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Robust Estimators of Errors-In-Variables Models Part I

by

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Robust Estimators of Errors-In-Variables Models Part I

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August 2004

Abstract

It is well known that consistent estimators of errors-in-variables models require knowledge of the ratio of error variances. What is not well known is that a Joint Least Squares estimator is robust to a wide misspecification of that ratio. Through a series of Monte Carlo experiments we show that an easy-to-implement estimator produces estimates that are nearly unbiased for a wide range of the ratio of error variances. These MC analyses encompass linear and nonlinear specifications and also a system on nonlinear equations where all the variables are measured with errors.

Keywords: Robust Estimators, Ratio of error variances, errors-in-variables, Joint least squares, concentrated joint least squares, Monte Carlo experiments.

JEL classification: C130, C200, C310.

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Robust Estimators of Errors-In-Variables Models – Part I

1. Introduction

The errors-in-variables (EIV) problem has a long history and a vast literature. It all began in 1878, when R. J. Adcock presented a one-page paper in *The Analyst* (published in Des Moines, Iowa) entitled “*A Problem in Least Squares.*” In that paper, he shows that, when a sample of bivariate information is measured with error, the regression line has an ambiguous meaning depending on which variable is taken as the dependent one. Since then, many distinguished statisticians have tackled the EIV problem and the conditions for consistent estimators are, by now, pretty well understood. Surprisingly, the same cannot be said for maximum likelihood estimators of the EIV model.

Briefly, one of the conditions for a consistent estimator of the EIV model is the knowledge of the ratio between the error variances. Since this knowledge cannot be ascertained from the sample information, consistent estimates of the EIV models are difficult to obtain.

Over the past forty years, the treatment of errors-in-variables models has received a progressively waning consideration in econometric textbooks: From the twenty eight pages of the 1963 edition of Johnson’s *Econometric Methods* to the two pages of the 2004 edition of Davidson and MacKinnon’s *Econometric Theory and Methods*. In recent years, the EIV problem has been revisited by several econometricians but, in general, there remains the impression that the EIV problem is rather intractable because no generally consistent estimator exists that is also easy to implement.

In this paper we explore an aspect of the EIV problem that seems to have gone unnoticed in the available literature. We argue that a weighted least-squares estimator is robust to misspecifications of the ratio of the error variances. In other words, even though that crucial piece of information is unavailable from the given sample information, a weighted least squares specification that uses misspecified values of the error variances' ratio produces estimates that are rather close to the true values of the parameters. In turn, and in linear models, the use of these parameter estimates produces estimates of the error variances that are close to their true values. All these assessments are deduced from a rather wide spectrum of Monte Carlo (MC) experiments that constitute the main body of this paper. We emphasize that the MC results presented in this paper constitute only a *prima facie* evidence of an empirical regularity that, if confirmed in other contexts, may renew the interest in dealing with EIV models, given the simplicity of the proposed estimators.

In section 2 we give a brief chronological review of the main papers on the EIV problem. In section 3 we discuss the simplest linear model with a single independent variable that is measured with error. In this section we will also review the condition for a consistent estimator. Extensive Monte Carlo evidence of the robustness of the proposed estimator is also presented. In section 4 we extend the linear EIV model to three independent variables and carry out the corresponding Monte Carlo analyses. In section 5 we tackle the nonlinear EIV problem by discussing two nonlinear specifications with two and three independent variables, respectively. By way of Monte Carlo analyses, it is shown again that the misspecified weighted least-squares estimator produces estimates that are close to their true values. In section 6 we deal with a nonlinear system of three

equations where all the variables are measured with error. We choose a linear expenditure system (LES) with three commodities as an example of a nonlinear system of equations that produces MC results remarkably close to the true values of the parameters.

In Part II of this study, the linear and nonlinear models analyzed in this Part I will be extended to include a substantially larger number of parameters.

2. Brief Review of the Literature

Adcock (1878) appears to be the first scientist to draw attention to the problem within the context of fitting a straight line. He suggested the minimization of the squared sum of orthogonal deviations of the observed points from the line. In a remarkable paper that appeared soon after, Kummell (1879) analyzed the nonlinear EIV problem within the context of a least-squares framework where the errors are weighted by the corresponding variables' precisions. He concluded, for the first time, that the solution of the EIV problem requires the assumption "that the precisions of the x and y which enter the same observation equation are considered to have a constant ratio." Surprisingly, Pearson (1901) followed the approach of Adcock by minimizing the orthogonal distance of the sample points from a line. Gini (1921), who may not have been aware of Kummell's work, recognizes the importance of the ratio of the error variance of the regressor to that of the dependent variable. He gave the quadratic formula for estimating the slope of a line that contains the error variance ratio.

By 1940, a distinction between functional and structural relations had been introduced in the literature. In the EIV problem, a functional relation specifies that the latent variables (associated with the regressor which is measured with error) be

considered fixed values. In a structural relation, these latent quantities are assumed to be random variables with a given probability distribution.

In a lengthy paper, Lindley (1947) gives a clear statement of the state of the art for that time about EIV problems. In particular, he discusses the non-existence of the likelihood function for a functional relation because the approach produces the implausible result that the square of the slope of the regression line must be equal to the error variances' ratio. The absence of a finite maximum will be made clearer by Solari (1969) who demonstrated that the likelihood function has a saddle point. Lindley then assumed that the latent variables are distributed normally together with the knowledge of the ratio between the error variances. These two assumptions produce consistent maximum likelihood estimates of the desired parameters except that the assumption of normality of the latent variables will soon be recognized to lead to the non-identification of the model.

The discussion about model identification under EIV began with a paper by Thomson (1916) who noticed that the joint distribution of the sample variates based only upon first and second moments cannot identify the parameters of a linear model. Neyman (1936) demonstrated that a necessary condition for identification is that the latent variables not be normally distributed because the joint distribution of two (or more) normal variates is itself normal and admits different values of the model's parameters. Hence, such a model is not identified. The discussion was concluded by Reiersol (1950) with the demonstration that non-normality is also a sufficient condition for identification. Neyman (1951) and Neyman and Scott (1948) presented significant results involving the

non-consistency of maximum likelihood estimators even when a consistent estimator exists.

The quest for consistent estimators of EIV models continued with a paper by Wald (1949) who presented conditions under which maximum likelihood estimates are consistent. These conditions, however, are difficult to verify. His results were re-elaborated by Kiefer and Wolfowitz (1956) in an important paper about mixed distributions that provided another set of conditions for consistent maximum likelihood estimators. Both papers are very theoretical and their existence theorems do not seem to provide operative indications easy to implement as to how consistent maximum likelihood estimates can be obtained.

The current literature on EIV models is vast and growing. The bottom line can be stated as follows. Consistent estimators seem to require one of the following conditions: either knowledge of the ratio of the error variances or the availability of replicate measurements of the latent variables. In general, both of these conditions are difficult to implement. For this reason, we assess the primal-dual estimator proposed in this paper by performing a Monte Carlo analysis using a sample of real data that we assume as the benchmark observations of the latent variables and compare the estimates obtained with the specification of the true ratio of the error variances with those obtained with misspecified ratios. The interesting conclusion is that the estimator is robust to misspecification of the true ratio of the error variances.

3. A Linear EIV Model With One Independent Variable

In this section we analyze the simplest linear model without intercept and with only one independent variable. This streamlined specification will allow a clear understanding of many issues pertaining to the EIV problem including the process of obtaining a consistent estimate of the model's parameter. The linear model is specified as follows

$$(1) \quad y_i = \beta x_i^* + u_i$$

$$(2) \quad x_i = x_i^* + \varepsilon_i$$

where X^* is considered a latent variable with unknown realizations x_i^* , $i = 1, \dots, N$ that are measured with error. These unknown entities can be considered either as fixed values or as draws from a given distribution. In the first case, model (1)-(2) is regarded as a *functional* specification while, in the second case, equations (1) and (2) are regarded as a *structural* model. The observed sample information is given by the N pairs (y_i, x_i) . The error terms are assumed to be iid random variables. A more definite specification, utilized also in the Monte Carlo analyses, will regard the error terms to be distributed as normal random variable as $u_i \sim N(0, \sigma_u)$, $\varepsilon_i \sim N(0, \sigma_\varepsilon)$, where σ_u and σ_ε are the standard deviations of the corresponding errors. The independence hypothesis implies that $\text{cov}(x_i^*, \varepsilon_i) = 0$, $\text{cov}(y_i, \varepsilon_i) = 0$, $\text{cov}(x_i^*, u_i) = 0$, $i = 1, \dots, N$. In order to satisfy the identification requirements stated by Riersol (1950), we assume that the latent variable is distributed according to the uniform distribution.

It is known that the likelihood function (based upon normal distributions of the measurement errors in a functional model) of the EIV model specified in equations (1) and (2) is not a feasible estimator because the likelihood function does not possess a finite maximum as a consequence of its saddle function structure (Solari, 1969). For the

structural EIV model, Kiefer and Wolfowitz (1956) demonstrated the consistency of the maximum likelihood estimator but the empirical implementation of their results has never appeared in the literature. The difficulty encountered in a feasible specification of the proper likelihood approach has induced a re-visitation of the least-squares estimator. It is, thus, known that a consistent and feasible estimator is the following weighted Joint Least Squares (JLS) estimator, where the two sums of squared residuals are weighted by the corresponding error variances

$$(3) \quad \min_{\beta, x_i^*} JLS = \sum_{i=1}^N (y_i - \beta x_i^*)^2 / \sigma_u^2 + \sum_{i=1}^N (x_i - x_i^*)^2 / \sigma_\varepsilon^2.$$

In order to implement the JLS consistent estimator given by equation (3), the ratio of the error variances must be known a priori. Hence, we posit the following parameter

$\lambda \stackrel{\text{def}}{=} \sigma_\varepsilon^2 / \sigma_u^2$ that transforms equation (3) into the following expression

$$(4) \quad \min_{\beta, x_i^*} JLS = \sum_{i=1}^N (y_i - \beta x_i^*)^2 + \sum_{i=1}^N (x_i - x_i^*)^2 / \lambda$$

where $\lambda > 0$. The first order necessary conditions of problem (4) are as follows

$$(5) \quad \frac{\partial JLS}{\partial \beta} = -2 \sum_{i=1}^N (y_i - \beta x_i^*) x_i^* = 0,$$

$$(6) \quad \frac{\partial JLS}{\partial x_i^*} = -2(y_i - \beta x_i^*)\beta - 2(x_i - x_i^*)/\lambda = 0.$$

The second order necessary conditions require that the following Hessian matrix be positive definite:

$$H = \begin{bmatrix} \sum_i (x_i^*)^2 & -(y_i - 2\beta x_i^*) & -(y_N - 2\beta x_N^*) \\ -(y_i - 2\beta x_i^*) & \beta^2 + 1/\lambda & \\ -(y_N - 2\beta x_N^*) & & \beta^2 + 1/\lambda \end{bmatrix}.$$

Notice that it is possible to rewrite equation (4) in concentrated form by exploiting the information of equation (6) which gives

$$(7) \quad x_i^* = \frac{\lambda\beta y_i + x_i}{1 + \lambda\beta^2}$$

and by replacing the latent variable x_i^* into equation (4) with the following result

$$(8) \quad \min_{\beta} CJLS = \frac{1}{(1 + \lambda\beta^2)} \sum_{i=1}^N (y_i - \beta x_i)^2$$

where *CJLS* stands for concentrated joint least squares. The solution of problems (4) and (8), for the same value of the λ parameter, gives identical estimates of the parameter β . The CJLS estimator does not have a closed form solution and, therefore, it requires access to an efficient mathematical programming package such as, for example, GAMS.

