2 = C^+_{N(r)-1}.$$

Focusing on Eq. (49) for $k \in \{2, 3, ..., N(r)-2\}$ and solving for C^{max}_k yields:

$$C^{max}_k = \frac{\sqrt{N^+_k b^{max}_k - \frac{N^-_{k-1}}{N}}}{N^+_k} \frac{C^{max}_{k+1} N^+_{k+1}}{\sqrt{N^+_{k+1} b^{max}_k - \frac{N^-_k}{N}}}. \tag{52}$$

Plugging (52) into Eq. (48) we obtain:

$$C^{max}_{k+1}\sqrt{N^+_{k+1}b^{max}_k - \frac{N^-_k}{N}} \left(1 - \frac{N^+_{k+1}}{N^+_k} \cdot \frac{N^+_k b^{max}_k - \frac{N^-_{k-1}}{N}}{N^+_{k+1} b^{max}_k - \frac{N^-_k}{N}}\right) = \bar{u}^+_k - \bar{u}^+_{k+1}. \tag{53}$$

After some algebra, the left-hand-side of Eq. (53) can be simplified so that:

$$\begin{aligned}
& \frac{C^{max}_{k+1} N^+_{k+1}}{N} \frac{\frac{N^-_{k-1}}{N^+_k} - \frac{N^-_k}{N^+_{k+1}}}{\sqrt{N^+_{k+1} b^{max}_k - \frac{N^-_k}{N}}} = \frac{\sum_{n\in\mathcal{N}^+_k} u_n}{N^+_k} - \frac{\sum_{n\in\mathcal{N}^+_{k+1}} u_n}{N^+_{k+1}} \\
\Rightarrow \quad & -C^{max}_{k+1} \frac{N_k}{N^+_k \sqrt{N^+_{k+1} b^{max}_k - \frac{N^-_k}{N}}} = \frac{\left(u_{(k)} N^+_{k+1} - \sum_{n\in\mathcal{N}^+_{k+1}} u_n\right) N_k}{N^+_k N^+_{k+1}} \\
\Rightarrow \quad & -\frac{C^{max}_{k+1}}{\sqrt{N^+_{k+1} b^{max}_k - \frac{N^-_k}{N}}} = \widehat{u}^+_k
\end{aligned} \tag{54}$$

Combining Eqs. (49) and (54) obtains for $k = 2, 3, ..., N(r)-2$:

$$C^{max}_k \quad = \quad -\frac{N^+_{k+1}}{N^+_k} \widehat{u}^+_k \sqrt{N^+_k b^{max}_k - \frac{N^-_{k-1}}{N}} \tag{55}$$

$$b_k^{max} = \frac{\left(\frac{C_{k+1}^{max}}{\widehat{u}_k^+}\right)^2 + \frac{N_k^-}{N}}{N_{k+1}^+}, \tag{56}$$

which after some simple algebra leads to the following nonhomogeneous linear recursion for the squares of the C_k^{max}'s:

$$(C_k^{max})^2 = \frac{N_{k+1}^+}{N_k^+}(C_{k+1}^{max})^2 + \frac{N_{k+1}^+}{N_k^+}\left(1 - \frac{N_{k+1}^+}{N_k^+}\right)\left(\widehat{u}_k^+\right)^2, \quad k = 2, 3, ..., N(r) - 2. \tag{57}$$

Solving recursion (57) backwards with (previously derived) initial value

$$\left(C_{N(r)-1}^{max}\right)^2 = \frac{N_{N(r)}}{N_{N(r)-1}} \cdot \left(\widehat{u}_{N(r)}^-\right)^2,$$

taking square roots, and recalling the positive sign of the C_k^{max}'s, establishes that $C_k^{max} = C_k^+$ for $k = 2, 3, ..., N(r)-2$. Subsequently applying Eq. (56) establishes $b_k^{max} = b_k^+$ for $k = 2, 3, ..., N(r)-2$. Finally, $C_1^{max} = C_1^+$ and $b_1^{max} = b_k^+$ is obtained by applying $C_2^{max} = C_2^+$ to Eq. (47) and solving Eqs. (46) and (47) for b_1^{max} and C_1^{max}. ■

Corollary 1: Characterization of $\mathbf{P^{max}}(r, b)$ and $\mathbf{P^{min}}(r, b)$. We distinguish between different cases.

1. $\mathbf{P^{max}}(r, b)$.

