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Abstract

The introduction of real-cash balances into the neoclassical model of the consumer wrecks havoc, in general, on the empirically observable refutable comparative statics properties of the model. We provide the most general solution of this problem to date by deriving a symmetric and negative semidefinite generalized Slutsky matrix that is empirically observable and which contains all other such comparative statics results as a special case. In addition, we clarify and correct two aspects of Samuelson and Sato's (1984) treatment of this problem.

Keywords: Money-Goods models; Weak Separability; Slutsky Matrix; Comparative Statics

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I. Introduction

The most general representation of a consumer's utility function includes commodity prices alongside the quantities of real goods. The justification for including commodity prices was advanced more than fifty years ago by Samuelson (1947) and Patinkin (1948), both of whom also suggested the introduction of real cash balances into the consumer's preferences. About the same time, Scitovsky (1945) suggested that prices may also be choice variables when they are perceived as an index of a commodity's quality. In spite of this idea's old vintage, it remains an open question as to how to derive empirically verifiable hypotheses under the most general formulation of the consumer's optimization problem. Most recently, Samuelson and Sato (1984) formulated this challenge in explicit terms and provided their own analysis of the problem.

In this paper, we have taken up Samuelson and Sato's (1984, p. 603) challenge and presented the most general comparative statics solution of the price-dependent utility maximization problem to date by deriving a symmetric and negative semidefinite generalized Slutsky matrix that is empirically observable and which contains all other such comparative statics results as special cases. Our analysis is of theoretical and empirical importance because it provides a testable alternative to the standard consumer model when it is rejected by the sample data. For better or worse, this event has occurred more often than not. For example, using data from Holland and from Germany, Barten and Geyskens (1975) found that the Dutch sample did not reject any of the theoretical restrictions implied by the archetype model, while the German sample rejected the hypothesis that the Slutsky matrix is symmetric and negative semidefinite. Similarly, Horney and McElroy (1988) found evidence against the neoclassical model, and Altonji, et. al. (1989) rejected the household pooling hypothesis implied by the standard consumer model. Using a Bayesian approach and an almost ideal demand system specification, Chalfant et. al. (1991) found that the posterior probability for concavity of the expenditure function to hold is only 0.16.

The rejection of symmetry and negative semidefiniteness of the Slutsky matrix weakens our confidence in a theory that assumes the maximization of the utility of real goods subject to a linear budget constraint. But, as Kuznets (1963) wisely wrote: "… (an) empirically irrelevant theory lies on until an empirically relevant theory replaces it; just testing is not enough." The mounting evidence against the prototype consumer demand model suggests the need for revisions and extensions that include the prototype specification as a special case. In this way, more powerful tests of the traditional theory can be performed and Kuznets' (1963) appeal for an empirically relevant theory may become a reality.

To this end, our paper extends the standard consumer model by deriving the most general set of conditions that may serve as a scaffolding for empirically verifiable hypotheses of the corresponding theories when real-cash balances and commodity prices enter the direct utility function of the neoclassical consumer in an explicit fashion. As noted above, this line of research is of old vintage, dating back to Samuelson (1947), Patinkin (1948), Lloyd (1971), Berglas and Razin (1974), and Samuelson and Sato (1984). In none of these works, however, have the authors achieved the most general set of empirically verifiable and refutable comparative statics properties possible under the stated specifications and assumptions.

At the macro level, economists seem to have accepted the view that consumers do not derive utility directly from money, but rather from the consumption of goods that money can buy [Marschak (1950) and Clower (1963)]. Hence, according to these economists, money does not explicitly enter the direct utility function. At the micro level, however, there is a compelling reason for considering a specification of the consumer problem that includes real-cash balances in the utility function, scilicet the increasing tendency of empirical studies to refute the implications of the standard consumer model. Lloyd (1971) has discussed this issue rather thoroughly, and it seems economical to quote him at length:

> "… it is important to recognize that even if one can leave money out of the consumer's utility function with accuracy, one need not. If the consumer pattern of choices conforms to certain axioms, then his preference ordering may be represented by a utility function. These axioms assert certain properties of monotonicity, continuity, and transitivity. If the axioms apply to a consumer's preferences over a commodity space which does not include money, there seems to be little reason to doubt that they would also apply were

> the commodity space augmented to include the money commodity. Roughly speaking, if a consumer can choose between bundles which do not include amounts of the money commodity, then he can likely choose between bundles which do. Moreover, if he prefers more commodities to less, he will probably prefer more money to less. If his preferences were transitive without money, they will be likely to remain transitive with its introduction, etc. To include money in the utility function, we need not claim that the consumer gains any real satisfaction from it; only that he can make consistent choices over bundles that include quantities of it, an almost gallantly innocuous assumption.

The analytical conclusion of Lloyd's (1971) discussion is that the utility function $U[\;]$ of a consumer exhibits the property of weak separability between the set of real goods $\mathbf{X}$ and the real-cash balances M, thereby implying that $(M, \mathbf{X}) \mapsto U[M, g(\mathbf{X})]$. Lloyd (1971) attempted, without success, to derive the empirical implications of a consumer model based upon a utility function that includes real-cash balances. It remained for Berglas and Razin (1974) to show how to derive those implications under the assumptions of weak separability and a unitary interest rate on money. Their specification of the utility function is slightly different from that of Lloyd (1971), in that the real-cash balances are resolved in the nominal quantity of money M and a price index p, as in the definition $M \overset{\text{def}}{=} M/p$. The utility function of Berglas and Razin therefore takes the form $(M, \mathbf{X}, p) \mapsto U[M, g(\mathbf{X}), p]$. They derive comparative statics relations using a two-stage maximization procedure. Although correct, their derivation produces only sufficient conditions and lacks the elegance of a general approach, and as such, may explain the neglect of this paper by authors that analyzed the subject in subsequent years.

Among these authors, Samuelson and Sato (1984) presented an interesting analysis of money-goods models that can be considered heretofore the most complete discussion of the subject. They elaborated on two main specifications that admit refutable hypotheses. Both formulations hinge upon the assumption of weak separability of the utility function with respect to some subset of the arguments. They also obtain only a sufficient set of conditions for problem (1) below. Because the work of Samuelson and Sato (1984) is closely related to our work, it is sagacious at this juncture to pause briefly and take stock of their basic assumptions, notation, method of attack, and results.