3.1 Consistent Estimates of the β parameter

We now show that, with knowledge of the true value of the λ ratio, equation (8) is a consistent estimator of parameter β . We begin by taking the derivative of equation (8), that is, the first order necessary condition of the minimization problem (8):

$$(9) \quad \frac{\partial CJLS}{\partial \beta} = -2 \frac{1}{(1 + \lambda\beta^2)} \sum_{i=1}^N (y_i - \beta x_i) x_i - 2 \frac{\lambda\beta}{(1 + \lambda\beta^2)} \sum_{i=1}^N (y_i - \beta x_i)^2 = 0$$

which can be simplified as the following quadratic equation in the β parameter:

$$(10) \quad -\sum_{i=1}^N y_i x_i + \beta \left(\sum_{i=1}^N x_i^2 - \lambda \sum_{i=1}^N y_i^2 \right) + \beta^2 \lambda \sum_{i=1}^N y_i x_i = 0$$

with solution

$$(11) \quad \tilde{\beta} = \frac{-(\sum_{i=1}^N x_i^2 - \lambda \sum_{i=1}^N y_i^2) \pm \sqrt{(\sum_{i=1}^N x_i^2 - \lambda \sum_{i=1}^N y_i^2)^2 + 4\lambda (\sum_{i=1}^N y_i x_i)^2}}{2\lambda \sum_{i=1}^N y_i x_i}.$$

Hence, the probability limit of the $\tilde{\beta}$ estimator can be stated as follows:

$$P \lim_{N \rightarrow \infty} \tilde{\beta} = \frac{-[\sum_i (x_i^* + \varepsilon_i)^2 - \lambda \sum_i (\beta x_i^* + u_i)^2] + \sqrt{[\sum_i (x_i^* + \varepsilon_i)^2 - \lambda \sum_i (\beta x_i^* + u_i)^2]^2 + 4\lambda [\sum_i (\beta x_i^* + u_i)(x_i^* + \varepsilon_i)]^2}}{2\lambda \sum_i (\beta x_i^* + u_i)(x_i^* + \varepsilon_i)}$$

$$P \lim_{N \rightarrow \infty} \tilde{\beta} = \frac{-[\sum_i x_i^{*2} + \sigma_\varepsilon^2 - \lambda \beta^2 \sum_i x_i^{*2} - \lambda \sigma_u^2] + \sqrt{[\sum_i x_i^{*2} + \sigma_\varepsilon^2 - \lambda \beta^2 \sum_i x_i^{*2} - \lambda \sigma_u^2]^2 + 4\lambda \beta^2 [\sum_i x_i^{*2}]^2}}{2\lambda \beta \sum_i x_i^{*2}}$$

$$P \lim_{N \rightarrow \infty} \tilde{\beta} = \frac{-[\sum_i x_i^{*2} - \frac{\sigma_\varepsilon^2}{\sigma_u^2} \beta^2 \sum_i x_i^{*2}] + \sqrt{[\sum_i x_i^{*2} - \frac{\sigma_\varepsilon^2}{\sigma_u^2} \beta^2 \sum_i x_i^{*2}]^2}}{2 \frac{\sigma_\varepsilon^2}{\sigma_u^2} \beta \sum_i x_i^{*2}}$$

$$P \lim_{N \rightarrow \infty} \tilde{\beta} = \frac{-\sum_i x_i^{*2} + \frac{\sigma_\varepsilon^2}{\sigma_u^2} \beta^2 \sum_i x_i^{*2} + [\sum_i x_i^{*2} + \frac{\sigma_\varepsilon^2}{\sigma_u^2} \beta^2 \sum_i x_i^{*2}]}{2 \frac{\sigma_\varepsilon^2}{\sigma_u^2} \beta \sum_i x_i^{*2}} = \frac{2 \frac{\sigma_\varepsilon^2}{\sigma_u^2} \beta^2 \sum_i x_i^{*2}}{2 \frac{\sigma_\varepsilon^2}{\sigma_u^2} \beta \sum_i x_i^{*2}}$$

$$(12) \quad P \lim_{N \rightarrow \infty} \tilde{\beta} = \beta$$

after using the fact that $\lambda \stackrel{\text{def}}{=} \sigma_\varepsilon^2 / \sigma_u^2$ and all the distributional assumptions stated above.

Knowledge of the true error variances' ratio is, thus, crucial for obtaining consistent estimates of the β parameter.

3.2 Monte Carlo Experiments for Model (1)-(2)

The empirical results of this section will show that the CJLS estimator, given by equation (8), produces estimates of the β parameter that are very close to the true value of β even though the ratio of the error variances is grossly misspecified. The definition of “very close” is admittedly subjective but it will be taken to be a discrepancy from the true value of the β parameter smaller than a few percentage points. We appeal to Monte Carlo experiments in order to gauge the performance of the CJLS estimator under a variety of error specifications.

In this initial set of Monte Carlo experiments the choice of parameters and distributions is reported in Table 1.

Table 1. Parameter and distribution choice for the MC experiments of model (1)-(2)

Set	β	x_i^*	σ_u	σ_ε	True λ	N
1	1.30	$U(2,20)$	2.0	3.0	2.25	200
2	1.30	$U(2,20)$	2.0	3.0	2.25	5000
3	1.30	$U(2,20)$	1.0	1.5	2.25	200
4	1.30	$U(2,20)$	1.0	1.5	2.25	5000
5	1.30	$U(2,20)$	1.0	1.0	1.0	200
6	1.30	$U(2,20)$	1.0	1.0	1.0	5000
7	1.30	$U(2,20)$	2.0	1.0	0.25	200
8	1.30	$U(2,20)$	2.0	1.0	0.25	5000
9	1.30	$U(2,20)$	1.5	1.0	0.4444	200
10	1.30	$U(2,20)$	2.0	1.0	0.25	200
11	1.30	$U(2,20)$	1.5	1.0	0.4444	200

The choice of two sample sizes is based on the desire to gauge the discrepancy from the consistency of the EIV estimator with a large sample size ($N = 5000$) and the conjecture that a more realistic sample size ($N = 200$), associated with 100 sample drawings, could reveal the extent of discrepancy from an unbiased estimate of the β parameter. The results of the Monte Carlo experiments are reported in Tables 2 and 3. In order to allow for a meaningful numerical estimation of the model even with very large sample sizes, all the (latent and error) variables were scaled by the squared root of N , the sample size.

The Monte Carlo computations were carried out for a range of the λ parameter that differs from the true value of the error variances' ratio up to a factor of five. In Table 2 the shaded rows indicate the average estimates (over 100 samples) corresponding to the true value of the error variances' ratio, under the first six sets of distributional

specifications as reported in Table 1. It can be seen that, for values of the error variances' ratio that differ considerably from the true value of the λ parameter, the average value of the estimates of the β parameter do not differ greatly.

Table 2. Average estimates (100 samples) of the β parameter for given λ , model (8)

Number	λ value	set 1	set 2	set 3	set 4	set 5	set 6
1	0.25	1.2446	1.2525	1.2874	1.2898	1.2969	1.2963
2	0.50	1.2636	1.2679	1.2924	1.2938	1.2995	1.2984
3	0.75	1.2758	1.2779	1.2955	1.2964	1.3013	1.2998
4	1.00	1.2843	1.2848	1.2977	1.2981	1.3025	1.3008
5	1.25	1.2906	1.2898	1.2992	1.2994	1.3035	1.3016
6	1.50	1.2953	1.2937	1.3004	1.3004	1.3042	1.3022
7	1.75	1.2991	1.2968	1.3014	1.3011	1.3048	1.3026
8	2.00	1.3021	1.2992	1.3021	1.3018	1.3052	1.3030
9	2.25	1.3047	1.3013	1.3028	1.2932	1.3056	1.3033
10	2.50	1.3068	1.3030	1.3033	1.3027	1.2998	1.2975
11	2.75	1.3085	1.3044	1.2909	1.3031	1.3063	1.3038
12	3.00	1.3101	1.3057	1.3041	1.3034	1.3065	1.3040
13	3.25	1.3114	1.2937	1.3045	1.3036	1.2938	1.3042
14	3.50	1.3126	1.3077	1.3048	1.3039	1.3069	1.3044
15	3.75	1.3136	1.3085	1.3050	1.3041	1.3071	1.3045
16	4.00	1.3146	1.3093	1.3052	1.3043	1.3072	1.3046
17	4.25	1.3154	1.3100	1.3054	1.3044	1.3074	1.3047
18	4.50	1.3162	1.3106	1.3056	1.3046	1.3075	1.3048
19	4.75	1.3168	1.3111	1.3058	1.3047	1.3076	1.3049
20	5.00	1.3175	1.3116	1.3060	1.3049	1.3077	1.3050

Table 3 reports the results of the Monte Carlo experiments when the true value of the λ parameter is smaller than one. The average estimates of the β parameter for the sets of distributional specifications 7, 8 and 9 of Table 1 are somewhat larger than the similar estimates of Table 2. It appears, therefore, that when the true value of the λ parameter is smaller than one, a more appropriate specification of the CJLS objective function may be a slightly modified CJLS of equation (8) that now reads

$$(13) \quad \min_{\beta} CJLS = \frac{\lambda}{(1 + \lambda\beta^2)} \sum_{i=1}^N (y_i - \beta x_i)^2.$$

Equation (13) corresponds to an original JLS objective function where the $\lambda \stackrel{\text{def}}{=} \sigma_{\varepsilon}^2 / \sigma_u^2$ parameter multiplies the first residual sum of squares, as indicated below

$$(14) \quad \min_{\beta, x_i^*} JLS = \lambda \sum_{i=1}^N (y_i - \beta x_i^*)^2 + \sum_{i=1}^N (x_i - x_i^*)^2.$$

The results of using equation (13) and the same distributional specifications of sets 7 and 9 of Table 1 are reported in Table 3 under sets 10 and 11, respectively. Hence, with this alternative specification of the CJLS objective function, the average estimate of the β parameter is closer to its true value for the entire spectrum of the λ parameter.

Table 3. Average estimates of the β parameter for given λ , models (8) and (13)

Number	λ value	set 7	set 8	set 9		set 10	set 11
1	0.05	1.0056	0.7653	0.8791		1.2938	1.2927
2	0.10	1.3113	1.3237	1.3161		1.2962	1.2944
3	0.15	1.3134	1.3254	1.3176		1.2983	1.2959
4	0.20	1.3153	1.3270	1.3189		1.3002	1.2972
5	0.25	1.3170	1.3283	1.3201		1.3019	1.2983
6	0.30	1.3185	1.3296	1.3211		1.3033	1.2993
7	0.35	1.3198	1.3306	1.3220		1.3046	1.3002
8	0.40	1.3210	1.3316	1.3229		1.3058	1.3010
9	0.45	1.3221	1.3325	1.3236		1.3069	1.3018
10	0.50	1.3231	1.3333	1.3243		1.3078	1.3024
11	0.55	1.3240	1.3340	1.3249		1.3087	1.3030
12	0.60	1.3248	1.3347	1.3255		1.3095	1.3036
13	0.65	1.3255	1.3353	1.3260		1.3103	1.3041
14	0.70	1.3262	1.3359	1.3265		1.3110	1.3046
15	0.75	1.3269	1.3364	1.3269		1.3116	1.3050
16	0.80	1.3275	1.3369	1.3273		1.3122	1.3054
17	0.85	1.3280	1.3374	1.3277		1.3127	1.3058
18	0.90	1.3285	1.3378	1.3281		1.3133	1.3062
19	0.95	1.3290	1.3382	1.3284		1.3137	1.3065
20	1.00	1.3295	1.3385	1.3287		1.3142	1.3068

3.3 Estimation of the Error Variances

Although the principal objective of this paper is the estimation of the β parameter, we now show how it is possible to obtain reasonable estimates of the error variances by using the estimate of the β parameter itself. In the structural interpretation of model (1)-(2), the latent variable X^* is assumed to have a uniform distribution (recall that for identification purposes its distribution cannot be normal). Under the assumptions of model (1)-(2), therefore, we have the following second moment relations

$$(15) \quad \text{var}(y) = \beta^2 \sigma_{X^*}^2 + \sigma_u^2$$

$$(16) \quad \text{var}(x) = \sigma_{X^*}^2 + \sigma_\varepsilon^2$$

$$(17) \quad \text{cov}(y, x) = \beta \sigma_{X^*}^2.$$

The terms on the left-hand-side of the equality sign can be approximated by the corresponding sample estimates of the variance and covariance quantities. This system of three equations has four unknowns $(\beta, \sigma_{X^*}^2, \sigma_u^2, \sigma_\varepsilon^2)$. However, given a “good” estimate of the β parameter, the other three unknowns $(\sigma_{X^*}^2, \sigma_u^2, \sigma_\varepsilon^2)$ can be determined univocally and with surprising precision. The results of applying this estimation procedure (using the estimates of the β parameter reported in Tables 2 and 3) are given in Tables 4, 5 and 6.

Given the chosen limits for the uniform distribution of the latent variable X^* , $U(a, b) = U(2, 20)$, the variance of X^* is equal to $\sigma_{X^*}^2 = (b - a)^2 / 12 = 27$. The estimated values of the variances of the error and latent variables are reasonably close to their true values especially for $N = 5000$.