Case (1a): $b < b_{N(r)-1}^+(r)$. In this case, by Proposition 2, we know that the optimal solution $\mathbf{p^{max}}(b)$ will be unique. Consider the vectors $(\boldsymbol{C}^+(r), \boldsymbol{b}^+(r))$ defined in Eqs. (16). Suppose expert n_k satisfies $n \in \mathcal{N}_k(r)$. Then, Proposition 3 and Theorem 2 and simple algebra establish that

$$p_{n_k}^{max}(r, b) = \begin{cases} \frac{1}{N} + \frac{\left(u_n(r) - \overline{u}_1(r)^+\right)}{C_1^+(r)}\sqrt{b} & b \in \left[0, b_1^+(r)\right) \\ \frac{1}{N_k^+(r)} + \frac{\left(u_{n_k}(r) - \overline{u}_k(r)^+\right)\sqrt{N_k^+(r)b - \frac{N_{k-1}^-(r)}{N}}}{C_k^+(r)N_k^+(r)} & b \in \left[b_{k-1}^+(r), b_k^+(r)\right), \ k = 2, 3, ..., k-1 \\ 0 & b \in \left[b_{k-1}^+(r), b_{N(r)-1}^+(r)\right). \end{cases}$$

Case (1b): $b \geq b_{N(r)-1}^+(r)$. Here, by Proposition 2 all vectors $\mathbf{p^{max}}(r, b)$ satisfying $p_n^{max}(r, b) = 0$ for $n \notin \mathcal{N}_{N(r)}(r)$ and $\sum_{n \in \mathcal{N}_{N(r)}} \left(p_n^{max}(r, b) - \frac{1}{N}\right)^2 \leq b - \frac{N_{N(r)-1}^-(r)}{N^2}$ will be optimal. This set is a singleton at $b = b_{N(r)-1}^+(r)$.

2. $\mathbf{P^{min}}(r, b)$.

Case (2a): $b < b_{N(r)-1}^-(r)$. In this case, by Proposition 2, we know that the optimal solution $\mathbf{p^{min}}(r, b)$ will be unique. Consider the vectors $(\boldsymbol{C}^-(r), \boldsymbol{b}^-(r))$ defined in Eqs. (17). Suppose expert n_k satisfies $n \in \mathcal{N}_k(r)$. Then, Proposition 3 and Theorem 2 and simple algebra establish that

$$p_{n_k}^{min}(r,b) \quad = \quad \begin{cases} \frac{1}{N} + \frac{\left(u_n(r) - \overline{u}_{N(r)}(r)^-\right)}{C_1^-(r)}\sqrt{b} & b \in \left[0, b_1^-(r)\right) \\ \frac{1}{N^-_{N(r)-k+1}(r)} + \frac{\left(u_{n_k}(r) - \overline{u}_{N(r)-k+1}(r)^-\right)\sqrt{N^-_{N(r)-k+1}(r)b - \frac{N^+_{N(r)-k+2}(r)}{N}}}{C_k^-(r)N^-_{N(r)-k+1}(r)} & b \in \left[b^-_{k-1}(r), b^-_k(r)\right), \\ \quad k = 2, 3, ..., N(r) - k + 10 & b \in \left[b^-_{k-1}(r), b^-_{N(r)-1}(r)\right). \end{cases}$$

Case (2b): $b \geq b^-_{N(r)-1}(r)$. Here, by Proposition 2 all vectors $\mathbf{p^{min}}(r,b)$ satisfying $p_n^{min}(r,b) = 0$ for $n \notin \mathcal{N}_1(r)$ and $\sum_{n \in \mathcal{N}_1(r)} \left(p_n^{min}(r,b) - \frac{1}{N}\right)^2 \leq b - \frac{N_2^+(r)}{N^2}$ will be optimal. This set is a singleton at $b = b^-_{N(r)-1}(r)$.