II. Samuelson and Sato

The first formulation of Samuelson and Sato [1984, Eq.(23)] is

$$V(r, \mathbf{P}, Y) \stackrel{\text{def}}{=} \max_{M,\mathbf{X}} \left\{ U[M, g(\mathbf{X}); \mathbf{P}] \quad \text{s.t.} \quad rM + \mathbf{P}'\mathbf{X} = Y \right\}, \tag{1}$$

where $M > 0$ is the nominal money balance, $r > 0$ is the interest rate, $Y > 0$ is the consumer's income, $\mathbf{X} \stackrel{\text{def}}{=} (x_1, x_2, \quad, x_n) \quad {}^{n}_{+}$ is the vector of real goods, $\mathbf{P} \stackrel{\text{def}}{=} (p_1, p_2, \quad, p_n) \quad {}^{n}_{++}$ is the vector of prices of the real goods, and $'$ denotes transposition. Note that we are following Samuelson and Sato's (1984) notation quite closely, diverging only in minor ways for the purpose of clarity. In particular, we adopt the convention that (i) the derivative of a scalar-valued function with respect to a column vector is a row vector, (ii) a double subscript on a scalar-valued function represents the Hessian matrix of that function, with the number of rows equal to the number of elements in the first subscript and the number of columns equal to the number of elements in the second subscript, and (iii) all vectors of variables are column vectors. In model (1) the real goods **X** are assumed to be weakly separable with respect to both *M* and **P**. To eliminate money illusion, the utility function is further assumed to be homogeneous of degree zero in the nominal money balance *M* and the prices **P**.

Samuelson and Sato (1984) used a two-stage maximization process to obtain a modified Slutsky matrix involving the uncompensated demand functions for money and real goods, which we now briefly outline. The first-stage maximization problem is

$$\max_{\mathbf{X}} \left\{ g(\mathbf{X}) \quad \text{s.t.} \quad \mathbf{P}'\mathbf{X} = \hat{Y} \right\}, \tag{2}$$

where $\hat{Y}$ is an arbitrary allocation of income for the goods. The solution of problem (2) yields the *conditional* demand functions $\mathbf{H}[\;]$, with values $\mathbf{H}[\mathbf{P}, \hat{Y}]$. These demand functions obey all the prototypical properties of standard demand functions. The second-stage maximization problem is

$$\max_{M,\hat{Y}} \left\{ U\left[M, g\left(\mathbf{H}[\mathbf{P}, \hat{Y}]\right); \mathbf{P}\right] \quad \text{s.t.} \quad rM + \hat{Y} = Y \right\}. \tag{3}$$

The solution of problem (3) yields the *uncompensated* demand function for money, to wit $M(\,)$, with value $M(r,\mathbf{P},Y)$, and the optimal allocation of income for the purchase of the real goods, namely $\hat{Y} = Y - rM(r,\mathbf{P},Y)$. The *uncompensated* demand functions $\mathbf{X}(\,)$ for the real goods, with values $\mathbf{X}(r,\mathbf{P},Y)$, are the solution to problem (1) and the ultimate objects of interest, along with $M(r,\mathbf{P},Y)$. The former's values are related to the values of the conditional demand functions by the identity

$$\mathbf{X}(r,\mathbf{P},Y) \equiv \mathbf{H}[\mathbf{P}, Y - rM(r,\mathbf{P},Y)], \tag{4}$$

as shown by Samuelson and Sato [1984, Eq. (31a)]. Using identity (4) and the fact that the conditional demand functions $\mathbf{H}[\,]$ obey the archetype Slutsky properties, Samuelson and Sato [1984, Eq. (32b)] derived the following modified Slutsky matrix

$$\mathbf{S}^1 \stackrel{\text{def}}{=} \left[\frac{\partial \mathbf{X}}{\partial \mathbf{P}} + \frac{\partial \mathbf{X}}{\partial Y}\mathbf{X}'\right] - \frac{\left[\dfrac{\partial \mathbf{X}}{\partial r} + \dfrac{\partial \mathbf{X}}{\partial Y}M\right]\left[\dfrac{\partial M}{\partial \mathbf{P}} + \dfrac{\partial M}{\partial Y}\mathbf{X}'\right]}{\left[\dfrac{\partial M}{\partial r} + \dfrac{\partial M}{\partial Y}M\right]}, \tag{5}$$

and showed that, under their solution procedure, it is symmetric and negative semidefinite almost everywhere. Samuelson and Sato (1984, p. 595) also showed that the compensated slope of the money demand function is strictly negative almost everywhere.

The modified Slutsky matrix $\mathbf{S}^1$ is remarkable in at least two respects. First, all the terms of Eq. (5) are observable and thus estimable in principle. Hence the matrix $\mathbf{S}^1$ provides a fundamental scaffolding for testing the hypothesis that consumers maximize utility with respect to their choices of real goods and cash balances. Second, it clearly contains the Slutsky matrix of the traditional consumer model as a special case. The modified Slutsky matrix $\mathbf{S}^1$ therefore provides the basis for additional statistical power for the test of the standard model.

The second formulation of Samuelson and Sato [1984, Eq.(24)] is

$$\max_{M,\mathbf{X}} \left\{ U[M/p(\mathbf{P}),\mathbf{X}] \quad \text{s.t.} \quad rM + \mathbf{P}'\mathbf{X} = Y \right\}, \tag{6}$$

where $p(\mathbf{P})$ is a price index that satisfies $p(\lambda\mathbf{P}) \equiv \lambda p(\mathbf{P})$. Note that the utility function in this formulation is assumed to be homogeneous of degree zero in money M and the price index $p(\mathbf{P})$.