Table 4. Estimates of the Error Variances for Model (1)-(2), Sets 1 and 2 of Table 1

Number	λ value	set 1 of table 1, N=200			set 2 of table 1, N=5000		
		$\sigma_u^2 = 4$	$\sigma_\varepsilon^2 = 9$	$\sigma_{X^*}^2 = 27$	$\sigma_u^2 = 4$	$\sigma_\varepsilon^2 = 9$	$\sigma_{X^*}^2 = 27$
1	0.25	6.0987	7.6677	25.4913	5.7068	7.9787	27.7935
2	0.50	5.4993	8.0487	25.1102	5.1695	8.3173	27.4549
3	0.75	5.1118	8.2891	24.8698	4.8238	8.5308	27.2413
4	1.00	4.8423	8.4536	24.7053	4.5837	8.6772	27.0950
5	1.25	4.6444	8.5730	24.5859	4.4076	8.7835	26.9886
6	1.50	4.4933	8.6635	24.4954	4.2731	8.8642	26.9080
7	1.75	4.3741	8.7344	24.4246	4.1671	8.9275	26.8447
8	2.00	4.2777	8.7914	24.3676	4.0814	8.9784	26.7938
9	2.25	4.1983	8.8382	24.3208	4.0107	9.0202	26.7520
10	2.50	4.1317	8.8773	24.2817	3.9515	9.0552	26.7170
11	2.75	4.0750	8.9105	24.2485	3.9010	9.0849	26.6873
12	3.00	4.0262	8.9389	24.22	3.8576	9.1104	26.6618
13	3.25	3.9837	8.9637	24.1953	3.8199	9.1326	26.6396
14	3.50	3.9465	8.9853	24.1736	3.7867	9.1520	26.6202
15	3.75	3.9135	9.0045	24.1545	3.7574	9.1691	26.6030
16	4.00	3.8841	9.0215	24.1375	3.7312	9.1844	26.5878
17	4.25	3.8578	9.0367	24.1222	3.7078	9.1981	26.5741
18	4.50	3.8341	9.0505	24.1085	3.6866	9.2104	26.5618
19	4.75	3.8125	9.0629	24.0961	3.6675	9.2216	26.5506
20	5.00	3.7930	9.0742	24.0848	3.6500	9.2317	26.5404

Table 5. Estimates of the Error Variances for Model (1)-(2), Sets 5 and 6 of Table 1

Number	λ value	set 5 of table 1, N=200			set 6 of table 1, N=5000		
		$\sigma_u^2 = 1$	$\sigma_\varepsilon^2 = 1$	$\sigma_{X^*}^2 = 27$	$\sigma_u^2 = 1$	$\sigma_\varepsilon^2 = 1$	$\sigma_{X^*}^2 = 27$
1	0.25	1.2428	0.8535	24.6163	1.1626	0.9091	26.8578
2	0.50	1.1533	0.9067	24.5632	1.0835	0.9562	26.8108
3	0.75	1.0971	0.9400	24.5299	1.0337	0.9857	26.7813
4	1.00	1.0585	0.9628	24.507	0.9996	1.0059	26.7610
5	1.25	1.0303	0.9794	24.4904	0.9747	1.0206	26.7463
6	1.50	1.0089	0.9920	24.4778	0.9557	1.0318	26.7351
7	1.75	0.9920	1.0020	24.4679	0.9408	1.0406	26.7263
8	2.00	0.9784	1.0100	24.4599	0.9288	1.0477	26.7192
9	2.25	0.9672	1.0166	24.4533	0.9189	1.0536	26.7134
10	2.50	0.9579	1.0221	24.4478	0.9105	1.0585	26.7085
11	2.75	0.9499	1.0268	24.4431	0.9035	1.0626	26.7043
12	3.00	0.9430	1.0308	24.4391	0.8974	1.0662	26.7007
13	3.25	0.9370	1.0343	24.4355	0.8921	1.0693	26.6976
14	3.50	0.9318	1.0374	24.4325	0.8875	1.0721	26.6949
15	3.75	0.9271	1.0401	24.4297	0.8834	1.0745	26.6925
16	4.00	0.9230	1.0425	24.4273	0.8797	1.0766	26.6903
17	4.25	0.9193	1.0447	24.4251	0.8764	1.0786	26.6884
18	4.50	0.9160	1.0467	24.4232	0.8734	1.0803	26.6866
19	4.75	0.9129	1.0484	24.4214	0.8708	1.0819	26.6851
20	5.00	0.9102	1.0501	24.4198	0.8683	1.0833	26.6836

Table 6. Estimates of the Error Variances for Model (1)-(2), Sets 7 and 8 of Table 1

Number	λ value	set 7 of table 1, N=200			set 8 of table 1, N=5000		
		$\sigma_u^2 = 4$	$\sigma_\varepsilon^2 = 1$	$\sigma_{X^*}^2 = 27$	$\sigma_u^2 = 4$	$\sigma_\varepsilon^2 = 1$	$\sigma_{X^*}^2 = 27$
1	0.05	4.30787	0.82181	24.648	4.2438	0.8729	26.8940
2	0.10	4.22969	0.86842	24.6014	4.1746	0.9141	26.8528
3	0.15	4.16181	0.90876	24.5611	4.1146	0.9498	26.8171
4	0.02	4.10233	0.94399	24.5259	4.0620	0.9810	26.7859
5	0.25	4.04982	0.97501	24.4948	4.0156	1.0085	26.7585
6	0.30	4.00313	1.00253	24.4673	3.9744	1.0328	26.7341
7	0.35	3.96135	1.0271	24.4428	3.9375	1.0546	26.7123
8	0.40	3.92376	1.04916	24.4207	3.9043	1.0742	26.6928
9	0.45	3.88976	1.06909	24.4008	3.8742	1.0918	26.6751
10	0.50	3.85887	1.08716	24.3827	3.8470	1.1078	26.6591
11	0.55	3.83068	1.10363	24.3662	3.8221	1.1224	26.6445
12	0.60	3.80485	1.11871	24.3511	3.7992	1.1358	26.6311
13	0.65	3.78111	1.13255	24.3373	3.7782	1.1481	26.6188
14	0.70	3.7592	1.1453	24.3245	3.7589	1.1594	26.6075
15	0.75	3.73893	1.15709	24.3128	3.7410	1.1699	26.5970
16	0.80	3.72012	1.16802	24.3018	3.7244	1.1796	26.5873
17	0.85	3.70261	1.17819	24.2917	3.7089	1.1886	26.5783
18	0.90	3.68629	1.18766	24.2822	3.6944	1.1971	26.5699
19	0.95	3.67102	1.19651	24.2733	3.6810	1.2049	26.5620
20	1.00	3.65672	1.20479	24.2651	3.6683	1.2123	26.5547

3.4 A Linear EIV Model With One Independent Variable and Intercept

We repeat the Monte Carlo analysis with the addition of an α intercept to the linear model (1)-(2) to read

$$(18) \quad y_i = \alpha + \beta x_i^* + u_i$$

$$(19) \quad x_i = x_i^* + \varepsilon_i.$$

The specifications for the various Monte Carlo experiments associated with model (18)-(19) are stated in Table 7.

Table 7. Parameter and distribution choice for the MC experiments of model (18)-(19)

Set	α	β	x_i^*	σ_u	σ_ε	True λ	N
12	1.0	0.80	$U(2,10)$	1.0	1.5	2.25	200
13	1.0	0.80	$U(2,10)$	1.0	1.5	2.25	5000
14	1.0	0.80	$U(2,20)$	1.0	1.5	2.25	200
15	1.0	0.80	$U(2,30)$	1.0	1.5	2.25	200

As the regression line goes through the centroid of model (18)-(19), that is $\bar{y} = \alpha + \beta\bar{x}$, where \bar{y} and \bar{x} are the sample averages of the corresponding variables and where $\bar{x} = \bar{x}^*$, it is convenient to transform model (18)-(19) in deviations from the mean

$$(20) \quad \tilde{y}_i = \beta\tilde{x}_i^* + u_i$$

$$(21) \quad \tilde{x}_i = \tilde{x}_i^* + \varepsilon_i$$

where $\tilde{y}_i = y_i - \bar{y}$ and $\tilde{x}_i = x_i - \bar{x}$. Then, model (20)-(21) can be estimated with the same CJLS estimator given by equation (8), using the additional step for $\tilde{\alpha} = \bar{y} - \tilde{\beta}\bar{x}$. The results of the Monte Carlo experiments associated with Table 7 are reported in Tables 8, 9, 10, and 11.

The information produced by this set of Monte Carlo experiments confirms the robustness of the CJLS estimator for Tables 10 and 11, while for Tables 8 and 9 the estimates exhibit a rate of departure from the true values that is higher than in Tables 2-6, 10 and 11. As the distribution of the latent variable widens, the CJLS estimator regains its original level of robustness. It appears, therefore, that in these MC experiments the introduction of an intercept perturbs the robustness of the CJLS estimator with one independent variable to an unexpected degree. In a subsequent linear model with three independent variables, the introduction of an intercept has less significant consequences.

The reason why the introduction of an intercept term disrupts the robustness of the CJLS estimator (for linear models) may have to do with the non identifiability of the intercept parameter: “If a parameter is not identifiable, no consistent (near consistent, in our case) estimate of the parameter will exist” (Reiersol, 1950). Notice, however, that in a translog model analyzed in section 5.2, the introduction of an intercept does not disrupts the robustness of the corresponding weighted least squares estimator.

Table 8. Average estimates of $\alpha, \beta, \sigma_u^2, \sigma_\varepsilon^2, \sigma_{X^*}^2$ parameters for given λ , model (20)-(21)

		set 12 of table 7, $X^* \sim U(2,10)$, $N = 200$				
Number	λ value	β slope	$\sigma_u^2 = 1$	$\sigma_\varepsilon^2 = 2.25$	$\sigma_{X^*}^2 = 5.33$	α intercept
1	0.25	0.5896	1.8290	0.4573	6.6257	2.1468
2	0.50	0.6265	1.6863	0.8431	6.2399	1.9468
3	0.75	0.6609	1.5530	1.1648	5.9182	1.7600
4	1.00	0.6925	1.4312	1.4312	5.6518	1.5891
5	1.25	0.7209	1.3214	1.6518	5.4312	1.4349
6	1.50	0.7337	1.2034	1.8051	5.1271	1.2553
7	1.75	0.7689	1.1360	1.9880	5.0950	1.1744
8	2.00	0.7890	1.0584	2.1168	4.9662	1.0653
9	2.25	0.8069	0.9894	2.2260	4.8570	0.9683
10	2.50	0.8229	0.9278	2.3195	4.7635	0.8818
11	2.75	0.8302	0.8649	2.3785	4.6248	0.7864
12	3.00	0.8499	0.8234	2.4703	4.6127	0.7351
13	3.25	0.8614	0.7790	2.5317	4.5513	0.6727
14	3.50	0.8719	0.7388	2.5858	4.4972	0.6162
15	3.75	0.8813	0.7024	2.6338	4.4492	0.5650
16	4.00	0.8899	0.6692	2.6767	4.4063	0.5183
17	4.25	0.8978	0.6388	2.7151	4.3679	0.4757
18	4.50	0.9050	0.6111	2.7498	4.3332	0.4366
19	4.75	0.9116	0.5855	2.7812	4.3018	0.4007
20	5.00	0.9177	0.5619	2.8097	4.2733	0.3676

Table 9. Average estimates of $\alpha, \beta, \sigma_u^2, \sigma_\varepsilon^2, \sigma_{X^*}^2$ parameters, given λ , model (20)-(21)

		set 13 of table 7, $X^* \sim U(2,10), N=5000$				
Number	λ value	β parameter	$\sigma_u^2 = 1$	$\sigma_\varepsilon^2 = 2.25$	$\sigma_{X^*}^2 = 5.33$	α intercept
1	0.25	0.5989	1.8649	0.4662	7.0800	2.2082
2	0.50	0.6341	1.7163	0.8582	6.6881	1.9985
3	0.75	0.6666	1.5789	1.1841	6.3621	1.8044
4	1.00	0.6961	1.4539	1.4539	6.0923	1.6281
5	1.25	0.7226	1.3418	1.6773	5.8690	1.4698
6	1.50	0.7462	1.2419	1.8629	5.6834	1.3288
7	1.75	0.7672	1.1532	2.0182	5.5281	1.2036
8	2.00	0.7858	1.0745	2.1490	5.3972	1.0925
9	2.25	0.8023	1.0046	2.2602	5.2860	0.9938
10	2.50	0.8170	0.9422	2.3555	5.1907	0.9058
11	2.75	0.8302	0.8865	2.4379	5.1084	0.8271
12	3.00	0.8420	0.8365	2.5095	5.0367	0.7566
13	3.25	0.8527	0.7915	2.5724	4.9739	0.6930
14	3.50	0.8623	0.7508	2.6278	4.9184	0.6356
15	3.75	0.8710	0.7139	2.6771	4.8692	0.5835
16	4.00	0.8789	0.6803	2.7211	4.8252	0.5360
17	4.25	0.8862	0.6496	2.7606	4.7856	0.4927
18	4.50	0.8929	0.6214	2.7963	4.7500	0.4529
19	4.75	0.8811	0.5838	2.7732	4.6236	0.4078
20	5.00	0.9046	0.5716	2.8581	4.6882	0.3827

Table 10. Average estimates of $\alpha, \beta, \sigma_u^2, \sigma_\varepsilon^2, \sigma_{X^*}^2$ parameters, given λ , model (20)-(21)

set 14 of table 7, $X^* \sim U(2, 20)$, $N = 200$						
Number	λ value	β slope	$\sigma_u^2 = 1$	$\sigma_\varepsilon^2 = 2.25$	$\sigma_{X^*}^2 = 27$	α intercept
1	0.25	0.7486	2.0202	0.5051	26.3029	1.4959
2	0.50	0.7600	1.7963	0.8981	25.9098	1.3854
3	0.75	0.7693	1.6136	1.2102	25.5977	1.2953
4	1.00	0.7770	1.4626	1.4626	25.3453	1.2208
5	1.25	0.7835	1.3362	1.6703	25.1377	1.1584
6	1.50	0.7889	1.2291	1.8436	24.9643	1.1055
7	1.75	0.7936	1.1373	1.9902	24.8177	1.0603
8	2.00	0.7976	1.0579	2.1157	24.6922	1.0211
9	2.25	0.8011	0.9886	2.2243	24.5837	0.9869
10	2.50	0.8042	0.9276	2.3190	24.4890	0.9568
11	2.75	0.8070	0.8736	2.4023	24.4057	0.9301
12	3.00	0.8094	0.8254	2.4761	24.3318	0.9064
13	3.25	0.8116	0.7821	2.5420	24.2660	0.8850
14	3.50	0.8136	0.7432	2.6011	24.2069	0.8658
15	3.75	0.8154	0.7078	2.6544	24.1536	0.8484
16	4.00	0.8171	0.6757	2.7027	24.1053	0.8325
17	4.25	0.8186	0.6463	2.7467	24.0613	0.8180
18	4.50	0.8199	0.6193	2.7869	24.0210	0.8047
19	4.75	0.8212	0.5945	2.8238	23.9841	0.7924
20	5.00	0.8224	0.5716	2.8578	23.9501	0.7811