■

Theorem 3. Here we apply part (iii) of Corollary 4 in Milgrom and Segal [26] to functions $V_{max}(r|b)$ and $V_{min}(r|b)$ (we express the latter as a maximization problem $-V_{max}(-r|b)$). ■

Corollary 2. Follows by Proposition 2 and Theorem 3. ■

Proposition 4. Follows from Proposition 2 and Theorem 3. ■

Corollary 3. The statement of the Corollary implies that $\mathcal{N}_{N(r)}(r) = \{n_1\}$ and $\mathcal{N}_1(r) = \{n_2\}$ for all $r \in \mathcal{R}$. Hence, $b^*_{max}(r) = b^*_{min}(r) = \frac{N-1}{N}$ for all $r \in [r_m, r_M]$. Applying Corollary 2 establishes the result. ■

A2: Constructing expert pdfs for the three R&D Scenarios from ICARUS survey data

In the ICARUS survey, experts were asked to provide values for the 10th, 50th, and 90th percentile of their distributions for the 2030 cost of solar technology conditional on all three Scenarios. In addition, they were asked to provide values for the probability of this cost being less than or equal to the following three values: 11.3, 5.5, and 3c$/kWh. These "threshold" cost levels correspond to projections of the costs of electricity from fossil fuels or nuclear in 2030. The first (11.27 c$/kWh) corresponds to the 2030 projected cost of electricity from traditional coal power plants in the presence of a specific policy to control CO2 emissions (thus effectively doubling electricity costs from fossil sources). The second threshold cost (5.5 c$/kWh) is the projected cost of electricity from traditional fossil fuels in 2030, without considering any carbon tax. Finally, the third (3 c$/kWh) reflects a situation in which solar power becomes competitive with the levelized cost of electricity from nuclear power.

Asking experts the follow up question on the likelihood of reaching threshold cost targets allowed the survey authors to guard against the cognitive pitfalls associated with direct elicitation of subjective probabilities, to increase the amount of elicited information, and to deepen the discussion with the expert, hence improving their perception of his/her beliefs. In cases where the two sets of answers (percentile values and threshold probabilities) were inconsistent, we contacted the expert in order to obtain coherent estimates. Moreover, we asked all experts to give values for the upper and lower limits of their distribution's support in order to pinpoint the intervals over which their implied probability distributions range.

Such corrected estimates were obtained from 14 out of the original 16 experts, and therefore the analysis of Section 4 focuses solely on them. Among the respondents, not all provided values on the left and right endpoints of their distributions' support. As a result, we deduced between 6 and 8 points of 14 experts' cumulative distribution functions (cdf) of the 2030 cost of solar electricity, given the aforementioned three R&D investment Scenarios. From these points a probability distribution function (pdf) was constructed using linear interpolation in the following way. First of all, and in accordance with the experts' answers, we considered cost levels c lying in [2c$/kWh, 30c$/kWh] and discretized this interval on a scale of 0.5 (30c$/kWh represents an estimate of the technology's current cost). Now, suppose an expert reported the values of his/her cdf F_n at two successive points c_1 and c_2 where $c_2 > c_1$ and gave no further information on cost levels between c_1 and c_2. Assuming right-continuity of F_n we took the probability mass $F_n(c_2) - F_n(c_1)$ to be distributed uniformly among the cost levels $\{c_1 + .5, c_1 + 1, ..., c_2\}$. For experts who did not provide values for the lower limit of their distribution's support we assumed that whatever probability mass remained to be allocated (always less than .1) was distributed uniformly between the smallest argument of the cdf and two cost levels below it. For example, if an expert only indicated that c_l was his y'th percentile and gave no further points of the cdf below this, we assumed that a probability mass of y was distributed evenly across $\{c_l - 1, c_l - .5, c_l\}$. In the case of an unknown upper limit, if an expert only indicated that c_u was his yth percentile and gave no further arguments for the cdf above it, we assumed that a probability mass of $1-y$ was distributed evenly across $\{c_u + .5, c_u + 1\}$.

Following this procedure we arrived at probability distribution functions for all 14 experts conditional on all three Scenarios. The implied cumulative distribution functions are depicted in Figure 1.

A3: Tables and Figures

2030 solar-technology cost c (2005 USc$/kWh)	Benefit $B(c)$ (US$ 10^9)
2	189.90
2.5	170.76
3	151.26
3.5	131.74
4	112.12
4.5	92.29
5	71.47
5.5	50.64
6	29.27
6.5	23.59
7	12.32
7.5	3.67
8	1.76
> 8	0

Table 1: EU discounted consumption improvement as a function of 2030 solar-power cost

