A Weak Bargaining Set for Contract Choice Problems

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Summary

In this paper, we consider the problem of choosing a set of multi-party contracts, where each coalition of agents has a non-empty finite set of feasible contracts to choose from. We call such problems, contract choice problems. The main result of this paper states that every contract choice problem has a non-empty weak bargaining set. The need for such a solution concept which is considerably weaker than the core arises, since it is well known that even for very simple contract choice problems, the core may be empty. We also show by means of an example that the bargaining set due to Mas-Colell (1989), as well as a weaker version of it, may be empty for contract choice problems, thereby implying that the weakening we suggest is in some ways "tight"

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Abstract

In this paper, we consider the problem of choosing a set of multi-party contracts, where each coalition of agents has a non-empty finite set of feasible contracts to choose from. We call such problems, contract choice problems. The main result of this paper states that every contract choice problem has a non-empty weak bargaining set. The need for a such a solution concept which is considerably weaker than the core arises, since it is well known that even for very simple contract choice problems, the core may be empty. We also show by means of an example that the bargaining set due to Mas-Colell (1989), as well as a weaker version of it, may be empty for contract choice problems, thereby implying that the weakening we suggest is in some ways "tight".

1. Introduction:

In this paper, we consider the problem of choosing a set of multi-party contracts, where each coalition of agents has a non-empty finite set of feasible contracts to choose from. We call such problems, contract choice problems. The economic motivation behind the problem, arises from several real world "commons problems", where agents can pool their initial resources and produce a marketable surplus, which needs to be shared among themselves. There are clearly, two distinct problems that arise out of such real world possibilities: (i) Coalition Formation: Which are the disjoint coalitions that will form in order to pool in their resources? (ii) Distribution: How will a coalition distribute the surplus within itself? While, the possibility of an aggregate amount of surplus being generated by a coalition is fairly common, there are many situations where more than one aggregate surplus results from a cooperative activity, and the distribution of the surplus depends on the particular aggregate that a coalition chooses to share.
In our model, which has been developed in a related paper [Lahiri (2003)] each non-empty subset of agents has a non-empty finite set of pay-off vectors to choose from. An outcome comprises a partition of the set of agents, and an assignment for each coalition in the partition a feasible pay-off vector. Our model is therefore a special kind of cooperative game with non-transferable utility. In the

context of our contract choice model, the Shapley-Scarf (1974) housing market corresponds to a situation, where each individual assigns a monetary worth to each object, and a feasible pay-off vector for a coalition, is the set of utility vectors available to the coalition, when it re-allocates objects within itself, without any one in the coalition retaining his initial endowment, unless the coalition is a singleton. Lahiri (2004) proves the existence and Weak Pareto Optimality of 'stable' outcomes in a two-sided contract choice problem. The model studied in Lahiri (2004) is originally due to Roth and Sotomayor (1996). Zhou (1994) introduced a concept of the bargaining set, which is a slight variation of the original one due to Aumann and Maschler (1964). Yet another notion of a bargaining set is due to Mas-Colell (1989). The Zhou(1994) bargaining set of a marriage problem always contains its non-empty core. Klijn and Masso(2003) introduced the concept of the weakly stable set for a marriage problem and showed that the set of efficient and weakly stable matchings coincided with its bargaining set as defined by Zhou (1994).
We introduce the concepts of the weak bargaining set for contract choice problems. The basic idea behind the weak bargaining set is a set of feasible allocations, which do not admit a credible objection (i.e. every strong objection has a strong counter-objection). Our definition of a credible objection is somewhat different from that of Zhou (1994) or Mas-Colell (1989), in that we require a strong counter-objection to make none of its proponents worse off than what they were at the time when the objection was raised. We further require that no sub-coalition of an objecting coalition can block the objecting pay-off.
The main result of this paper states that every contract choice problem has a non-empty weak bargaining set. We show with the help of a three-agent example, that a natural analog of the bargaining set due to Mas-Colell (1989), and hence the bargaining set due to Mas-Colell (1989), may well be empty for room-mates problems. This, in particular suggest that the weakening of the bargaining set we suggest here, is indeed “tight”.