Table 11. Average estimates of $\alpha, \beta, \sigma_u^2, \sigma_\varepsilon^2, \sigma_{X^*}^2$ parameters, given λ , model (20)-(21)

set 15 of table 7, $X^* \sim U(2,30)$, $N = 200$						
Number	λ value	β slope	$\sigma_u^2 = 1$	$\sigma_\varepsilon^2 = 2.25$	$\sigma_{X^*}^2 = 65.33$	α intercept
1	0.25	0.7781	2.0547	0.5137	61.2148	1.3040
2	0.50	0.7831	1.8147	0.9073	60.8211	1.2335
3	0.75	0.7872	1.6233	1.2175	60.5110	1.1774
4	1.00	0.7904	1.4676	1.4676	60.2608	1.1317
5	1.25	0.7931	1.3387	1.6733	60.0551	1.0938
6	1.50	0.7954	1.2302	1.8453	59.8832	1.0620
7	1.75	0.7974	1.1377	1.9910	59.7374	1.0348
8	2.00	0.7990	1.0580	2.1161	59.6124	1.0115
9	2.25	0.8005	0.9887	2.2245	59.5039	0.9911
10	2.50	0.8018	0.9278	2.3194	59.4090	0.9732
11	2.75	0.8029	0.8739	2.4032	59.3253	0.9574
12	3.00	0.8039	0.8259	2.4776	59.2509	0.9433
13	3.25	0.8048	0.7828	2.5441	59.1843	0.9307
14	3.50	0.8056	0.7440	2.6040	59.1245	0.9193
15	3.75	0.8064	0.7088	2.6581	59.0703	0.9090
16	4.00	0.8070	0.6768	2.7073	59.0211	0.8996
17	4.25	0.8077	0.6476	2.7522	58.9762	0.8910
18	4.50	0.8082	0.6207	2.7934	58.9351	0.8831
19	4.75	0.8087	0.5960	2.8312	58.8973	0.8758
20	5.00	0.8092	0.5732	2.8661	58.8624	0.8691

An indirect way to assess the quality of the CJLS estimates of model (20)-(21) is to consider the ordinary least squares estimates resulting from the use of the same sample information. The results are reported in Table 12 and they indicate that the OLS estimates of the intercept and of the slope are considerably more biased than the corresponding estimates of the CJLS estimator under a wide spectrum of the λ parameter's values.

Table 12. OLS estimates using the sample information of sets 12 and 15 of Table 7

Set	$\alpha=1.0$	$\beta=0.80$	x_i^*	σ_u	σ_ε	N
12	2.3517	0.55033	$U(2,10)$	1.0	1.5	200
15	1.3946	0.77161	$U(2,30)$	1.0	1.5	200

4. A Linear EIV Model With Three Independent Variables

Next, we analyze a linear model with three independent variables exhibiting the following structure

$$(22) \quad y_i = \beta_1 x_{1i}^* + \beta_2 x_{2i}^* + \beta_3 x_{3i}^* + u_i$$

$$(23) \quad x_{ji} = x_{ji}^* + \varepsilon_{ji}$$

$j=1, \dots, 3$. As in section 3, we assume that the error terms are iid variables. More specifically, $u_i \sim N(0, \sigma_u)$, $\varepsilon_{ji} \sim N(0, \sigma_{j\varepsilon})$, where σ_u and $\sigma_{j\varepsilon}$ are the standard deviations of the corresponding errors, $j=1, \dots, 3$. The independence hypothesis implies that $\text{cov}(x_{ji}^*, \varepsilon_{ji}) = 0$, $\text{cov}(y_i, \varepsilon_{ji}) = 0$, $\text{cov}(x_{ji}^*, u_i) = 0$, $i = 1, \dots, N$, $j = 1, \dots, 3$. Furthermore, the latent variables X_j^* are assumed to be identically, independently, and uniformly distributed. Initially, model (22)-(23) has no intercept. An intercept will be introduced later in this section.

Under these specifications, the CJLS objective function assumes the following structure:

$$(24) \quad \min_{\beta_1, \beta_2, \beta_3} CJLS = \frac{1}{(1 + \sum_j \lambda_j \beta_j^2)} \sum_{i=1}^N (y_i - \beta_1 x_{1i} - \beta_2 x_{2i} - \beta_3 x_{3i})^2$$

where $\lambda_j \stackrel{\text{def}}{=} \sigma_{\varepsilon_j}^2 / \sigma_u^2$, $j = 1, 2, 3$.

The second moment relationships among the variables of model (22)-(23) can be stated explicitly as

$$(25) \quad \text{var}(y) = \beta_1^2 \sigma_{x_1^*}^2 + \beta_2^2 \sigma_{x_2^*}^2 + \beta_3^2 \sigma_{x_3^*}^2 + 2\beta_1 \beta_2 \sigma_{x_1^* x_2^*} + 2\beta_1 \beta_3 \sigma_{x_1^* x_3^*} + 2\beta_2 \beta_3 \sigma_{x_2^* x_3^*} + \sigma_u^2$$

$$(26) \quad \text{var}(x_1) = \sigma_{x_1^*}^2 + \sigma_{\varepsilon_1}^2$$

$$(27) \quad \text{var}(x_1) = \sigma_{x_1^*}^2 + \sigma_{\varepsilon_1}^2$$

$$(28) \quad \text{var}(x_3) = \sigma_{X_3^*}^2 + \sigma_{\varepsilon_3}^2$$

$$(29) \quad \text{cov}(x_1, x_1) = \sigma_{X_1^* X_2^*} + \sigma_{\varepsilon_1 \varepsilon_2}$$

$$(30) \quad \text{cov}(x_1, x_3) = \sigma_{X_1^* X_3^*} + \sigma_{\varepsilon_1 \varepsilon_3}$$

$$(31) \quad \text{cov}(x_2, x_3) = \sigma_{X_2^* X_3^*} + \sigma_{\varepsilon_2 \varepsilon_3}$$

$$(32) \quad \text{cov}(y, x_1) = \beta_1 \sigma_{X_1^*}^2 + \beta_2 \sigma_{X_1^* X_2^*} + \beta_3 \sigma_{X_1^* X_3^*} + \sigma_{u \varepsilon_1}$$

$$(33) \quad \text{cov}(y, x_2) = \beta_1 \sigma_{X_1^* X_2^*} + \beta_2 \sigma_{X_2^*}^2 + \beta_3 \sigma_{X_2^* X_3^*} + \sigma_{u \varepsilon_2}$$

$$(34) \quad \text{cov}(y, x_3) = \beta_1 \sigma_{X_1^* X_3^*} + \beta_2 \sigma_{X_2^* X_3^*} + \beta_3 \sigma_{X_3^*}^2 + \sigma_{u \varepsilon_3}.$$

Using the initial assumptions of this model, which imply $\sigma_{\varepsilon_1 \varepsilon_2} = \sigma_{\varepsilon_1 \varepsilon_3} = \sigma_{\varepsilon_2 \varepsilon_3} = 0$ and $\sigma_{u \varepsilon_2} = \sigma_{u \varepsilon_3} = 0$, the system of ten equations (25)-(34) can be solved for the remaining ten unknown variances and covariances, given good estimates of the three β parameters. This procedure is an extension of a similar procedure developed in section 3.

Table 13 gives the specification of the Monte Carlo experiments associated with model (22)-(23). The choice of the β parameter values intends to represent positive and negative coefficients with magnitude greater and smaller than unity. The choice of the variances' level intends to create an unfavorable environment for the CJLS estimator using a wider distribution of the measurement errors of the independent variables relative to that of the dependent variable. We chose the same variance for the errors of the independent variables in order to simplify slightly the reporting of results.

Table 13. Parameter specification for the Monte Carlo experiments of model (22)-(23)

Set	β_1	β_2	β_3	x_i^*	σ_u	$\sigma_{\varepsilon_j}, j = 1, 2, 3$	True λ	N
16	1.30	-1.0	0.70	$U(2, 20)$	2.0	3.0	2.25	200
17	1.30	-1.0	0.70	$U(2, 20)$	2.0	3.0	2.25	5000
18	1.30	-1.0	0.70	$U(10, 20)$	2.0	3.0	2.25	200
19	1.30	-1.0	0.70	$U(2, 30)$	2.0	3.0	2.25	200
20	1.30	-1.0	0.70	$U(2, 20)$	1.0	1.0	1.00	200
21	1.30	-1.0	0.70	$U(2, 20)$	2.0	1.0	0.25	200
22	Same as set 16 with an intercept							

The results of this section are presented in Table 14 through Table 20. This Monte Carlo evidence confirms that the CJLS estimator is robust with respect to the λ parameter also in the case of a linear model with three independent variables. The average estimates (over 100 samples) of each of the three β parameters are rather close to their true values for a wide spectrum of the misspecified ratio of the error variances. Although the main objective of this paper was a robust estimation of the β parameters, we carried out also an exploratory study of estimating the error variances. Using the estimates of the β parameters, the variances of the various measurement errors were computed with less accurate results: it is worth to point out, however, that the values of the estimated variances for the measurement errors of the independent variables are within a relatively small range of the true values; the estimated variance values of the measurement error of the dependent variable are much less accurate. This is undoubtedly due to the complexity of relation (25) where the estimate of the error variance σ_u^2 depends on several other parameter estimates.

The introduction of an intercept into model (22)-(23) produces the results reported in Table 20. They do not differ greatly from those already commented above.

Table 14. Results of Set 16, $N = 200$, of Table 13.

Number	λ ratio	$\beta_1 = 1.30$	$\beta_2 = -1.0$	$\beta_3 = 0.70$	$\sigma_u^2 = 4$	$\sigma_{\varepsilon_1}^2 = 9$	$\sigma_{\varepsilon_2}^2 = 9$	$\sigma_{\varepsilon_3}^2 = 9$
1	0.25	1.1629	-0.8241	0.6519	15.1664	5.5609	2.2553	6.8551
2	0.50	1.2139	-0.8899	0.6719	11.0466	6.8522	4.9187	7.6488
3	0.75	1.2434	-0.9283	0.6835	8.6552	7.5528	6.2953	8.0867
4	1.00	1.2623	-0.9529	0.6909	7.1182	7.9857	7.1229	8.3598
5	1.25	1.2755	-0.9700	0.6960	6.0536	8.2779	7.6719	8.5454
6	1.50	1.2850	-0.9825	0.6997	5.2749	8.4878	8.0616	8.6793
7	1.75	1.2923	-0.9921	0.7026	4.6814	8.6456	8.3522	8.7803
8	2.00	1.2981	-0.9996	0.7048	4.2144	8.7686	8.5770	8.8591
9	2.25	1.3027	-1.0057	0.7066	3.8376	8.8670	8.7559	8.9224
10	2.50	1.3065	-1.0107	0.7081	3.5273	8.9475	8.9017	8.9742
11	2.75	1.3097	-1.0148	0.7094	3.2673	9.0146	9.0228	9.0174
12	3.00	1.3125	-1.0184	0.7104	3.0464	9.0713	9.1249	9.0540
13	3.25	1.3148	-1.0215	0.7113	2.8563	9.1199	9.2122	9.0855
14	3.50	1.3168	-1.0241	0.7121	2.6912	9.1621	9.2877	9.1127
15	3.75	1.3186	-1.0265	0.7128	2.5462	9.1989	9.3535	9.1365
16	4.00	1.3202	-1.0285	0.7134	2.4181	9.2314	9.4116	9.1575
17	4.25	1.3216	-1.0304	0.7140	2.3040	9.2603	9.4630	9.1762
18	4.50	1.3228	-1.0320	0.7145	2.2017	9.2861	9.5090	9.1930
19	4.75	1.3240	-1.0335	0.7149	2.1095	9.3093	9.5503	9.2080
20	5.00	1.3250	-1.0348	0.7153	2.0260	9.3303	9.5876	9.2217

Table 15. Results of Set 17, $N = 5000$, of Table 13.