By defining real cash balances as $x_0 \stackrel{\text{def}}{=} M/p(\mathbf{P})$ and letting $p_0 \stackrel{\text{def}}{=} rp(\mathbf{P})$, Samuelson and Sato (1984, p. 592) showed that model (6) is formally equivalent to the standard specification of the consumer problem, to wit

$$\max_{x_0,\mathbf{X}}\left\{U[x_0,\mathbf{X}] \quad \text{s.t.} \quad p_0 x_0 + \mathbf{P}'\mathbf{X} = Y\right\}. \tag{7}$$

Let the value of the demand functions associated with problem (7) be $x_i = \hat{H}^i(p_0, p_1, \ldots, p_n, Y)$, $i = 0,1,\ldots,n$, to which there corresponds the symmetric and negative semidefinite Slutsky matrix

$$[\hat{H}^i_j + \hat{H}^j \hat{H}^i_{n+1}],\ i,j = 0,1,\ldots,n, \tag{8}$$

where the $(n+1)$-st parameter of the demand functions is the given level of income, and the subscripts on the function $\hat{H}^i(\,)$ indicate partial differentiation. Note that we use $\hat{H}^i(\,)$ for the demand functions of this model rather than $H^i(\,)$ so as to distinguish them from the conditional demand functions of model (1).

About problem (7) Samuelson and Sato (1984, p. 593) issued a "*Warning*: if $p(\mathbf{P})$ is not known to us in advance–and why would it be?–and why even be known to exist?–the (demand functions) observations do not seem to be sufficient to 'identify' the form or even the existence of the $p(\mathbf{P})$ function and the (8) tests cannot be performed!"

In the present paper, we generalize and extend Samuelson and Sato's (1984) results in two directions. First, by abandoning their two-stage maximization scheme and applying the primal-dual formalism of Silberberg (1974) to model (1) directly, we obtain an empirically observable generalized Slutsky matrix that exhibits a *more general structure* than that obtained by Samuelson and Sato (1984). That is, we produce the most general modified Slutsky matrix to date that is empirically verifiable and which contains the Samuelson and Sato (1984) modified Slutsky matrix as a special case. This is the central result of our paper.

Second, Samuelson and Sato's (1984) aforementioned "*Warning*" regarding model (7) is unwarranted. We will show that only the existence of the price index $p(\mathbf{P})$ is required, while its form need not be known. That is, we show that model (6) produces observable and verifiable comparative statics relations without explicit knowledge of the form of the price index $p(\mathbf{P})$.

In models (1) and (6) there are $n+1$ decision variables $(M,\mathbf{X})$, $n+2$ parameters $(r,\mathbf{P},Y)$, and one constraint, thereby implying that there are *n* degrees of freedom in the decision space and $n+1$ degrees of freedom in the parameter space. As a result the maximal rank of a comparative statics matrix in either model cannot exceed the smaller of these two numbers, scilicet *n*. We can therefore limit our search to $n \times n$ comparative statics matrices without loss of information.

Finally, we should note the assumptions upon which the results of Samuelson and Sato (1984), and consequently ours, rest. Briefly, the central assumptions are the existence of a $C^{(2)}$ direct utility function which is strictly increasing and strictly quasi-concave in $(M,\mathbf{X})$ for given $\mathbf{P}$. In addition, an interior solution to the utility maximization problem is assumed, a sufficient condition for which is the classical Inada-type condition.

III. Comparative Statics Without Two-Stage Maximization

In this section we prove that the solution of problem (1) without using a two-stage maximization approach generates comparative statics relations which encompass the modified Slutsky matrix $\mathbf{S}^1$ of Samuelson and Sato (1984). The resulting generalized Slutsky matrix constitutes a set of necessary and sufficient conditions for the consumer's problem (1). This is the central result of our paper. The discussion, therefore, is in the spirit of Samuelson and Sato's (1984, p. 603) open question to search for the most general specification of the money-goods model that yields empirically observable refutable implications on the money and real goods demand functions.

A convenient analytical framework for dealing *directly* with problem (1) is the primal-dual method of Silberberg (1974). We thus intend to solve the ensuing primal-dual problem:

$$\min_{M,\mathbf{X},r,\mathbf{P},Y} \left\{ V(r,\mathbf{P},Y) - U[M,g(\mathbf{X});\mathbf{P}] \quad \text{s.t.} \quad rM + \mathbf{P}'\mathbf{X} = Y \right\}. \tag{9}$$

Defining the Lagrangian for problem (9) as

$$L(M,\mathbf{X},r,\mathbf{P},Y) \stackrel{\text{def}}{=} V(r,\mathbf{P},Y) - U[M,g(\mathbf{X});\mathbf{P}] + \lambda[rM + \mathbf{P}'\mathbf{X} - Y],$$

the first-order necessary conditions for problem (9) include

$$L_M(M,\mathbf{X},r,\mathbf{P},Y) = -U_M[M,g(\mathbf{X});\mathbf{P}] + \lambda r = 0, \tag{10}$$

$$L_{\mathbf{X}}(M,\mathbf{X},r,\mathbf{P},Y) = -U_g[M,g(\mathbf{X});\mathbf{P}]g_{\mathbf{X}}(\mathbf{X}) + \lambda \mathbf{P}' = \mathbf{0}_n', \tag{11}$$

$$L_r(M, \mathbf{X}, r, \mathbf{P}, Y) = V_r(r, \mathbf{P}, Y) + \lambda M = 0\,, \tag{12}$$

$$L_{\mathbf{P}}(M, \mathbf{X}, r, \mathbf{P}, Y) = V_{\mathbf{P}}(r, \mathbf{P}, Y) - U_{\mathbf{P}}[M, g(\mathbf{X}); \mathbf{P}] + \lambda \mathbf{X}' = \mathbf{0}_n'\,, \tag{13}$$

$$L_Y(M, \mathbf{X}, r, \mathbf{P}, Y) = V_Y(r, \mathbf{P}, Y) - \lambda = 0\,, \tag{14}$$

and the budget constraint. By making use of Eqs. (10)–(14) and the second-order necessary conditions of problem (9), we prove the following theorem in the appendix. It is the central result of our paper.