2. Contract Choice Problems:

Let X be a non-empty finite subset of $\aleph$(: the set of natural numbers), denoting the set of participating agents. Let $\Re$ denote the set of all real numbers and $\Re_+$ the set of non-negative real numbers. Let [X] denote the set of all non-empty subsets of X. Members of [X] are called coalitions. Further let $\Re^S$ denote the set of all functions from S to $\Re$ and $[\Re^S]$ denote the collection of all non-empty finite subsets of $\Re^S$. Given $S\in[X]$, let #S denote the number of elements of S.
Given $S\in[X]$, let $C(S) = \{\mu / \mu$ is a bijection on X with $\mu(S) = S\}$ and $C^0(S) = \{\mu\in C(S)/$ T is a non-empty proper subset of S implies $\mu(T) \neq T\}$.
Thus, if $\#S \geq 2$, then the function $\mu: X\to X$, such that $\mu(a) = a$ for all $a\in S$, belongs to $C(S)\setminus C^0(S)$.
A Contract Choice Problem (CCP) is a function $G: [X]\to(\bigcup_{S\in[X]}[\Re^S])\cup\{\phi\}$ such that for all $S\in[X]$: (i) $G(S) \subset \Re^S$; (ii) $G(\{a\}) = \{0\}$ for all $a\in X$.
G(S) is the set of all feasible allocations of pay-offs for agents in S.

Given a CCP G, a coalition structure for G is a partition of X.
A pay-off function is a function $v : X \to \Re_{+}$. If v is a pay-off function and $S \in [X]$, then $v|S$ denotes the restriction of v to the set S.
An outcome for a CCP G is a pair (f, v), where f is a coalition structure for G and v is a pay-off function such that (i) for all $a \in X$: $v(a) \geq 0$; (ii) for all $S \in f$: $v|S \in G(S)$.
The pair (f, v), where $f = \{\{a\}/ a \in X\}$ and $v(a) = 0$ for all $a \in X$, is an outcome for every CCP. Hence the set of outcomes is always non-empty.

Given an outcome (f, v) for a CCP G, a coalition $S \in [X]$ is said to block (f, v) if there exists $x \in G(S)$: $x(a) > v(a)$ for all $a \in S$.
An outcome (f, v) for a CCP G is said to belong to the core of G, if it does not admit any blocking coalition. Let Core(G) denote the set of outcomes in the core of G.
An outcome (f, v) for a CCP G is said to be Weakly Pareto Optimal if it does not admit X as a blocking coalition.

Given a CCP G, an outcome (f, v) is said to be weakly blocked by a coalition $T \in [X]$, if there exists $x \in G(T)$: $x(a) \geq v(a)$ for all $a \in T$, with strict inequality for at least one $a \in T$. If an outcome (f,v) is weakly blocked by a coalition $T \in [X]$, via $x \in G(T)$, then $a \in T$ is said to be an active member of the weakly blocking coalition T, if $x(a) > v(a)$.
An outcome (f,v) is said to be Pareto Optimal if it does not admit X as a weakly blocking coalition.

A special case of a CCP is the room-mates problem of Gale and Shapley (1962), where $G(S) = \phi$, whenever $\#S > 2$. The marriage problem of Gale and Shapley (1962) is in turn a special case of their room-mates problem. If $G(S) = \phi$, whenever $\#S > 3$, then we have a possible generalization of the man, woman and child problem of Alkan (1988).
The following example due to Gale and Shapley (1962) shows that the core of a room-mate problem may be empty.

Example 1 (Gale Shapley (1962)) : Let X = {1,2,3,4}. For $a \in X$, let $u^a: X \to \Re$ be defined as follows:
u^1: $u^1(2) = 3$, $u^1(3) = 2$, $u^1(4) = 1$, $u^1(1) = 0$;
u^2: $u^2(3) = 3$, $u^2(1) = 2$, $u^2(4) = 1$, $u^2(2) = 0$;
u^3: $u^3(1) = 3$, $u^3(2) = 2$, $u^3(4) = 1$, $u^3(3) = 0$;
u^4: $u^4(1) = 3$, $u^4(2) = 2$, $u^4(3) = 1$, $u^4(4) = 0$.
Let, G be a CCP such that for all $S \in [X]$: (i)$G(S) = \{x \in \Re^S /$ for some $\mu \in C^0(S)$, $x(a) = u^a(\mu(a))$ for all $a \in S\}$, if $\#S \in \{1,2\}$; (ii) $G(S) = \phi$, otherwise.
Suppose (f,v) is an outcome such that $v(4) \neq 0$. If $v(4) = 1$, then $\{3,4\} \in f$ and $v(3) = 1$. Thus,{2,3} blocks (f,v), since 2 can get 3 units and 3 can get 2 units in G({2,3}); if $v(4) = 2$, then $\{2,4\} \in f$ and $v(2) = 1$. Thus, {1,2} blocks (f,v) since 1 can get 3 units and 2 can get 2 units in G({1,2}); if $v(4) = 3$, then $\{1,4\} \in f$ and