Number	λ ratio	$\beta_1 = 1.30$	$\beta_2 = -1.0$	$\beta_3 = 0.70$	$\sigma_u^2 = 4$	$\sigma_{\varepsilon_1}^2 = 9$	$\sigma_{\varepsilon_2}^2 = 9$	$\sigma_{\varepsilon_3}^2 = 9$
1	0.25	1.1531	-0.8059	0.6394	15.5244	5.5291	2.4048	6.5731
2	0.50	1.2065	-0.8758	0.6611	11.3641	6.8727	5.0982	7.4955
3	0.75	1.2378	-0.9168	0.6737	8.9258	7.6048	6.4898	8.0067
4	1.00	1.2579	-0.9434	0.6818	7.3520	8.0575	7.3249	8.3259
5	1.25	1.2719	-0.9618	0.6875	6.2596	8.3630	7.8780	8.5428
6	1.50	1.2821	-0.9754	0.6916	5.4595	8.5825	8.2701	8.6993
7	1.75	1.2899	-0.9857	0.6948	4.8493	8.7475	8.5620	8.8173
8	2.00	1.2961	-0.9938	0.6972	4.3690	8.8759	8.7877	8.9094
9	2.25	1.3010	-1.0003	0.6992	3.9813	8.9787	8.9672	8.9833
10	2.50	1.3051	-1.0057	0.7009	3.6618	9.0627	9.1134	9.0438
11	2.75	1.3085	-1.0103	0.7022	3.3942	9.1328	9.2346	9.0943
12	3.00	1.3114	-1.0141	0.7034	3.1667	9.1920	9.3369	9.1370
13	3.25	1.3139	-1.0174	0.7044	2.9710	9.2427	9.4243	9.1737
14	3.50	1.3161	-1.0203	0.7053	2.8009	9.2866	9.4997	9.2054
15	3.75	1.3180	-1.0228	0.7061	2.6516	9.3251	9.5656	9.2332
16	4.00	1.3197	-1.0251	0.7068	2.5196	9.3590	9.6236	9.2578
17	4.25	1.3212	-1.0271	0.7074	2.4021	9.3891	9.6751	9.2796
18	4.50	1.3225	-1.0288	0.7079	2.2967	9.4160	9.7210	9.2991
19	4.75	1.3238	-1.0305	0.7084	2.2017	9.4402	9.7623	9.3166
20	5.00	1.3249	-1.0319	0.7088	2.1157	9.4621	9.7996	9.3325

Table 16. Results of Set 18, $N = 200$, of Table 13.

Number	λ ratio	$\beta_1 = 1.30$	$\beta_2 = -1.0$	$\beta_3 = 0.70$	$\sigma_u^2 = 4$	$\sigma_{\varepsilon_1}^2 = 9$	$\sigma_{\varepsilon_2}^2 = 9$	$\sigma_{\varepsilon_3}^2 = 9$
1	0.25	0.9460	-0.5211	0.5705	13.0238	5.6452	0.1453	7.0822
2	0.50	1.0524	-0.6662	0.6122	10.3087	6.8080	4.0103	7.6969
3	0.75	1.1283	-0.7702	0.6420	8.3687	7.5060	5.8807	8.0891
4	1.00	1.1819	-0.8439	0.6631	6.9964	7.9459	6.9254	8.3457
5	1.25	1.2208	-0.8975	0.6784	5.9996	8.2412	7.5766	8.5218
6	1.50	1.2500	-0.9378	0.6899	5.2515	8.4507	8.0163	8.6486
7	1.75	1.2725	-0.9690	0.6988	4.6727	8.6061	8.3312	8.7435
8	2.00	1.2904	-0.9937	0.7059	4.2131	8.7255	8.5672	8.8170
9	2.25	1.3049	-1.0138	0.7117	3.8401	8.8200	8.7502	8.8755
10	2.50	1.3169	-1.0304	0.7165	3.5316	8.8965	8.8961	8.9229
11	2.75	1.3269	-1.0444	0.7205	3.2724	8.9597	9.0151	8.9622
12	3.00	1.3355	-1.0563	0.7239	3.0518	9.0127	9.1139	8.9953
13	3.25	1.3429	-1.0665	0.7269	2.8617	9.0578	9.1972	9.0235
14	3.05	1.3493	-1.0755	0.7294	2.6964	9.0966	9.2684	9.0477
15	3.75	1.3549	-1.0833	0.7317	2.5512	9.1304	9.3299	9.0689
16	4.00	1.3599	-1.0902	0.7337	2.4228	9.1600	9.3835	9.0874
17	4.25	1.3643	-1.0964	0.7355	2.3084	9.1862	9.4308	9.1038
18	4.50	1.3683	-1.1019	0.7370	2.2058	9.2095	9.4727	9.1185
19	4.75	1.3719	-1.1069	0.7385	2.1134	9.2305	9.5101	9.1316
20	5.00	1.3752	-1.1114	0.7398	2.0296	9.2493	9.5437	9.1434

Table 17. Results of Set 19, $N = 200$, of Table 13.

Number	λ ratio	$\beta_1 = 1.30$	$\beta_2 = -1.0$	$\beta_3 = 0.70$	$\sigma_u^2 = 4$	$\sigma_{\varepsilon_1}^2 = 9$	$\sigma_{\varepsilon_2}^2 = 9$	$\sigma_{\varepsilon_3}^2 = 9$
1	0.25	1.2408	-0.9242	0.6801	15.7796	5.5998	2.6470	6.8493
2	0.50	1.2642	-0.9543	0.6892	11.2178	6.8925	5.0874	7.6721
3	0.75	1.2770	-0.9708	0.6941	8.7086	7.5831	6.3669	8.1149
4	1.00	1.2850	-0.9812	0.6973	7.1312	8.0102	7.1498	8.3898
5	1.25	1.2906	-0.9883	0.6994	6.0503	8.2998	7.6769	8.5766
6	1.50	1.2946	-0.9935	0.7010	5.2643	8.5087	8.0556	8.7117
7	1.75	1.2976	-0.9975	0.7021	4.6673	8.6666	8.3406	8.8139
8	2.00	1.3000	-1.0006	0.7031	4.1987	8.7900	8.5628	8.8938
9	2.25	1.3019	-1.0031	0.7038	3.8210	8.8891	8.7409	8.9581
10	2.50	1.3035	-1.0051	0.7044	3.5103	8.9704	8.8868	9.0109
11	2.75	1.3048	-1.0068	0.7049	3.2501	9.0383	9.0085	9.0550
12	3.00	1.3060	-1.0083	0.7054	3.0292	9.0959	9.1116	9.0924
13	3.25	1.3069	-1.0096	0.7058	2.8391	9.1453	9.1999	9.1245
14	3.50	1.3078	-1.0106	0.7061	2.6740	9.1882	9.2766	9.1524
15	3.75	1.3085	-1.0116	0.7064	2.5292	9.2258	9.3436	9.1768
16	4.00	1.3092	-1.0125	0.7066	2.4011	9.2590	9.4029	9.1984
17	4.25	1.3097	-1.0132	0.7068	2.2871	9.2886	9.4555	9.2176
18	4.50	1.3103	-1.0139	0.7071	2.1849	9.3150	9.5026	9.2348
19	4.75	1.3107	-1.0145	0.7072	2.0928	9.3388	9.5450	9.2503
20	5.00	1.3112	-1.0150	0.7074	2.0094	9.3604	9.5834	9.2644

Table 18. Results of Set 20, $N = 200$, of Table 13.

Number	λ ratio	$\beta_1 = 1.30$	$\beta_2 = -1.0$	$\beta_3 = 0.70$	$\sigma_u^2 = 1$	$\sigma_{\varepsilon_1}^2 = 1$	$\sigma_{\varepsilon_2}^2 = 1$	$\sigma_{\varepsilon_3}^2 = 1$
1	0.25	1.2869	-0.9831	0.6966	2.0986	0.6985	0.3958	0.8294
2	0.50	1.2944	-0.9930	0.6995	1.4820	0.8626	0.6950	0.9355
3	0.75	1.2985	-0.9983	0.7010	1.1522	0.9495	0.8526	0.9918
4	1.00	1.3010	-1.0015	0.7020	0.9473	1.0033	0.9497	1.0267
5	1.25	1.3027	-1.0038	0.7027	0.8077	1.0398	1.0155	1.0504
6	1.50	1.3040	-1.0054	0.7032	0.7065	1.0662	1.0631	1.0676
7	1.75	1.3049	-1.0066	0.7035	0.6298	1.0861	1.0990	1.0805
8	2.00	1.3056	-1.0075	0.7038	0.5697	1.1018	1.1271	1.0907
9	2.25	1.3062	-1.0084	0.7040	0.5213	1.1143	1.1497	1.0989
10	2.50	1.3067	-1.0090	0.7042	0.4815	1.1247	1.1682	1.1056
11	2.75	1.3071	-1.0095	0.7044	0.4482	1.1333	1.1837	1.1112
12	3.00	1.3075	-1.0100	0.7045	0.4199	1.1406	1.1969	1.1160
13	3.25	1.3078	-1.0104	0.7046	0.3956	1.1469	1.2081	1.1201
14	3.50	1.3080	-1.0107	0.7047	0.3745	1.1524	1.2180	1.1237
15	3.75	1.3083	-1.0110	0.7048	0.3560	1.1572	1.2265	1.1268
16	4.00	1.3085	-1.0113	0.7049	0.3396	1.1615	1.2341	1.1295
17	4.25	1.3086	-1.0115	0.7050	0.3250	1.1652	1.2409	1.1320
18	4.50	1.3088	-1.0117	0.7050	0.3119	1.1686	1.2470	1.1342
19	4.75	1.3090	-1.0119	0.7051	0.3002	1.1716	1.2524	1.1362
20	5.00	1.3091	-1.0121	0.7051	0.2895	1.1744	1.2573	1.1380

Table 19. Results of Set 21, $N = 200$, of Table 13.

Number	λ ratio	$\beta_1 = 1.30$	$\beta_2 = -1.0$	$\beta_3 = 0.70$	$\sigma_u^2 = 4$	$\sigma_{\varepsilon_1}^2 = 1$	$\sigma_{\varepsilon_2}^2 = 1$	$\sigma_{\varepsilon_3}^2 = 1$
1	0.05	0.2778	-0.9720	0.6941	5.7234	0.5397	-0.0154	0.7754
2	0.10	0.2856	-0.9821	0.6971	5.0921	0.7096	0.2983	0.8847
3	0.15	0.2917	-0.9902	0.6995	4.5889	0.8435	0.5439	0.9711
4	0.20	0.2968	-0.9968	0.7015	4.1794	0.9516	0.7409	1.0410
5	0.25	0.3009	-1.0022	0.7031	3.8402	1.0404	0.9021	1.0986
6	0.30	0.3044	-1.0068	0.7044	3.5551	1.1147	1.0364	1.1468
7	0.35	0.3074	-1.0107	0.7056	3.3121	1.1777	1.1499	1.1877
8	0.40	0.3100	-1.0140	0.7066	3.1028	1.2317	1.2469	1.2228
9	0.45	0.3122	-1.0169	0.7074	2.9207	1.2785	1.3309	1.2533
10	0.50	0.3142	-1.0195	0.7082	2.7608	1.3195	1.4042	1.2800
11	0.55	0.3159	-1.0218	0.7089	2.6194	1.3556	1.4688	1.3036
12	0.60	0.3174	-1.0238	0.7094	2.4934	1.3878	1.5260	1.3245
13	0.65	0.3188	-1.0256	0.7100	2.3805	1.4165	1.5772	1.3433
14	0.70	0.3201	-1.0272	0.7105	2.2787	1.4423	1.6231	1.3601
15	0.75	0.3212	-1.0287	0.7109	2.1866	1.4657	1.6646	1.3754
16	0.80	0.3222	-1.0301	0.7113	2.1027	1.4869	1.7023	1.3893
17	0.85	0.3232	-1.0313	0.7117	2.0261	1.5062	1.7366	1.4019
18	0.90	0.3240	-1.0324	0.7120	1.9558	1.5240	1.7680	1.4135
19	0.95	0.3248	-1.0335	0.7123	1.8910	1.5403	1.7969	1.4242
20	1.00	0.3256	-1.0344	0.7126	1.8313	1.5553	1.8235	1.4340

Table 20. Results of Set 22, $N = 200$, of Table 13, with intercept.

Number	λ ratio	$\alpha_1 = 1.0$	$\beta_1 = 1.30$	$\beta_2 = -1.0$	$\beta_3 = 0.70$	$\sigma_u^2 = 4$	$\sigma_{\varepsilon_1}^2 = 9$	$\sigma_{\varepsilon_2}^2 = 9$	$\sigma_{\varepsilon_3}^2 = 9$
1	0.25	2.930	1.108	-0.875	0.590	16.556	4.139	4.139	4.139
2	0.50	2.271	1.177	-0.924	0.631	11.968	5.984	5.984	5.984
3	0.75	1.878	1.218	-0.952	0.655	9.289	6.967	6.967	6.967
4	1.00	1.623	1.245	-0.970	0.671	7.565	7.565	7.565	7.565
5	1.25	1.445	1.263	-0.982	0.682	6.373	7.966	7.966	7.966
6	1.50	1.314	1.276	-0.991	0.690	5.501	8.251	8.251	8.251
7	1.75	1.214	1.286	-0.998	0.696	4.837	8.465	8.465	8.465
8	2.00	1.135	1.294	-1.003	0.700	4.315	8.630	8.630	8.630
9	2.25	1.071	1.301	-1.008	0.704	3.894	8.762	8.762	8.762
10	2.50	1.018	1.306	-1.011	0.708	3.548	8.870	8.870	8.870
11	2.75	0.974	1.310	-1.014	0.710	3.258	8.960	8.960	8.960
12	3.00	0.937	1.314	-1.017	0.713	3.012	9.035	9.035	9.035
13	3.25	0.904	1.317	-1.019	0.714	2.800	9.100	9.100	9.100
14	3.50	0.876	1.320	-1.021	0.716	2.616	9.156	9.156	9.156
15	3.75	0.852	1.323	-1.022	0.718	2.455	9.205	9.205	9.205
16	4.00	0.830	1.325	-1.024	0.719	2.312	9.248	9.248	9.248
17	4.25	0.810	1.327	-1.025	0.720	2.185	9.286	9.286	9.286
18	4.50	0.793	1.329	-1.026	0.721	2.071	9.320	9.320	9.320
19	4.75	0.777	1.330	-1.027	0.722	1.969	9.351	9.351	9.351
20	5.00	0.763	1.332	-1.028	0.723	1.876	9.379	9.379	9.379

5. Nonlinear EIV Models

When independent variables are measured with error in a nonlinear model, no easy concentration of the joint least-squares objective function is, in general, available. Furthermore, the set of nonlinear functions that may be admissible for representing regression models is large.