Theorem 1. *For the money-goods model* (1) *without two-stage maximization, the generalized Slutsky matrix takes the form*

$$\mathbf{S} \stackrel{\text{def}}{=} \left[\frac{\partial \mathbf{X}}{\partial \mathbf{P}} + \frac{\partial \mathbf{X}}{\partial Y}\mathbf{X}'\right] - \frac{\left[\frac{\partial \mathbf{X}}{\partial r} + \frac{\partial \mathbf{X}}{\partial Y}M\right]\left[\frac{\partial M}{\partial \mathbf{P}} + \frac{\partial M}{\partial Y}\mathbf{X}'\right]}{\left[\frac{\partial M}{\partial r} + \frac{\partial M}{\partial Y}M\right]} + \frac{\left[\frac{\partial M}{\partial \mathbf{P}} + \frac{\partial M}{\partial Y}\mathbf{X}'\right]'\left[\frac{\partial M}{\partial \mathbf{P}} + \frac{\partial M}{\partial Y}\mathbf{X}'\right]}{\left[\frac{\partial M}{\partial r} + \frac{\partial M}{\partial Y}M\right]},$$

and is symmetric and negative semidefinite.

A comparison of the generalized Slutsky matrices $\mathbf{S}^1$ and $\mathbf{S}$ reveals that $\mathbf{S}$ equals $\mathbf{S}^1$ plus a third term. Hence the two matrices differ only by the third term of $\mathbf{S}$, which is comprised of a positive semidefinite matrix in the numerator and a strictly negative scalar denominator. Since both $\mathbf{S}^1$ and $\mathbf{S}$ are derived under the assumption of weak separability of the real goods with respect to money and prices, the two-stage maximization procedure implies a curvature condition that is sufficient for obtaining the negative semidefiniteness of $\mathbf{S}^1$. Conversely, in the absence of two-stage maximization, that is, by attacking problem (1) directly via the primal-dual formalism, the negative semidefiniteness of $\mathbf{S}$ requires an additional matrix, to wit its third term, which is negative semidefinite by construction. Consequently, Samuelson and Sato's (1984) modified Slutsky matrix $\mathbf{S}^1$ is a special case of that derived here. In other words, the negative semidefiniteness of the modified Slutsky matrix $\mathbf{S}^1$ is a sufficient, but not a necessary, condition for the negative semidefiniteness of the generalized Slutsky matrix $\mathbf{S}$.

How is it then that we are able to derive a more general Slutsky matrix than Samuelson and Sato (1984) while employing exactly their assumptions and model formulation? The answer lies in the *method of derivation* of the generalized Slutsky matrix. The two-stage maximization approach, in effect, focuses one's attention on the main diagonal block matrices $\mathbf{S}_{11}$ and $\mathbf{S}_{22}$ of the symmetric and negative semidefinite matrix **S** defined in the Appendix, since one deals with the decision variables M and $\mathbf{X}$ in two separate stages. Hence, by employing the two-stage maximization approach one is essentially unaware of the off diagonal (or interaction) block matrices $\mathbf{S}_{12}$ and $\mathbf{S}_{21}$, and thus the fact that they obey the symmetry property $\mathbf{S}'_{12} = \mathbf{S}_{21}$. Moreover, inspection of the proof of Theorem 1 in the Appendix reveals that the symmetry $\mathbf{S}'_{12} = \mathbf{S}_{21}$ is basal in establishing it. The primal-dual method, in other words, explicitly focuses one's attention on the entire symmetric and negative semidefinite matrix **S**, thereby permitting the exploitation of the crucial symmetry property $\mathbf{S}'_{12} = \mathbf{S}_{21}$.

The generalized Slutsky matrices $\mathbf{S}^1$ and $\mathbf{S}$ extend the empirical relevance of the archetypal consumer model. To see this, suppose that a sample of observations on consumer choices of real goods and money balances made at different price vectors, interest rates, and income levels is available. The hypothesis that such information is consistent with the standard utility maximization model can be verified by testing whether the prototype Slutsky matrix is symmetric and negative semidefinite. Many empirical studies that have appeared in the literature have refuted the implications of the standard model. Under these circumstances the usual conclusions are that either (i) the consumers did not behave as utility maximizers, or (ii) the quality of the data is insufficient to perform a reliable test, or (iii) the tests were conditioned on the functional forms used in the empirical analysis. In the absence of an alternative specification the possibility that the standard model is the cause of the rejection would be an inoperative conclusion. The analysis presented in this section introduces precisely the needed alternative specification. Thus, it is possible that a sample of consumers who are not utility maximizers according to the prototypical model may behave rationally according to model (1). The estimation of the observable uncompensated demand functions $M(r,\mathbf{P},Y)$ and $\mathbf{X}(r,\mathbf{P},Y)$, and the verification of whether $\mathbf{S}^1$ and $\mathbf{S}$ are symmetric and negative semidefinite matrices constitute two separate tests that the sample information is consistent with the maximization of utility functions which include cash balances and commodity prices explicitly.

IV. The Observability of Model (6)

Contrary to Samuelson and Sato's "*Warning*" (1984, p. 593), the comparative statics implications of model (6) are observable and, therefore, can form the basis for a test of consumer rationality under the money-goods specification. The resolution of this issue proceeds by deriving explicit expressions for the derivatives appearing in the Slutsky matrix in Eq. (8). For this case the relevant demand functions are

$$M(r,\mathbf{P},Y) \overset{\text{def}}{=} p(\mathbf{P})\hat{H}^0\big(rp(\mathbf{P}),\mathbf{P},Y\big), \tag{15}$$

$$x_i(r,\mathbf{P},Y) \overset{\text{def}}{=} \hat{H}^i\big(rp(\mathbf{P}),\mathbf{P},Y\big),\ i = 1,2,\ldots,n. \tag{16}$$

while the associated symmetric and negative semidefinite Slutsky matrix in Eq. (8) can be restated more explicitly as

$$\hat{\mathbf{S}} \overset{\text{def}}{=} \begin{bmatrix} \left[\hat{H}_0^0 + \hat{H}^0\hat{H}_{n+1}^0\right] & \left[\hat{H}_j^0 + \hat{H}^j\hat{H}_{n+1}^0\right] \\ \left[\hat{H}_0^i + \hat{H}^0\hat{H}_{n+1}^i\right] & \left[\hat{H}_j^i + \hat{H}^j\hat{H}_{n+1}^i\right] \end{bmatrix},\ i,j = 1,2,\ldots,n. \tag{17}$$

Notice that up to this point, we have followed Samuelson and Sato's (1984) development scrupulously. The further step taken here to establish observability of the matrix $\left[\hat{H}_j^i + \hat{H}^j\hat{H}_{n+1}^i\right]$, $i,j = 1,2,\ldots,n$, is the use of the symmetry inherent in the matrix $\hat{\mathbf{S}}$, a step not taken by Samuelson and Sato (1984).