v(1) = 1. Thus, {1,3} blocks (f,v) since 3 can get 3 units and 1 can get 2 units in G({1,3}). Thus, $v(4) \neq 0$ implies (f,v) does not belong to Core(G). Hence suppose v(4) = 0. If v(3) = 0, then both {2,3} and {3,4} block (f,v); if v(2) = 0, then both {1,2}and {2,4} block (f,v); if v(1) = 0, then both {1,3} and {1,4} block (f,v). Since v(4) = 0 requires v(a) = a for at least one $a \in \{1,2,3\}$, Core(G) = ϕ.
Note that the outcome (f^*,v^*) such that $f^* = \{\{1,2,3\},\{4\}\}$ and $v^*(1) = v^*(2) = v^*(3) = 3$, $v^*(4) = 0$, belongs to the Core(G^*), where G^* is such that for all $S \in [X]$: (i)$G^*(S) = \{x \in \Re^S /$ for some $\mu \in C^0(S)$, $x(a) = u^a(\mu(a))$ for all $a \in S\}$, if #S $\in \{1,2,3\}$; (ii) $G^*(S) = \phi$, otherwise.

Given a CCP G and a Pareto Optimal outcome (f,v), the pair ((f',v'), T) where (f',v') is an outcome for G and T∈f' is said to be a strong objection against (f,v) if v'(a) > v(a) for all a ∈T and no subset of T is a blocking coalition for (f',v').

Given a CCP G, a Pareto Optimal outcome (f,v) and a strong objection ((f',v'),T) against (f,v), an ordered pair ((f'',v''),U) where (f'',v'') is an outcome for G and U∈ f'' is said to be a strong counter-objection against ((f',v'),T) if: (a)U\T, T\U and U∩T are all non-empty; (b)v''(a) > v'(a) for all a∈ U.
The strong objection ((f',v'),T) against the outcome (f,v) is said to be justified, if ((f',v'),T) has no strong counter-objection.

We define the weak bargaining set of a CCP G, to be the set WB(G) = {(f,v)/ (f,v) is Pareto Optimal and such that no strong objection against (f,v) is justified}.

Example 1 (due to Gale and Shapley (1962)) is one that has an empty core, but a non-empty weak bargaining set. We saw in Example 1, that Core(G) = ϕ. However, consider v(4) = 1, v(3) = 1, v(2) = 2, v(1) = 3, f = {{1,2},{3,4}}. The pair ((f', v'), {2,3}) is a strong objection against (f,v), where v'(2) = 3, v'(3) = 2, v'(1) = v'(4) = 0 and f' = {{1},{4},{2,3}}. Let f'' = {{2}, {4}, {1,3}}, v''(1) = 2, v''(3) = 3, v'' (2) = v''(4) = 0.Then the pair ((f'',v''), {1,3}) is a strong counter-objection against ((f',v'),{2,3}). Further, (f,v) admits no blocking coalition other than {1,3}. Since no subset of {1,3} blocks (f',v'), (f,v) belongs to WB(G).

Note that it is possible to provide a definition of the weak bargaining set modified along the lines suggested in Mas-Colell (1989).

Given a CCP G an outcome (f,v) and a strong objection ((f',v'),T) against (f,v), an ordered pair ((f'',v''),U) is said to be a classical strong counter-objection against ((f',v'),T) if: (a) U∈f''; (b) U\T, U\S and U∩T are all non-empty; (c) v''(a) ≥ v(a) for all a∈U|T; (d) v''(a) > v'(a) for all a∈U.
The strong objection ((f',v'),T) against the outcome (f,v) is said to be classically justified, if ((f',v'),T) has no classical strong counter-objection.
We define the classical weak bargaining set of a CCP G, to be the set $WB^*(G)$ = *{(f,v)/ (f,v) is Pareto Optimal, and such that no strong objection against (f,v) is classically justified}.*

However, the following example reveals that even for room-mates problems, $WB^*(G)$ may be empty.

Example 2: Let X = {1,2,3}. For $a \in X$, let $u^a: X \rightarrow \Re$ be defined as follows:
u^1: $u^1(2) = 2$, $u^1(3) = 1$, $u^1(1) = 0$;
u^2: $u^2(3) = 2$, $u^2(1) = 1$, $u^2(2) = 0$;
u^3: $u^3(1) = 3$, $u^3(2) = 2$, $u^3(3) = 0$.