5.1 A Cobb-Douglas Model

For this initial analysis, therefore, we choose a Cobb-Douglas functional form. Hence the nonlinear regression model with measurement errors has the following structure

$$(35) \quad y_i = A \prod_{j=1}^3 (x_{ji}^*)^{\beta_j} + u_i$$

$$(36) \quad x_{ji} = x_{ji}^* + \varepsilon_{ji}$$

where $j = 1, \dots, 3$. As in section 4, we assume that the error terms are iid variables. More specifically, $u_i \sim N(0, \sigma_u)$, $\varepsilon_{ji} \sim N(0, \sigma_{j\varepsilon})$, where σ_u and $\sigma_{j\varepsilon}$ are the standard deviations of the corresponding errors, $j = 1, \dots, 3$. The independence hypothesis implies that $\text{cov}(x_{ji}^*, \varepsilon_{ji}) = 0$, $\text{cov}(y_i, \varepsilon_{ji}) = 0$, $\text{cov}(x_{ji}^*, u_i) = 0$, $i = 1, \dots, N$, $j = 1, \dots, 3$. Furthermore, the latent variables X_j^* are assumed to be identically, independently, and uniformly distributed.

The nonlinear joint least-squares (NJLS) EIV estimator utilized in this section for estimating model (35)-(36) has the following structure:

$$(37) \quad NJLS = \min_{A, \beta_j, x_{ji}^*, u_i, \varepsilon_{ji}} \left\{ \sum_{i=1}^N [y_i - A \prod_{j=1}^3 (x_{ji}^*)^{\beta_j}]^2 + \sum_{j=1}^3 \sum_{i=1}^N (x_{ji} - x_{ji}^*)^2 / \lambda_j \right\}$$

where $\lambda_j > 0$, $j = 1, 2, 3$.

As before, the objective is to assess the performance of the NJLS EIV estimator by a sequence of Monte Carlo experiments. In this nonlinear case, the second moment relationships for estimating the error variances are not easily obtained. Hence, the empirical results will concentrate on the estimation of the A and β parameters of model (35)-(36). The choice of parameter values and error variances is given in Table 21. All the draws were scaled by a factor equal to \sqrt{N} .

Table 21. Specification for the Monte Carlo experiments of the C-D model (35)-(36)

Set	A	β_1	β_2	β_3	x_i^*	σ_u	$\sigma_{\varepsilon_j} j=1,2,3$	True λ	N
23	2.0	0.3	0.5	0.70	$U(10,20)$	2.0	3.0	2.25	200
24	2.0	0.3	0.5	0.70	$U(10,20)$	2.0	3.0	2.25	5000
25	2.0	0.3	0.5	0.70	$U(10,30)$	2.0	3.0	2.25	200
26	2.0	0.3	0.5	0.70	$U(10,20)$	1.0	2.0	4.00	200
27	2.0	0.3	0.5	0.70	$U(10,20)$	1.0	1.0	1.00	200
28	2.0	0.3	0.5	0.70	$U(10,20)$	2.0	1.0	0.25	200

The results in all the Tables (22)-(27) show a remarkable precision of the estimates (including the A parameter) for almost all the values of the error variances' ratio. An alternative MC run for specification set 26 with a scale factor equal to 10 (not reported here) gave equally accurate results.

Table 22. Results of Set 23, $N = 200$, of Table 21.

Number	λ ratio	$A = 2.0$	$\beta_1 = 0.3$	$\beta_2 = 0.5$	$\beta_3 = 0.7$
1	0.25	2.0033	0.1652	0.2911	0.3848
2	0.50	2.0737	0.2113	0.3642	0.4884
3	0.75	2.0654	0.2385	0.4043	0.5477
4	1.00	2.0600	0.2575	0.4312	0.5884
5	1.25	2.0562	0.2711	0.4503	0.6175
6	1.50	2.0535	0.2814	0.4644	0.6391
7	1.75	2.0515	0.2893	0.4752	0.6558
8	2.00	2.0499	0.2955	0.4837	0.6690
9	2.25	2.0486	0.3006	0.4906	0.6797
10	2.50	2.0476	0.3048	0.4963	0.6886
11	2.75	2.0467	0.3084	0.5011	0.6960
12	3.00	2.0460	0.3114	0.5051	0.7023
13	3.25	2.0453	0.3140	0.5086	0.7077
14	3.50	2.0448	0.3163	0.5117	0.7125
15	3.75	2.0443	0.3183	0.5143	0.7166
16	4.00	2.0439	0.3201	0.5167	0.7203
17	4.25	2.0435	0.3216	0.5188	0.7235
18	4.50	2.0432	0.3231	0.5206	0.7265
19	4.75	2.0429	0.3243	0.5222	0.7291
20	5.00	2.0426	0.3255	0.5238	0.7315

Table 23. Results of Set 24, $N = 5000$, of Table 21.

Number	λ ratio	$A = 2.0$	$\beta_1 = 0.3$	$\beta_2 = 0.5$	$\beta_3 = 0.7$
1	0.25	2.5240	0.1859	0.3186	0.4503
2	0.50	2.3440	0.2249	0.3810	0.5375
3	0.75	2.2453	0.2481	0.4174	0.5880
4	1.00	2.1860	0.2628	0.4400	0.6195
5	1.25	2.1471	0.2727	0.4552	0.6407
6	1.50	2.1197	0.2799	0.4661	0.6559
7	1.75	2.0993	0.2852	0.4743	0.6672
8	2.00	2.0837	0.2894	0.4806	0.6760
9	2.25	2.0714	0.2928	0.4857	0.6831
10	2.50	2.0613	0.2955	0.4898	0.6888
11	2.75	2.0530	0.2978	0.4933	0.6936
12	3.00	2.0460	0.2997	0.4962	0.6976
13	3.25	2.0400	0.3014	0.4987	0.7010
14	3.50	2.0349	0.3028	0.5008	0.7040
15	3.75	2.0304	0.3041	0.5027	0.7066
16	4.00	2.0264	0.3052	0.5043	0.7089
17	4.25	2.0229	0.3062	0.5058	0.7110
18	4.50	2.0198	0.3070	0.5071	0.7128
19	4.75	2.0170	0.3078	0.5083	0.7144
20	5.00	2.0145	0.3085	0.5094	0.7159

Table 24. Results of Set 25, $N = 200$, of Table 21.

Number	λ ratio	$A = 2.0$	$\beta_1 = 0.3$	$\beta_2 = 0.5$	$\beta_3 = 0.7$
1	0.25	2.1381	0.2614	0.4461	0.6167
2	0.50	2.0967	0.2745	0.4647	0.6476
3	0.75	2.0767	0.2811	0.4738	0.6630
4	1.00	2.0649	0.2850	0.4792	0.6723
5	1.25	2.0572	0.2876	0.4828	0.6784
6	1.50	2.0517	0.2895	0.4853	0.6827
7	1.75	2.0476	0.2909	0.4872	0.6860
8	2.00	2.0445	0.2920	0.4887	0.6885
9	2.25	2.0419	0.2928	0.4899	0.6906
10	2.50	2.0399	0.2936	0.4908	0.6922
11	2.75	2.0382	0.2942	0.4916	0.6936
12	3.00	2.0367	0.2947	0.4923	0.6947
13	3.25	2.0355	0.2951	0.4929	0.6957
14	3.50	2.0344	0.2954	0.4934	0.6966
15	3.75	2.0335	0.2958	0.4938	0.6973
16	4.00	2.0327	0.2961	0.4942	0.6980
17	4.25	2.0320	0.2963	0.4946	0.6986
18	4.50	2.0313	0.2965	0.4949	0.6991
19	4.75	2.0307	0.2967	0.4951	0.6996
20	5.00	2.0302	0.2969	0.4954	0.7000

Table 25. Results of Set 26, $N = 200$, of Table 21.

Number	λ ratio	$A = 2.0$	$\beta_1 = 0.3$	$\beta_2 = 0.5$	$\beta_3 = 0.7$
1	1.25	2.0260	0.2780	0.4658	0.6468
2	1.50	2.0246	0.2822	0.4717	0.6564
3	1.75	2.0235	0.2854	0.4763	0.6637
4	2.00	2.0226	0.2879	0.4799	0.6695
5	2.25	2.0219	0.2899	0.4827	0.6741
6	2.50	2.0213	0.2916	0.4851	0.6779
7	2.75	2.0209	0.2930	0.4871	0.6811
8	3.00	2.0205	0.2942	0.4888	0.6839
9	3.25	2.0201	0.2953	0.4902	0.6862
10	3.50	2.0198	0.2962	0.4915	0.6883
11	3.75	2.0195	0.2970	0.4926	0.6900
12	4.00	2.0193	0.2977	0.4936	0.6916
13	4.25	2.0191	0.2983	0.4945	0.6930
14	4.50	2.0189	0.2988	0.4953	0.6943
15	4.75	2.0188	0.2994	0.4960	0.6954
16	5.00	2.0186	0.2998	0.4966	0.6965
17	5.25	2.0185	0.3002	0.4972	0.6974
18	5.50	2.0183	0.3006	0.4977	0.6983
19	5.75	2.0182	0.3010	0.4982	0.6991
20	6.00	2.0181	0.3013	0.4986	0.6998

Table 26. Results of Set 27, $N = 200$, of Table 21.

Number	λ ratio	$A = 2.0$	$\beta_1 = 0.3$	$\beta_2 = 0.5$	$\beta_3 = 0.7$
1	0.25	2.0113	0.2831	0.4746	0.6622
2	0.50	2.0080	0.2909	0.4862	0.6811
3	0.75	2.0063	0.2952	0.4923	0.6912
4	1.00	2.0053	0.2978	0.4962	0.6975
5	1.25	2.0046	0.2996	0.4988	0.7017
6	1.50	2.0041	0.3009	0.5006	0.7048
7	1.75	2.0037	0.3019	0.5021	0.7071
8	2.00	2.0034	0.3027	0.5032	0.7090
9	2.25	2.0031	0.3033	0.5041	0.7104
10	2.50	2.0029	0.3038	0.5048	0.7116
11	2.75	2.0028	0.3042	0.5054	0.7126
12	3.00	2.0026	0.3046	0.5059	0.7135
13	3.25	2.0025	0.3049	0.5064	0.7142
14	3.50	2.0024	0.3052	0.5068	0.7148
15	3.75	2.0023	0.3054	0.5071	0.7154
16	4.00	2.0022	0.3056	0.5074	0.7159
17	4.25	2.0022	0.3058	0.5077	0.7163
18	4.50	2.0021	0.3060	0.5079	0.7167
19	4.75	2.0020	0.3062	0.5081	0.7171
20	5.00	2.0020	0.3063	0.5083	0.7174

Table 27. Results of Set 28, $N = 200$, of Table 21.

Number	λ ratio	$A = 2.0$	$\beta_1 = 0.3$	$\beta_2 = 0.5$	$\beta_3 = 0.7$
1	0.05	2.0162	0.2724	0.4589	0.6388
2	0.10	2.0128	0.2802	0.4706	0.6579
3	0.15	2.0100	0.2866	0.4801	0.6734
4	0.20	2.0077	0.2919	0.4878	0.6862
5	0.25	2.0058	0.2962	0.4942	0.6968
6	0.30	2.0043	0.3000	0.4996	0.7057
7	0.35	2.0030	0.3031	0.5042	0.7133
8	0.40	2.0018	0.3059	0.5082	0.7199
9	0.45	2.0009	0.3083	0.5116	0.7256
10	0.50	2.0000	0.3104	0.5146	0.7306
11	0.55	1.9993	0.3122	0.5173	0.7350
12	0.60	1.9986	0.3139	0.5197	0.7390
13	0.65	1.9980	0.3154	0.5218	0.7425
14	0.70	1.9975	0.3167	0.5237	0.7456
15	0.75	1.9970	0.3179	0.5254	0.7485
16	0.80	1.9966	0.3190	0.5270	0.7511
17	0.85	1.9962	0.3201	0.5284	0.7535
18	0.90	1.9958	0.3210	0.5297	0.7556
19	0.95	1.9955	0.3218	0.5309	0.7576
20	1.00	1.9952	0.3226	0.5320	0.7595

5.2 A Translog Model

The next model takes the form of a translog specification with two independent variables

$$(38) \quad \log(y_i) = \alpha + \sum_{j=1}^2 \beta_j \log(x_{ji}^*) + \sum_{j=1}^2 \sum_{k=1}^2 \gamma_{jk} \log(x_{ji}^*) \log(x_{ki}^*) + u_i$$

$$(39) \quad x_{ji} = x_{ji}^* + \varepsilon_{ji}$$

with symmetry of the γ parameters, that is, $\gamma_{jk} = \gamma_{kj}$, $j, k = 1, 2$, and with the same assumptions on the error terms and latent variables as stated above. We notice that the above translog model is nonlinear in the parameters because the latent variables are unknown. This specification clearly increases the difficulty of estimating a nonlinear EIV model and, thus, puts the *NJLS* estimator to a serious scrutiny.