To begin, differentiate Eq. (15) to get

$$\frac{\partial M}{\partial r} = p(\mathbf{P})^2\,\hat{H}_0^0\big(rp(\mathbf{P}),\mathbf{P},Y\big), \tag{18}$$

$$\frac{\partial M}{\partial p_j} = rp(\mathbf{P})\hat{H}_0^0\frac{\partial p(\mathbf{P})}{\partial p_j} + p(\mathbf{P})\hat{H}_j^0 + \hat{H}^0\frac{\partial p(\mathbf{P})}{\partial p_j},\ j = 1,2,\ldots,n \tag{19}$$

$$\frac{\partial M}{\partial Y} = p(\mathbf{P})\hat{H}_{n+1}^0\big(rp(\mathbf{P}),\mathbf{P},Y\big). \tag{20}$$

Similarly, differentiate Eq. (16) to get

$$\frac{\partial x_i}{\partial r} = p(\mathbf{P})\hat{H}_0^i\left(rp(\mathbf{P}),\mathbf{P},Y\right),\ i=1,2,\ldots,n, \tag{21}$$

$$\frac{\partial x_i}{\partial p_j} = r\hat{H}_0^i\left(rp(\mathbf{P}),\mathbf{P},Y\right)\frac{\partial p(\mathbf{P})}{\partial p_j} + \hat{H}_j^i\left(rp(\mathbf{P}),\mathbf{P},Y\right),\ i,j=1,2,\ldots,n, \tag{22}$$

$$\frac{\partial x_i}{\partial Y} = \hat{H}_{n+1}^i\left(rp(\mathbf{P}),\mathbf{P},Y\right),\ i=1,2,\ldots,n. \tag{23}$$

Using Eqs. (15), (16), and (18)–(23), we find that the elements of $\hat{\mathbf{S}}$ can be written as follows:

$$\hat{H}_0^0 + \hat{H}^0\hat{H}_{n+1}^0 = \frac{1}{p(\mathbf{P})^2}\left[\frac{\partial M}{\partial r} + \frac{\partial M}{\partial Y}M\right] \leq 0, \tag{24}$$

$$\hat{H}_j^0 + \hat{H}^j\hat{H}_{n+1}^0 = \frac{1}{p(\mathbf{P})}\left[\frac{\partial M}{\partial p_j} + \frac{\partial M}{\partial Y}x_j\right] - \frac{1}{p(\mathbf{P})^2}\frac{\partial p(\mathbf{P})}{\partial p_j}\left[M + r\frac{\partial M}{\partial r}\right],\ j=1,2,\ldots,n, \tag{25}$$

$$\hat{H}_0^i + \hat{H}^0\hat{H}_{n+1}^i = \frac{1}{p(\mathbf{P})}\left[\frac{\partial x_i}{\partial r} + \frac{\partial x_i}{\partial Y}M\right],\ i=1,2,\ldots,n, \tag{26}$$

$$\hat{H}_j^i + \hat{H}^j\hat{H}_{n+1}^i = \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial Y}x_j - r\frac{\partial x_i}{\partial r}\left[\frac{1}{p(\mathbf{P})}\frac{\partial p(\mathbf{P})}{\partial p_j}\right],\ i,j=1,2,\ldots,n. \tag{27}$$

By the symmetry of $\hat{\mathbf{S}}$, Eq. (25) is equal to Eq. (26) when $i=j$, a fact that allows elimination of the price index $p(\mathbf{P})$ and its derivatives:

$$\frac{1}{p(\mathbf{P})}\frac{\partial p(\mathbf{P})}{\partial p_j} = \frac{\left[\frac{\partial M}{\partial p_j} + \frac{\partial M}{\partial Y}x_j\right] - \left[\frac{\partial x_j}{\partial r} + \frac{\partial x_j}{\partial Y}M\right]}{\left[M + r\frac{\partial M}{\partial r}\right]},\ j=1,2,\ldots,n. \tag{28}$$

Substituting Eq. (28) into Eq. (27) and recognizing that the symmetry and negative semidefiniteness of $\hat{\mathbf{S}}$ implies the same for its submatrix $\tilde{\mathbf{S}} \stackrel{\text{def}}{=} \left[\hat{H}_j^i + \hat{H}^j\hat{H}_{n+1}^i\right]$, completes the proof of the result of this section, to wit

Proposition 1. *The comparative statics of problem* (6) *are summarized by the statement that the matrix* $\tilde{\mathbf{S}}$, *with typical element*

$$\tilde{S}_{ij} \stackrel{\text{def}}{=} \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial Y}x_j - r\frac{\partial x_i}{\partial r}\frac{\left[\frac{\partial M}{\partial p_j} + \frac{\partial M}{\partial Y}x_j\right] - \left[\frac{\partial x_j}{\partial r} + \frac{\partial x_j}{\partial Y}M\right]}{\left[M + r\frac{\partial M}{\partial r}\right]},\ i,j=1,2,\ldots,n,$$

is symmetric and negative semidefinite.

Proposition 1 demonstrates that the comparative statics tests of model (6) can be performed with the estimation of $M(r,\mathbf{P},Y)$ and $\mathbf{X}(r,\mathbf{P},Y)$, without the requirement of knowing explicitly the form of the price index $p(\mathbf{P})$. The empirical relevance of the discussion elaborated in this section is based upon the fact that it is not necessary to estimate the demand function for real balances $x_0(r,\mathbf{P},Y)$ but, rather, for nominal money balances $M(r,\mathbf{P},Y)$. The definition $x_0 \stackrel{\text{def}}{=} M/p(\mathbf{P})$ is posited only for the purpose of logical analysis and need not be estimated since its price slopes do not enter in Proposition 1. To achieve this result, we eliminated the unknown price index effects via a combination of observable price and income effects on the real goods and money demand functions using the symmetry of the full Slutsky matrix (17). This operation is exactly analogous to the estimation of the (directly unmeasurable) Hicksian substitution effects of the standard model by means of observable Marshallian price and income effects.