R&D Scenario r	Opportunity Cost $O(r)$ (US$ 10^9)
r_1	3.67
r_2	5.51
r_3	7.35

Table 2: Discounted opportunity cost of R&D Scenarios

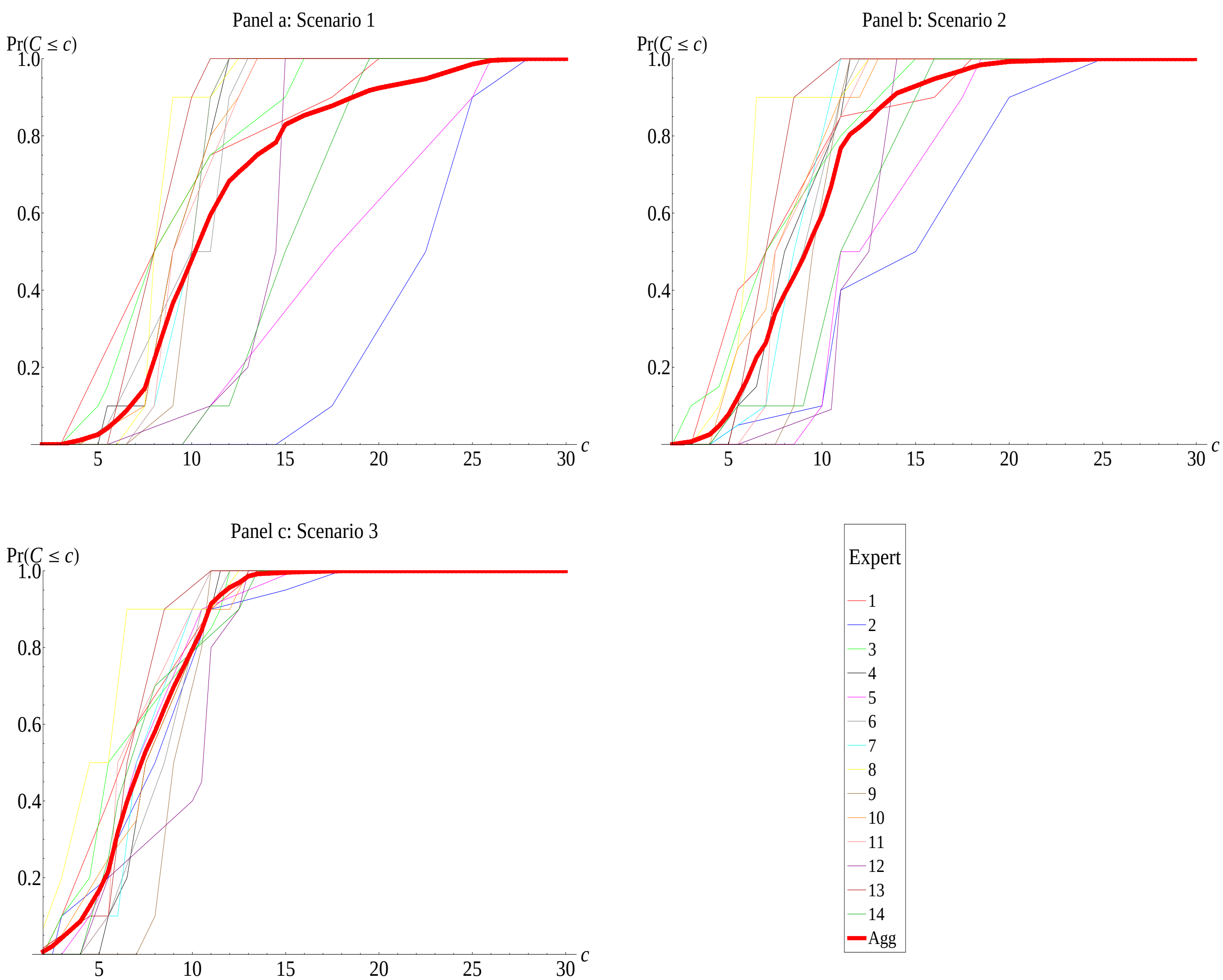

Figure 1: Expert and aggregate cdfs of the 2030 cost of solar technology under the three R&D Scenarios. Recall that the cdf's domain is $\{2, 2.5, ..., 29, 29.5, 30\}$. Cost is measured in 2005 USc\$/kWh.