The parameter specifications for the translog model are given in Table 28 while the results are reported in Table 29 through Table 34. The choice of parameter values mimics the structure of a generic function and of a production function. All the draws were scaled by a factor equal to 20.

Table 28. Specification for the Monte Carlo experiments of the translog model (38)-(39)

Set	α	β_1	β_2	γ_{11}	γ_{12}	γ_{22}	x_i^*	σ_u	σ_{ε_j}	True λ	N
29	-0.5	0.4	0.5	1.0	-0.2	0.8	$U(10,20)$	0.316	0.5	2.25	200
30	-0.5	0.4	0.5	1.0	-0.2	0.8	$U(10,20)$	0.5	0.5	1.00	200
31	-0.5	0.4	0.5	1.0	-0.2	0.8	$U(10,20)$	1.0	0.5	0.25	200
32	1.5	0.6	0.3	-1.0	-0.4	-0.7	$U(10,20)$	0.316	0.5	2.25	200
33	1.5	0.6	0.3	-1.0	-0.4	-0.7	$U(10,20)$	0.5	0.5	1.00	200
34	1.5	0.6	0.3	-1.0	-0.4	-0.7	$U(10,20)$	1.0	0.5	0.25	200

Notice that in all the translog models of Table 28, the introduction of an intercept term does not disrupt the robustness of the *NJLS* estimator.

Table 29. Results of Set 29, $N = 200$, of Table 28

Number	λ ratio	$\alpha = -0.5$	$\beta_1 = 0.4$	$\beta_2 = 0.5$	$\gamma_{11} = 1.0$	$\gamma_{12} = -0.2$	$\gamma_{22} = 0.8$
1	0.25	-0.4967	0.3950	0.4935	0.9169	-0.1884	0.7591
2	0.50	-0.4978	0.3964	0.4952	0.9323	-0.1903	0.7699
3	0.75	-0.4987	0.3976	0.4966	0.9444	-0.1920	0.7780
4	1.00	-0.4993	0.3986	0.4978	0.9542	-0.1935	0.7843
5	1.25	-0.4999	0.3995	0.4988	0.9623	-0.1950	0.7892
6	1.50	-0.5011	0.3960	0.5036	0.9808	-0.1892	0.7802
7	1.75	-0.5006	0.4010	0.5004	0.9748	-0.1976	0.7964
8	2.00	-0.5010	0.4016	0.5011	0.9798	-0.1988	0.7989
9	2.25	-0.5012	0.4021	0.5017	0.9841	-0.1999	0.8010
10	2.50	-0.5014	0.4027	0.5022	0.9879	-0.2010	0.8028
11	2.75	-0.5016	0.4031	0.5027	0.9913	-0.2020	0.8041
12	3.00	-0.5018	0.4035	0.5032	0.9944	-0.2029	0.8052
13	3.25	-0.5019	0.4039	0.5036	0.9971	-0.2039	0.8061
14	3.50	-0.5021	0.4043	0.5040	0.9996	-0.2048	0.8068
15	3.75	-0.5022	0.4046	0.5044	1.0019	-0.2057	0.8074
16	4.00	-0.5023	0.4050	0.5048	1.0039	-0.2065	0.8079
17	4.25	-0.5024	0.4053	0.5051	1.0058	-0.2073	0.8082
18	4.50	-0.5024	0.4055	0.5054	1.0075	-0.2081	0.8084
19	4.75	-0.5025	0.4058	0.5057	1.0091	-0.2089	0.8086
20	5.00	-0.5026	0.4061	0.5060	1.0106	-0.2096	0.8086

Table 30. Results of Set 30, $N = 200$, of Table 28

Number	λ ratio	$\alpha = -0.5$	$\beta_1 = 0.4$	$\beta_2 = 0.5$	$\gamma_{11} = 1.0$	$\gamma_{12} = -0.2$	$\gamma_{22} = 0.8$
1	0.25	-0.4975	0.3972	0.4943	0.9265	-0.1912	0.7664
2	0.50	-0.4992	0.3997	0.4972	0.9507	-0.1948	0.7823
3	0.75	-0.5005	0.4018	0.4996	0.9702	-0.1983	0.7944
4	1.00	-0.5016	0.4037	0.5016	0.9861	-0.2015	0.8036
5	1.25	-0.5024	0.4053	0.5034	0.9995	-0.2045	0.8108
6	1.50	-0.5035	0.3991	0.5086	1.0100	-0.2045	0.8135
7	1.75	-0.5037	0.4080	0.5062	1.0206	-0.2102	0.8210
8	2.00	-0.5042	0.4092	0.5075	1.0291	-0.2128	0.8247
9	2.25	-0.5046	0.4102	0.5086	1.0372	-0.2153	0.8267
10	2.50	-0.5050	0.4112	0.5096	1.0443	-0.2177	0.8284
11	2.75	-0.5053	0.4121	0.5106	1.0507	-0.2200	0.8296
12	3.00	-0.5056	0.4126	0.5115	1.0563	-0.2225	0.8287
13	3.25	-0.5058	0.4137	0.5124	1.0619	-0.2245	0.8308
14	3.50	-0.5060	0.4144	0.5132	1.0667	-0.2266	0.8310
15	3.75	-0.5062	0.4151	0.5140	1.0715	-0.2285	0.8303
16	4.00	-0.5064	0.4156	0.5147	1.0774	-0.2301	0.8279
17	4.25	-0.5065	0.4163	0.5154	1.0800	-0.2323	0.8288
18	4.50	-0.5066	0.4169	0.5161	1.0836	-0.2341	0.8281
19	4.75	-0.5067	0.4174	0.5167	1.0881	-0.2356	0.8261
20	5.00	-0.5069	0.4180	0.5172	1.0886	-0.2378	0.8284

Table 31. Results of Set 31, $N = 200$, of Table 28

Number	λ ratio	$\alpha = -0.5$	$\beta_1 = 0.4$	$\beta_2 = 0.5$	$\gamma_{11} = 1.0$	$\gamma_{12} = -0.2$	$\gamma_{22} = 0.8$
1	0.05	-0.4974	0.3965	0.4926	0.9358	-0.2015	0.7624
2	0.10	-0.4987	0.3984	0.4948	0.9548	-0.2042	0.7743
3	0.15	-0.5000	0.4003	0.4969	0.9736	-0.2069	0.7858
4	0.20	-0.5012	0.4022	0.4990	0.9920	-0.2096	0.7969
5	0.25	-0.5025	0.4040	0.5009	1.0102	-0.2123	0.8075
6	0.30	-0.5036	0.4057	0.5028	1.0279	-0.2150	0.8176
7	0.35	-0.5048	0.4074	0.5045	1.0454	-0.2178	0.8274
8	0.40	-0.5059	0.4090	0.5062	1.0625	-0.2205	0.8367
9	0.45	-0.5069	0.4106	0.5078	1.0792	-0.2233	0.8457
10	0.50	-0.5080	0.4121	0.5094	1.0957	-0.2260	0.8543
11	0.55	-0.5090	0.4136	0.5109	1.1120	-0.2288	0.8626
12	0.60	-0.5100	0.4151	0.5123	1.1280	-0.2316	0.8706
13	0.65	-0.5110	0.4165	0.5137	1.1437	-0.2344	0.8782
14	0.70	-0.5119	0.4179	0.5151	1.1592	-0.2372	0.8856
15	0.75	-0.5128	0.4193	0.5164	1.1744	-0.2400	0.8928
16	0.80	-0.5137	0.4206	0.5177	1.1892	-0.2428	0.8997
17	0.85	-0.5146	0.4220	0.5189	1.2037	-0.2456	0.9064
18	0.90	-0.5155	0.4233	0.5201	1.2178	-0.2484	0.9129
19	0.95	-0.5163	0.4246	0.5213	1.2318	-0.2512	0.9191
20	1.00	-0.5171	0.4258	0.5225	1.2456	-0.2539	0.9248

Table 32. Results of Set 32, $N = 200$, of Table 28

Number	λ ratio	$\alpha = 1.5$	$\beta_1 = 0.6$	$\beta_2 = 0.3$	$\gamma_{11} = -1.0$	$\gamma_{12} = -0.4$	$\gamma_{22} = -0.7$
1	0.25	1.4958	0.5837	0.2866	-0.8982	-0.3888	-0.6281
2	0.50	1.4981	0.5885	0.2897	-0.9415	-0.3926	-0.6464
3	0.75	1.4993	0.5916	0.2918	-0.9641	-0.3949	-0.6589
4	1.00	1.5001	0.5937	0.2933	-0.9779	-0.3965	-0.6686
5	1.25	1.5007	0.5953	0.2945	-0.9872	-0.3978	-0.6767
6	1.50	1.5012	0.5967	0.2957	-0.9944	-0.3996	-0.6839
7	1.75	1.5015	0.5978	0.2964	-0.9989	-0.3998	-0.6898
8	2.00	1.5017	0.5983	0.2970	-1.0007	-0.3997	-0.6958
9	2.25	1.5020	0.5996	0.2979	-1.0062	-0.4012	-0.7006
10	2.50	1.5024	0.6000	0.2986	-1.0065	-0.4023	-0.7080
11	2.75	1.5024	0.6011	0.2991	-1.0114	-0.4024	-0.7099
12	3.00	1.5026	0.6021	0.2998	-1.0141	-0.4032	-0.7151
13	3.25	1.5027	0.6024	0.3001	-1.0154	-0.4035	-0.7184
14	3.50	1.5028	0.6028	0.3003	-1.0192	-0.4017	-0.7220
15	3.75	1.5025	0.6034	0.3007	-1.0182	-0.4045	-0.7240
16	4.00	1.5028	0.6042	0.3010	-1.0169	-0.4038	-0.7222
17	4.25	1.5029	0.6053	0.3016	-1.0155	-0.4057	-0.7304
18	4.50	1.5034	0.6051	0.3024	-1.0229	-0.4055	-0.7371
19	4.75	1.5036	0.6056	0.3028	-1.0241	-0.4059	-0.7406
20	5.00	1.5037	0.6060	0.3032	-1.0253	-0.4062	-0.7441

Table 33. Results of Set 33, $N = 200$, of Table 28

Number	λ ratio	$\alpha = 1.5$	$\beta_1 = 0.6$	$\beta_2 = 0.3$	$\gamma_{11} = -1.0$	$\gamma_{12} = -0.4$	$\gamma_{22} = -0.7$
1	0.25	1.4970	0.5870	0.2878	-0.9173	-0.3922	-0.6404
2	0.50	1.4998	0.5933	0.2922	-0.9678	-0.3976	-0.6683
3	0.75	1.5015	0.5975	0.2953	-0.9944	-0.4011	-0.6892
4	1.00	1.5026	0.6006	0.2977	-1.0110	-0.4038	-0.7066
5	1.25	1.5034	0.6032	0.2997	-1.0225	-0.4060	-0.7218
6	1.50	1.5041	0.6053	0.3015	-1.0313	-0.4079	-0.7355
7	1.75	1.5047	0.6072	0.3030	-1.0384	-0.4095	-0.7483
8	2.00	1.5052	0.6090	0.3044	-1.0443	-0.4110	-0.7604
9	2.25	1.5056	0.6105	0.3057	-1.0486	-0.4118	-0.7699
10	2.50	1.5063	0.6121	0.3064	-1.0550	-0.4116	-0.7808
11	2.75	1.5064	0.6138	0.3084	-1.0584	-0.4163	-0.7919
12	3.00	1.5071	0.6146	0.3096	-1.0608	-0.4170	-0.8059
13	3.25	1.5075	0.6162	0.3106	-1.0668	-0.4173	-0.8161
14	3.50	1.5077	0.6176	0.3115	-1.0696	-0.4149	-0.8258
15	3.75	1.5078	0.6196	0.3138	-1.0727	-0.4235	-0.8380
16	4.00	1.5086	0.6196	0.3148	-1.0723	-0.4212	-0.8506
17	4.25	1.5083	0.6209	0.3144	-1.0744	-0.4198	-0.8496
18	4.50	1.5095	0.6227	0.3164	-1.0861	-0.4234	-0.8704
19	4.75	1.5099	0.6240	0.3176	-1.0898	-0.4246	-0.8812
20	5.00	1.5103	0.6253	0.3188	-1.0932	-0.4258	-0.8919