V. Implications

The major economic implication of both models is that, now, the traditional Slutsky matrix is neither symmetric nor negative semidefinite. Giffen-type commodities are admissible among real goods even in the case of positive income effects. Indeed, the above models liberate demand analysis from the shackles of normal goods, and the set of Giffen goods need not be confined to a few, improbable examples. Sloping upward demand curves are admissible even for normal goods because the burden of assuring a negative substitution term may fall upon a complex combination of all marginal responses of the demand functions for money and real goods.

The above discussion also invalidates the well known proposition that, if the uncompensated cross-price effects of the demand for real goods are equal, then the goods weighted income effects are also equal [see, e.g., Silberberg (1990, p. 343)]. No such conclusion is possible under the assumptions of the two models discussed in this paper.

The notion of commodity substitutes and complements must also be revised, accordingly. In the standard model, two commodities are said to be substitutes (complements) if the cross-price derivative of the compensated demand function is positive (negative). For normal goods, the positivity of the uncompensated cross-price derivative implies that the two commodities are substitutes. Conversely, for inferior goods, the negativity of the uncompensated cross-price derivative implies that the commodities are complements. In the two money-goods models discussed above, these conclusions are invalidated and no more shortcuts can be taken in the determination of whether two commodities are either substitutes or complements.

Another implication derives from model (1) when attacked directly with the primal-dual formalism. The derivation of the generalized Slutsky matrix $\mathbf{S}$ of Theorem 1 required expressing a combination of derivatives of the utility function in terms of the observable compensated price slopes of the demand functions for money and real goods, as in Eq. (43) of the Appendix. A close scrutiny of Eqs. (40) and (43) of the Appendix shows that the same generalized Slutsky matrix $\mathbf{S}$ could be obtained with the assumption $U_{\mathbf{PX}} \equiv \mathbf{0}_{n\times n}$. In other words, the assumption that goods $\mathbf{X}$ are weakly separable with respect to money balances M and the prices of the goods $\mathbf{P}$, generates the same set of observable refutable implications as the assumption that goods $\mathbf{X}$ are additively separable from their prices $\mathbf{P}$. The most general form of the utility function that satisfies $U_{\mathbf{PX}} \equiv \mathbf{0}_{n\times n}$ is

$$U(M,\mathbf{X};\mathbf{P}) = F(M,\mathbf{P}) + G(M,\mathbf{X}). \tag{29}$$

The additively separable utility function in Eq. (29) is more general than the weakly separable utility function $U[M,g(\mathbf{X});\mathbf{P}]$ in model (1) in the sense that goods $\mathbf{X}$ are not weakly separable from money balances M. On the other hand, the additively separable utility function in Eq. (29) is less general than the weakly separable utility function $U[M,g(\mathbf{X});\mathbf{P}]$ in model (1) in the sense that goods $\mathbf{X}$ are additively separable from the good prices $\mathbf{P}$. In order to clarify how two utility functions which are not monotone increasing transformations of one another can correspond to the same empirically verifiable comparative statics relations as expressed by the generalized Slutsky matrix $\mathbf{S}$ of Theorem 1, it is sufficient to note that $\mathbf{S}$ was obtained by exploiting the particular structure of the primal first-order necessary conditions (10) and (11) of the primal-dual problem (9), a step that is not required under the utility function in Eq. (29).

A final implication concerns the theory of revealed preference. It is well known that the weak axiom of revealed preference implies that the matrix of substitution effects of the standard model is negative semidefinite, and that the strong axiom is equivalent to the assertion of utility maximization in the prototype consumer model. The results of Samuelson and Sato (1984), however, as well as the generalization established here, show that the prototype Slutsky matrix no longer need be symmetric and negative semidefinite to define rational behavior. This result, in turn, means that the strong axiom of revealed preference also need no longer hold to define rational economic behavior. In other words, data which reject the symmetry and negative semidefiniteness of the archetype Slutsky matrix and, therefore, the strong axiom of revealed preference, may be consistent with the modified Slutsky matrix developed by Samuelson and Sato (1984) and generalized here. Thus there exists, in principle, a generalized version of revealed preference theory that is equivalent to the money-goods models discussed in this paper.

VI. Conclusions

We accepted Samuelson and Sato's (1984, p. 603) challenge to find the most general set of conditions which correspond to observable and empirically verifiable relations of the money-goods model. Such relations were obtained in Theorem 1, which encompass Samuelson and Sato's (1984) sufficient conditions. The observability of the generalized Slutsky matrix $\mathbf{S}$ hinges upon the structure of the matrix of cross-partial derivatives of the utility function $U_{\mathbf{PX}}$. By attacking problem (1) directly, we have shown that two alternative specifications of the utility function give rise to the generalized Slutsky matrix $\mathbf{S}$, videlicet $U[M, g(\mathbf{X}), \mathbf{P}]$ or $F(M,\mathbf{P}) + G(M,\mathbf{X})$. In the first case $U_{\mathbf{PX}} = U_{\mathbf{P}g} g_{\mathbf{X}}$, while in the second case $U_{\mathbf{PX}} = \mathbf{0}_{n\times n}$. A priori, neither of these utility functions can be judged to be more general than the other.

Finally, by using symmetry conditions, which were disregarded by Samuelson and Sato (1984), we were able to show that model (6) produces observable and verifiable relations without the necessity of knowing explicitly the form of the price index $p(\mathbf{P})$. This finding voids Samuelson and Sato's "*Warning*" (1984, p. 593) issued in relation to this model.