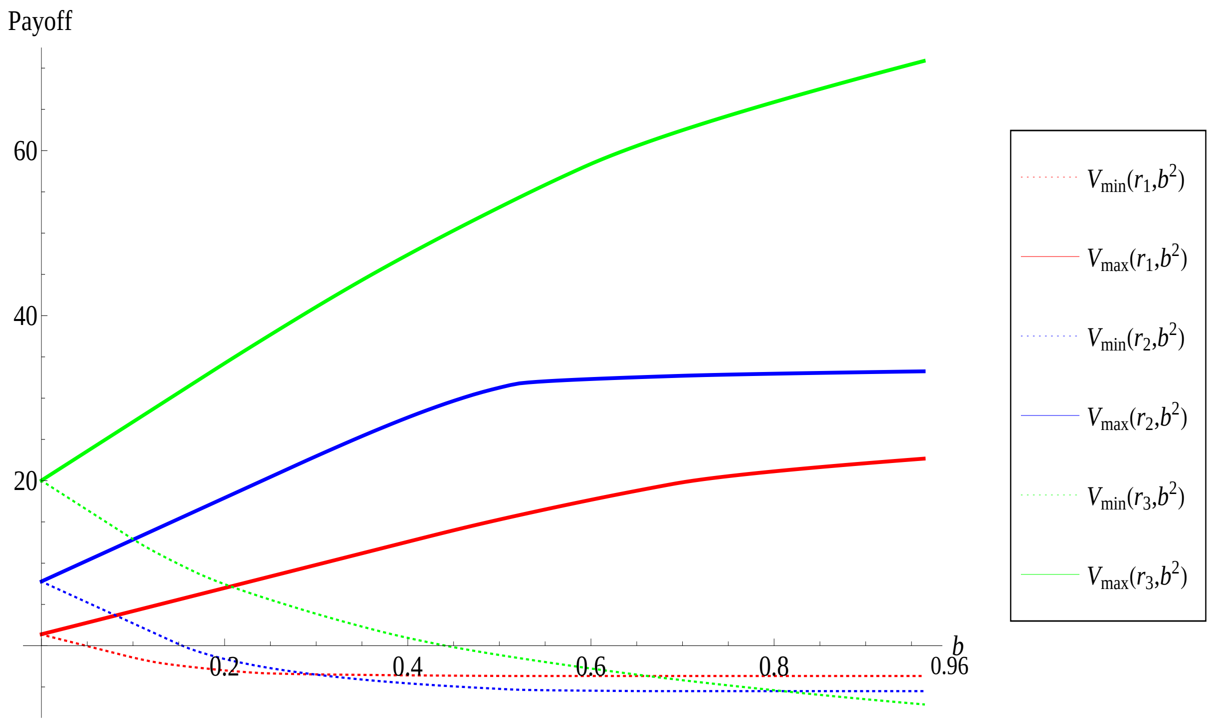

Figure 2: Worst and Best-Case net payoffs (benefits minus opportunity cost) for the three R&D scenarios. Net payoffs are measured in US$$10^9$.