Let, G be the CCP such that for all $S \in [X]$: (i)$G(S) = \{x \in \Re^S /$ for some $\mu \in C^0(S)$, $x(a) = u^a(\mu(a))$ for all $a \in S\}$, if $\#S \in \{1,2\}$; (ii) $G(S) = \phi$, otherwise.
Since (f,v) such that v(a) = 0 for all $a \in X$ is not Pareto Optimal, it cannot belong to $WB^*(G)$.
Let (f,v) be the outcome such that f = {{1,3},{2}}, v(1) = 1, v(2) = 0, v(3) = 2 and (f',v') be the outcome such that f' = {{1,2},{3}}, v'(1) = 2, v'(2) = 1, v'(3) = 0. Thus, ((f',v'), {1,2}) is a strong objection against (f,v). Any strong counter-objection or classical strong counter-objection cannot contain agent 1, since agent 1 gets 2 units of pay-off at (f',v'). The only possibility is (({{2,3}, {1}}, v''), {2,3}) where v''(1) = 0, v''(3) = 1, v''(2) = 2,which is a strong counter-objection though not a classical strong counter-objection, since agent 3 is worse off at (f'',v'') than at (f,v). Thus, $(f,v) \notin WB^*(G)$.
Let (f,v) be the outcome such that f = {{1},{2,3}}, v(1) = 0, v(2) = 2, v(3) = 1 and (f',v') be the outcome such that f' = {{1,3},{2}}, v'(1) = 1, v'(2) = 0, v'(3) = 2. Thus, ((f',v'), {1,3}) is a strong objection against (f,v). Any strong counter-objection or classical strong counter-objection cannot contain agent 3, since agent 3 gets 2 units of pay-off at (f',v'). The only possibility is (({{1,2}, {3}}, v''), {1,3}) where v''(1) = 2, v''(3) = 0, v''(2) = 1,which is a strong counter-objection though not a classical strong counter-objection, since agent 2 is worse off at (f'',v'') than at (f,v). Thus, $(f,v) \notin WB^*(G)$. Thus, $WB^*(G) = \phi$.
Hence $Bar^*(G) = \phi$.

3. The non-emptiness of the weak bargaining set:

Theorem 1: Let G be a CCP. Then, $WB(G) \neq \phi$.

Proof: Let G be a CCP and let (f,v) be a Pareto Optimal outcome for G. If (f,v) does not admit a strong objection then clearly, $(f,v) \in WB(G)$. Suppose $((f^1,v^1), S^1)$ is a strong objection against (f,v) which further does not admit a strong counter-objection. Then, no member of S^1 is part of a strong objection against (f^1,v^1). Since $((f^1,v^1), S^1)$ is a strong objection against (f,v) which further does not admit a strong counter-objection, there can be no strong objection $((f^2,v^2), S^2)$ against (f^1,v^1) such that $S^2 \cap S^1 \neq \phi$. If (f^1,v^1) does not admit any strong objection, then $(f^1,v^1) \in WB(G)$. Suppose $((f^2,v^2), S^2)$ is a strong objection against (f^1,v^1) which further does not admit a strong counter-objection. Thus, $S^2 \cap S^1 = \phi$. Without loss of generality suppose $S^1 \in f^2$ and $v^2(a) = v^1(a)$ for all $a \in S^1$. This is possible, since $S^2 \cap S^1 = \phi$. Then, no member of $S^1 \cup S^2$ is part of a strong objection against (f^2,v^2).

Having constructed a strong objections $((f^p,v^p), S^p)$ against (f^{p-1},v^{p-1}) for p = 1,…., k, where $(f^0,v^0) = (f,v)$, such that no member of $\bigcup_{p=1}^{k} S^p$ is part of a blocking coalition against (f^k,v^k) there are two possibilities: there does exist a strong objection against (f^k,v^k) in which case $(f^k,v^k)\in WB(G)$; there exists a strong objection $((f^{k+1}, v^{k+1}), S^{k+1})$ against (f^k,v^k). If every such strong objection admits a strong counter-objection, then $(f^k,v^k)\in WB(G)$. If not then there exists a strong objection $((f^{k+1},v^{k+1}), S^{k+1})$, which further does not admit a strong counter-objection. Clearly, $S^{k+1}\cap(\bigcup_{p=1}^{k} S^p) = \phi$. Without loss of generality suppose, $S^p\in f^{k+1}$ for p = 1,…,k and $v^{k+1}(a) = v^k(i)$ for all $a\in\bigcup_{p=1}^{k} S^p$. Then no member of $\bigcup_{p=1}^{k+1} S^p$ is part of a strong objection against (f^{k+1}, v^{k+1}).
Since X is a finite set, there is a smallest positive integer K, such that either every objection ((f',v'), T) against (f^K,v^K) admits a strong counter-objection, or [$\bigcup_{p=1}^{K} S^p$ = X or no member of X is part of a blocking coalition against (f^K,v^K). In either case, $(f^K,v^K)\in WB(G)$. Q.E.D.

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