Appendix

Proof of Theorem 1. The second-order necessary conditions of problem (9) require that

$$\mathbf{u}'L_{\alpha\alpha}(M,\mathbf{X},r,\mathbf{P},Y)\mathbf{u} \geq 0 \quad \forall\, \mathbf{u} \in \mathbb{R}^{n+2} \ni h_{\alpha}(\alpha)\mathbf{u} = 0, \tag{30}$$

where $\alpha' \stackrel{\text{def}}{=} (r,\mathbf{P}',Y) \in \mathbb{R}^{n+2}_{++}$, $h(\alpha) \stackrel{\text{def}}{=} Y - rM - \mathbf{P}'\mathbf{X}$, and $h_{\alpha}(\alpha) = (-M,-\mathbf{X}',1) \in \mathbb{R}^{n+2}$ is the normal vector to the level curve of the constraint function in the $n+2$-dimensional parameter space. Since $h_{\alpha}(\alpha) \neq \mathbf{0}'_{n+2}$ the implicit function theorem implies that the level curve of the constraint function in the $n+2$-dimensional parameter space is of dimension $n+1$. Thus we seek $n+1$ vectors that form a basis for the tangent hyperplane to the level set of the constraint function in parameter space. It is relatively straightforward to verify that a suitable set of such basis vectors is given by $\mathbf{t}^1 \stackrel{\text{def}}{=} (1,\mathbf{0}_n,M)'$ and $\mathbf{t}^k \stackrel{\text{def}}{=} (0_1,0_2,\ldots,0_{k-1},1_k,0_{k+1},\ldots,0_{n+1},X^{k-1})'$, $k = 2,3,\ldots,n+1$. Define the $(n+2)\times(n+1)$ matrix $\mathbf{A}$ by placing the basis vector $\mathbf{t}^m$ in the mth column, $m = 1,2,\ldots,n+1$. We can then define the following matrix

$$\begin{aligned}
\mathbf{S} &\stackrel{\text{def}}{=} -\tfrac{1}{V_Y}\mathbf{A}'L_{\alpha\alpha}\mathbf{A} = -\frac{1}{V_Y}\begin{bmatrix} 1 & \mathbf{0}'_n & M \\ \mathbf{0}_n & \mathbf{I}_n & \mathbf{X} \end{bmatrix} \begin{bmatrix} V_{rr} & V_{r\mathbf{P}} & V_{rY} \\ V_{\mathbf{P}r} & \left[V_{\mathbf{PP}} - U_{\mathbf{PP}}\right] & V_{\mathbf{P}Y} \\ V_{Yr} & V_{Y\mathbf{P}} & V_{YY} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0}'_n \\ \mathbf{0}_n & \mathbf{I}_n \\ M & \mathbf{X}' \end{bmatrix} \\
&= -\frac{1}{V_Y}\begin{bmatrix} \left[V_{rr} + 2MV_{rY} + M^2V_{YY}\right] & \left[V_{r\mathbf{P}} + V_{rY}\mathbf{X}' + MV_{Y\mathbf{P}} + M\mathbf{X}'V_{YY}\right] \\ \left[V_{\mathbf{P}r} + MV_{\mathbf{P}Y} + \mathbf{X}V_{Yr} + M\mathbf{X}V_{YY}\right] & \left[V_{\mathbf{PP}} - U_{\mathbf{PP}} + V_{\mathbf{P}Y}\mathbf{X}' + \mathbf{X}V_{Y\mathbf{P}} + \mathbf{X}\mathbf{X}'V_{YY}\right] \end{bmatrix} \\
&\stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix}
\end{aligned}$$

where all the terms are evaluated at the solution to problem (1), namely $M(r,\mathbf{P},Y)$ and $\mathbf{X}(r,\mathbf{P},Y)$, and where $\mathbf{S}$ is a symmetric and negative semidefinite matrix in view of Eq. (30) and $V_Y(r,\mathbf{P},Y) > 0$.

Now solve Eqs. (12) and (14) to get $M = -V_r(r,\mathbf{P},Y)/V_Y(r,\mathbf{P},Y)$, and then differentiate:

$$\frac{\partial M}{\partial r} = -\frac{V_{rr}}{V_Y} - M\frac{V_{Yr}}{V_Y}, \tag{31}$$

$$\frac{\partial M}{\partial \mathbf{P}} = -\frac{V_{r\mathbf{P}}}{V_Y} - M\frac{V_{Y\mathbf{P}}}{V_Y}, \tag{32}$$

$$\frac{\partial M}{\partial Y} = -\frac{V_{rY}}{V_Y} - M\frac{V_{YY}}{V_Y}. \tag{33}$$

Next, compensate Eq. (31) with $\frac{\partial M}{\partial Y} M$ and (32) with $\frac{\partial M}{\partial Y} \mathbf{X}'$ to produce

$$\frac{\partial M}{\partial r} + \frac{\partial M}{\partial Y} M = -\frac{1}{V_Y}\left[V_{rr} + 2MV_{rY} + M^2 V_{YY}\right] \stackrel{\text{def}}{=} \mathbf{S}_{11} \leq 0\,, \tag{34}$$

$$\frac{\partial M}{\partial \mathbf{P}} + \frac{\partial M}{\partial Y} \mathbf{X}' = -\frac{1}{V_Y}\left[V_{r\mathbf{P}} + V_{rY}\mathbf{X}' + MV_{Y\mathbf{P}} + M\mathbf{X}'V_{YY}\right] \stackrel{\text{def}}{=} \mathbf{S}_{12}\,. \tag{35}$$