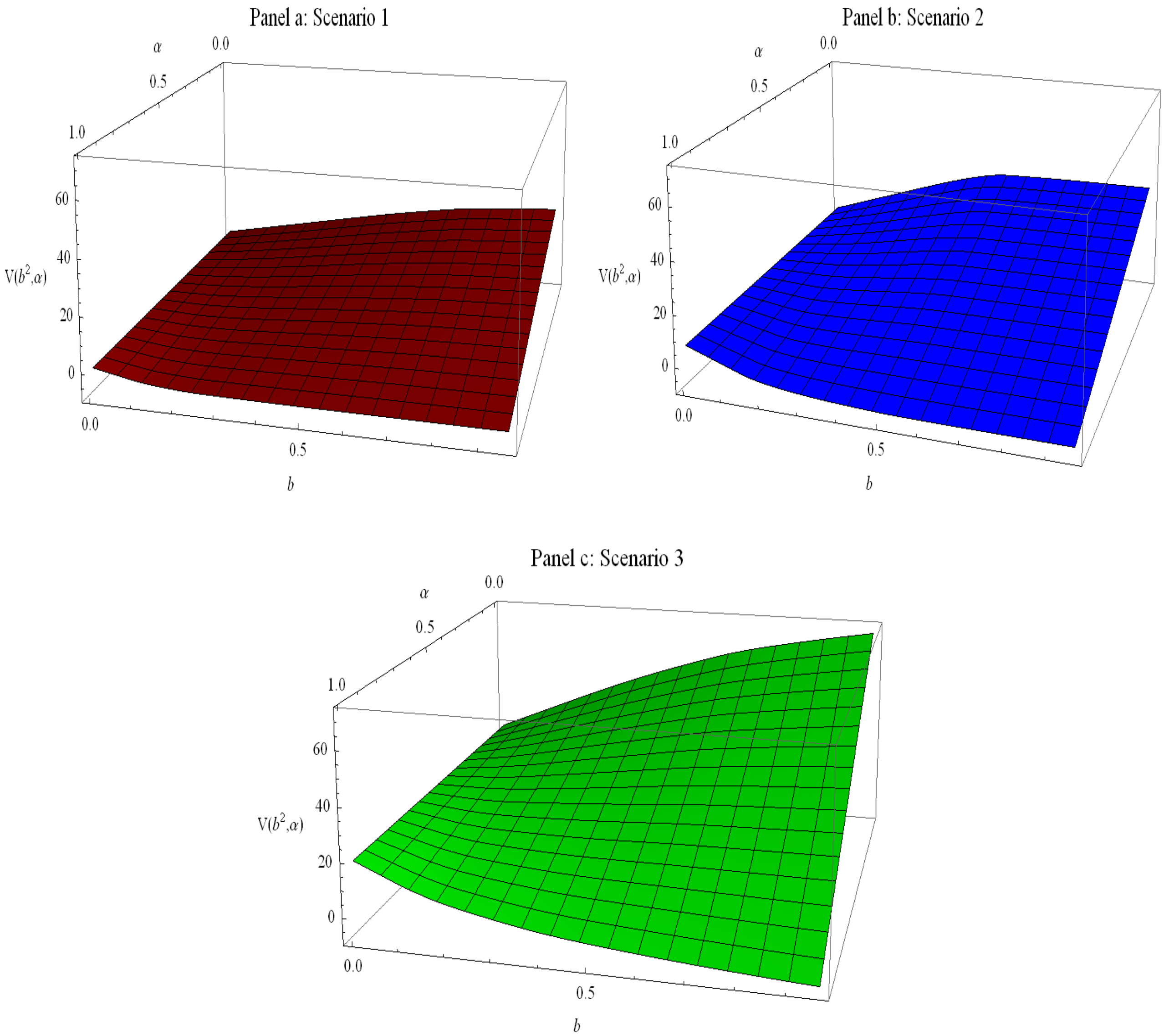

Figure 3: Net payoff (benefits minus opportunity cost) for the three R&D scenarios, as a function of ambiguity b and ambiguity attitude α. Net payoffs are measured in US\$$10^9$.

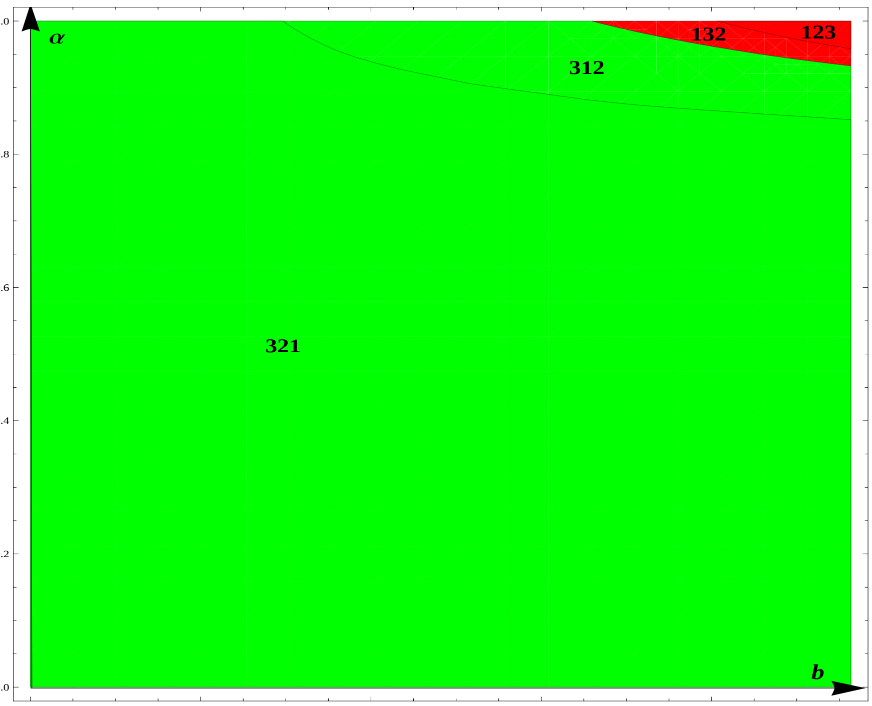

Figure 4: Comparison of the three scenarios over all values of b and α. A

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