Similarly, solve Eqs. (13) and (14) to get

$$\mathbf{X} = \frac{-V_{\mathbf{P}}(r,\mathbf{P},Y)'}{V_Y(r,\mathbf{P},Y)} + \frac{U_{\mathbf{P}}\left[M(r,\mathbf{P},Y), g\big(\mathbf{X}(r,\mathbf{P},Y)\big), \mathbf{P}\right]'}{V_Y(r,\mathbf{P},Y)}\,,$$

and then differentiate:

$$\frac{\partial \mathbf{X}}{\partial r} = -\frac{V_{\mathbf{P}r}}{V_Y} - \mathbf{X}\frac{V_{Yr}}{V_Y} + \frac{U_{\mathbf{P}M}}{V_Y}\frac{\partial M}{\partial r} + \frac{U_{\mathbf{P}g}}{V_Y} g_{\mathbf{X}} \frac{\partial \mathbf{X}}{\partial r}\,, \tag{36}$$

$$\frac{\partial \mathbf{X}}{\partial \mathbf{P}} = -\frac{V_{\mathbf{PP}}}{V_Y} - \mathbf{X}\frac{V_{Y\mathbf{P}}}{V_Y} + \frac{U_{\mathbf{PP}}}{V_Y} + \frac{U_{\mathbf{P}M}}{V_Y}\frac{\partial M}{\partial \mathbf{P}} + \frac{U_{\mathbf{P}g}}{V_Y} g_{\mathbf{X}} \frac{\partial \mathbf{X}}{\partial \mathbf{P}}\,, \tag{37}$$

$$\frac{\partial \mathbf{X}}{\partial Y} = -\frac{V_{\mathbf{P}Y}}{V_Y} - \mathbf{X}\frac{V_{YY}}{V_Y} + \frac{U_{\mathbf{P}M}}{V_Y}\frac{\partial M}{\partial Y} + \frac{U_{\mathbf{P}g}}{V_Y} g_{\mathbf{X}} \frac{\partial \mathbf{X}}{\partial Y}\,. \tag{38}$$

Now compensate Eq. (36) with $\frac{\partial \mathbf{X}}{\partial Y} M$ and (37) with $\frac{\partial \mathbf{X}}{\partial Y} \mathbf{X}'$ to produce

$$\left[\frac{\partial \mathbf{X}}{\partial r} + \frac{\partial \mathbf{X}}{\partial Y} M\right] - \frac{U_{\mathbf{P}M}}{V_Y}\left[\frac{\partial M}{\partial r} + \frac{\partial M}{\partial Y} M\right] - \frac{U_{\mathbf{P}g}}{V_Y} g_{\mathbf{X}} \left[\frac{\partial \mathbf{X}}{\partial r} + \frac{\partial \mathbf{X}}{\partial Y} M\right] = \mathbf{S}_{21}\,, \tag{39}$$

$$\left[\frac{\partial \mathbf{X}}{\partial \mathbf{P}} + \frac{\partial \mathbf{X}}{\partial Y} \mathbf{X}'\right] - \frac{U_{\mathbf{P}M}}{V_Y}\left[\frac{\partial M}{\partial \mathbf{P}} + \frac{\partial M}{\partial Y} \mathbf{X}'\right] - \frac{U_{\mathbf{P}g}}{V_Y} g_{\mathbf{X}} \left[\frac{\partial \mathbf{X}}{\partial \mathbf{P}} + \frac{\partial \mathbf{X}}{\partial Y} \mathbf{X}'\right] = \mathbf{S}_{22}\,. \tag{40}$$

The next step in the proof is to apply the compensated derivatives $\frac{\partial}{\partial r} + \frac{\partial}{\partial Y} M$ and $\frac{\partial}{\partial \mathbf{P}} + \frac{\partial}{\partial Y} \mathbf{X}'$ to the budget constraint in identity form, scilicet $rM(r,\mathbf{P},Y) + \mathbf{P}'\mathbf{X}(r,\mathbf{P},Y) \equiv Y$, to get

$$\mathbf{P}'\left[\frac{\partial \mathbf{X}}{\partial r} + \frac{\partial \mathbf{X}}{\partial Y} M\right] = -r\left[\frac{\partial M}{\partial r} + \frac{\partial M}{\partial Y} M\right], \tag{41}$$

$$\mathbf{P}'\left[\frac{\partial \mathbf{X}}{\partial \mathbf{P}} + \frac{\partial \mathbf{X}}{\partial Y} \mathbf{X}'\right] = -r\left[\frac{\partial M}{\partial \mathbf{P}} + \frac{\partial M}{\partial Y} \mathbf{X}'\right]. \tag{42}$$

Note that the symmetry of $\mathbf{S}$ implies that $\mathbf{S}_{12}' = \mathbf{S}_{21}$, and that $g_{\mathbf{X}} = \frac{U_M}{U_g}\frac{\mathbf{P}'}{r}$ follows from the first-order necessary conditions (10) and (11). Use these results in Eqs. (35) and (39), and then use Eq. (41) to simplify the resulting expression to obtain

$$\left[\frac{U_M}{V_Y}\frac{U_{\mathbf{P}g}}{U_g}-\frac{U_{\mathbf{P}M}}{V_Y}\right]=\frac{\left[\frac{\partial M}{\partial \mathbf{P}}+\frac{\partial M}{\partial Y}\mathbf{X}'\right]'-\left[\frac{\partial \mathbf{X}}{\partial r}+\frac{\partial \mathbf{X}}{\partial Y}M\right]}{\left[\frac{\partial M}{\partial r}+\frac{\partial M}{\partial Y}M\right]}. \tag{43}$$

Now substitute $g_{\mathbf{X}}=\frac{U_M}{U_g}\frac{\mathbf{P}'}{r}$ into Eq. (40), then use Eq. (42) to rewrite the resulting expression, and finally substitute Eq. (43) into that result to get

$$\mathbf{S}_{22}=\left[\frac{\partial \mathbf{X}}{\partial \mathbf{P}}+\frac{\partial \mathbf{X}}{\partial Y}\mathbf{X}'\right]-\frac{\left[\frac{\partial \mathbf{X}}{\partial r}+\frac{\partial \mathbf{X}}{\partial Y}M\right]\left[\frac{\partial M}{\partial \mathbf{P}}+\frac{\partial M}{\partial Y}\mathbf{X}'\right]}{\left[\frac{\partial M}{\partial r}+\frac{\partial M}{\partial Y}M\right]}+\frac{\left[\frac{\partial M}{\partial \mathbf{P}}+\frac{\partial M}{\partial Y}\mathbf{X}'\right]'\left[\frac{\partial M}{\partial \mathbf{P}}+\frac{\partial M}{\partial Y}\mathbf{X}'\right]}{\left[\frac{\partial M}{\partial r}+\frac{\partial M}{\partial Y}M\right]}.$$

Defining $\mathbf{S}\overset{\text{def}}{=}\mathbf{S}_{22}$ completes the proof. *Q.E.D.*

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