Table 34. Results of Set 34, $N = 200$, of Table 28

Number	λ ratio	$\alpha = 1.5$	$\beta_1 = 0.6$	$\beta_2 = 0.3$	$\gamma_{11} = -1.0$	$\gamma_{12} = -0.4$	$\gamma_{22} = -0.7$
1	0.05	1.4935	0.5835	0.2822	-0.8568	-0.3884	-0.6175
2	0.10	1.4966	0.5889	0.2861	-0.9090	-0.3946	-0.6434
3	0.15	1.4991	0.5936	0.2895	-0.9507	-0.3993	-0.6663
4	0.20	1.5011	0.5977	0.2926	-0.9847	-0.4033	-0.6875
5	0.25	1.5029	0.6013	0.2955	-1.0128	-0.4068	-0.7074
6	0.30	1.5045	0.6046	0.2981	-1.0365	-0.4100	-0.7264
7	0.35	1.5059	0.6077	0.3005	-1.0568	-0.4129	-0.7448
8	0.40	1.5071	0.6105	0.3029	-1.0746	-0.4157	-0.7626
9	0.45	1.5083	0.6131	0.3050	-1.0903	-0.4183	-0.7799
10	0.50	1.5094	0.6155	0.3071	-1.1045	-0.4208	-0.7969
11	0.55	1.5104	0.6178	0.3091	-1.1173	-0.4232	-0.8135
12	0.60	1.5113	0.6200	0.3110	-1.1292	-0.4255	-0.8298
13	0.65	1.5123	0.6221	0.3129	-1.1402	-0.4277	-0.8459
14	0.70	1.5131	0.6242	0.3146	-1.1506	-0.4299	-0.8619
15	0.75	1.5140	0.6261	0.3164	-1.1605	-0.4320	-0.8777
16	0.80	1.5148	0.6280	0.3181	-1.1699	-0.4341	-0.8935
17	0.85	1.5156	0.6299	0.3197	-1.1790	-0.4361	-0.9092
18	0.90	1.5164	0.6317	0.3213	-1.1877	-0.4381	-0.9249
19	0.95	1.5172	0.6334	0.3229	-1.1963	-0.4400	-0.9407
20	1.00	1.5181	0.6346	0.3244	-1.2014	-0.4398	-0.9578

6. A Nonlinear EIV System of Equations

The next model has the structure of a simultaneous system of three nonlinear equations with all the variables measured with error. In general, the formulation of a Monte Carlo specification for a simultaneous nonlinear equations system is rather difficult because the choice of parameter values and random draws of the sample variables may rarely fit the desired relations. Fortunately, there exists a nonlinear system of equations that can serve as the framework for a Monte Carlo experiment. Such a system is the Linear Expenditure System (LES) of demand analysis. We define a LES model with three commodities

$$(40) \quad x_{ji}^* = \gamma_j + \beta_j (y_i^* - \sum_{k=1}^3 \gamma_k p_{ki}^*) / p_{ji}^*$$

$$(41) \quad y_i^* = \sum_{j=1}^3 p_{ji}^* x_{ji}^*$$

$$(42) \quad y_i = y_i^* + u_i$$

$$(43) \quad x_{ji} = x_{ji}^* + \varepsilon_{ji}$$

$$(44) \quad p_{ji} = p_{ji}^* + v_{ji}$$

where $j=1,2,3$ and $\sum_j \beta_j = 1$. The x_{ji}^* variables are interpreted as commodity quantities while p_{ji}^* variables are the corresponding prices. Hence, $y_i^* = \sum_j p_{ji}^* x_{ji}^*$ is disposable income. The γ_j parameters are interpreted as subsistence levels of the corresponding commodities with $\gamma_j \leq \min_i x_{ji}^*$. We assume that quantities, prices and income are iid uniformly random variables. The measurement errors are iid normal random variables.

To generate the Monte Carlo sample information we first draw the prices p_{ji}^* and income y_i^* from the uniform distribution and define the quantities x_{ji}^* using relations (40). The structure of the LES model guarantees that the adding up condition on

disposable income $y_i^* = \sum_j p_{ji}^* x_{ji}^*$ is satisfied. Then, errors ε_{ij} and v_{ij} are drawn from the normal distribution according to the desired specification. The “observed” quantities, prices and disposable income are, thus, defined as $x_{ij} = x_{ij}^* + \varepsilon_{ij}$, $p_{ij} = p_{ij}^* + v_{ij}$, and $y_i = \sum_j x_{ij} p_{ij}$.

The empirical solution of the nonlinear model (40)-(44) requires access to a very efficient mathematical programming code. In all the computations reported in this paper we used the commercial package GAMS. We notice that the Monte Carlo experiments associated with model (40)-(44) necessitated the definition of upper and lower bounds on all the unknowns $(\beta_j, \gamma_j, x_{ij}^*, p_{ij}^*, y_i^*)$ without which the optimal solution is difficult to obtain. The solution of each problem required 1500 iterations, on average. In these Monte Carlo experiments we limited the sample repetitions to 10.

The specification of the parameter and variance values is given in Table 35. All the draws were measured in the original units.

Table 35. Specification of the parameter values for the LES Model (40)-(44)

Parameter	Set 1	Set 2	Set 3	Set 4	Set 5	Set 6
β_1	0.3	0.3	0.3	0.3	0.3	0.3
β_2	0.5	0.5	0.5	0.5	0.5	0.5
β_3	0.2	0.2	0.2	0.2	0.2	0.2
γ_1	1.2	1.2	1.2	1.2	1.2	1.2
γ_2	0.9	0.9	0.9	0.9	0.9	0.9
γ_3	0.5	0.5	0.5	0.5	0.5	0.5
σ_{ε_1}	0.3	0.4	0.5	0.6	0.4	0.6
σ_{v_1}	0.3	0.4	0.5	0.6	0.6	0.4
Latent Y^*	U(10,20)	U(10,20)	U(10,20)	U(10,20)	U(10,20)	U(10,20)
Latent $P_j^*, j = 1, 2, 3$	U(1,4)	U(1,4)	U(1,4)	U(1,4)	U(1,4)	U(1,4)
Sample size	N = 200	N = 200	N = 200	N = 200	N = 200	N = 200

The choice of the beta parameters values is dictated by the necessity that these parameters be positive and add to unity. The choice of the gamma parameters tries to approximate the theoretical restriction $\gamma_j \leq \min_i x_{ji}^*$, although it may not be achievable for all the sample values. The choice of the standard deviations of the error terms attempts to strike a balance between large errors and the level of the observed variables. In order to judge the magnitude of the errors in relation to the level of the various variables we computed the ratio between the average largest absolute value of the errors and the average largest value of the corresponding variable. Hence, in the sample the relative errors can be even larger for some observations. These computations are reported in Table 35B. It shows that the choice of the standard deviations of the error terms reported in Table 35 produces errors that can be as large as the ratio values given in Table 35B. Thirty and forty percent errors are large error and, therefore, the estimator is put to a serious test.

Table 35B. Ratios of errors to levels of the variables

Table 35 Set	$\frac{\max \varepsilon_1 }{\max x_1}$	$\frac{\max \varepsilon_2 }{\max x_2}$	$\frac{\max \varepsilon_3 }{\max x_3}$	$\frac{\max v_1 }{\max p_1}$	$\frac{\max v_2 }{\max p_2}$	$\frac{\max v_3 }{\max p_3}$
1	0.160	0.119	0.284	0.200	0.253	0.166
2	0.213	0.158	0.359	0.251	0.326	0.212
3	0.261	0.197	0.427	0.297	0.393	0.253
4	0.306	0.236	0.489	0.338	0.455	0.300
5	0.213	0.158	0.359	0.338	0.455	0.291
6	0.306	0.236	0.489	0.251	0.326	0.212

The Monte Carlo results are reported in Tables 36 through 41. The suggested EIV estimator produces very accurate estimates of the β parameters regardless of the error variances' ratio and of the distribution of the error terms. On the contrary, the γ

parameters are estimated with a lesser level of accuracy as the distribution of the error term widens. Also, when the error of the quantities is relatively small and the error of the prices is relatively large (set 5 of Table 35), the estimates of the γ parameters overshoot the true values up to 30 percent. The opposite is true when the relative magnitude of the errors is reversed (set 6 of Table 35). We would like to underline again that the error distributions selected in these Monte Carlo experiments have a significant dispersion as reported in Table 35B.

Table 36. Monte Carlo results, LES model, 3 equations, Set 1, $N = 200$, of Table 35

Number	λ ratio	$\beta_1 = 0.30$	$\beta_2 = 0.50$	$\beta_3 = 0.20$	$\gamma_1 = 1.20$	$\gamma_2 = 0.90$	$\gamma_3 = 0.50$
1	0.25	0.2997	0.4996	0.2007	1.1893	0.9092	0.4860
2	0.50	0.3005	0.4989	0.2006	1.1878	0.9098	0.4884
3	0.75	0.3006	0.4988	0.2007	1.1873	0.9096	0.4879
4	1.00	0.3006	0.4987	0.2007	1.1871	0.9095	0.4876
5	1.25	0.3006	0.4987	0.2007	1.1869	0.9095	0.4874
6	1.50	0.3006	0.4986	0.2008	1.1868	0.9094	0.4873
7	1.75	0.3006	0.4986	0.2008	1.1867	0.9094	0.4872
8	2.00	0.3006	0.4986	0.2008	1.1867	0.9094	0.4872
9	2.25	0.3006	0.4986	0.2008	1.1866	0.9093	0.4871
10	2.50	0.3006	0.4986	0.2008	1.1866	0.9093	0.4871
11	2.75	0.3006	0.4986	0.2008	1.1866	0.9093	0.4871
12	3.00	0.3006	0.4986	0.2008	1.1866	0.9093	0.4870
13	3.25	0.3006	0.4986	0.2008	1.1865	0.9093	0.4870
14	3.50	0.3006	0.4986	0.2008	1.1865	0.9093	0.4870
15	3.75	0.3006	0.4986	0.2008	1.1865	0.9093	0.4870
16	4.00	0.2994	0.4981	0.2026	1.1930	0.9144	0.4839
17	4.25	0.3006	0.4986	0.2008	1.1865	0.9093	0.4869
18	4.50	0.3006	0.4985	0.2008	1.1865	0.9093	0.4869
19	4.75	0.3006	0.4985	0.2008	1.1865	0.9093	0.4869
20	5.00	0.3006	0.4985	0.2008	1.1864	0.9093	0.4869

Table 37. Monte Carlo results, LES model, 3 equations, Set 2, $N = 200$, of Table 35

Number	λ ratio	$\beta_1 = 0.30$	$\beta_2 = 0.50$	$\beta_3 = 0.20$	$\gamma_1 = 1.20$	$\gamma_2 = 0.90$	$\gamma_3 = 0.50$
1	0.25	0.3017	0.4973	0.2011	1.1751	0.9126	0.4787
2	0.50	0.3018	0.4967	0.2014	1.1728	0.9115	0.4763
3	0.75	0.3019	0.4966	0.2016	1.1720	0.9111	0.4754
4	1.00	0.3019	0.4965	0.2016	1.1715	0.9108	0.4749
5	1.25	0.3019	0.4964	0.2017	1.1712	0.9107	0.4747
6	1.50	0.3019	0.4964	0.2017	1.1710	0.9106	0.4745
7	1.75	0.3019	0.4963	0.2017	1.1709	0.9105	0.4744
8	2.00	0.3020	0.4963	0.2017	1.1708	0.9104	0.4743
9	2.25	0.3020	0.4963	0.2017	1.1707	0.9104	0.4742
10	2.50	0.3020	0.4963	0.2017	1.1706	0.9103	0.4742
11	2.75	0.3020	0.4963	0.2017	1.1706	0.9103	0.4741
12	3.00	0.3020	0.4963	0.2017	1.1705	0.9103	0.4741
13	3.25	0.3020	0.4963	0.2018	1.1705	0.9103	0.4740
14	3.50	0.3020	0.4963	0.2018	1.1704	0.9102	0.4740
15	3.75	0.3020	0.4963	0.2018	1.1704	0.9102	0.4740
16	4.00	0.3042	0.4959	0.2000	1.1648	0.9114	0.4800
17	4.25	0.3020	0.4963	0.2018	1.1704	0.9102	0.4739
18	4.50	0.3020	0.4962	0.2018	1.1703	0.9102	0.4739
19	4.75	0.3020	0.4962	0.2018	1.1703	0.9102	0.4739
20	5.00	0.3020	0.4962	0.2018	1.1703	0.9102	0.4739

Table 38. Monte Carlo results, LES model, 3 equations, Set 3, $N = 200$, of Table 35

Number	λ ratio	$\beta_1 = 0.30$	$\beta_2 = 0.50$	$\beta_3 = 0.20$	$\gamma_1 = 1.20$	$\gamma_2 = 0.90$	$\gamma_3 = 0.50$
1	0.25	0.3049	0.4966	0.1985	1.1492	0.9064	0.4782
2	0.50	0.3055	0.4961	0.1984	1.1436	0.9028	0.4771
3	0.75	0.3058	0.4959	0.1983	1.1408	0.9013	0.4768
4	1.00	0.3060	0.4958	0.1982	1.1392	0.9003	0.4765
5	1.25	0.3061	0.4957	0.1982	1.1381	0.8997	0.4764
6	1.50	0.3062	0.4957	0.1981	1.1373	0.8992	0.4762
7	1.75	0.3063	0.4956	0.1981	1.1367	0.8988	0.4761
8	2.00	0.3063	0.4956	0.1981	1.1362	0.8985	0.4760
9	2.25	0.3064	0.4956	0.1981	1.1358	0.8982	0.4760
10	2.50	0.3064	0.4956	0.1981	1.1355	0.8980	0.4759
11	2.75	0.3064	0.4955	0.1980	1.1352	0.8978	0.4758
12	3.00	0.3064	0.4955	0.1980	1.1350	0.8976	0.4758
13	3.25	0.3065	0.4955	0.1980	1.1348	0.8975	0.4758
14	3.50	0.3065	0.4955	0.1980	1.1346	0.8974	0.4757
15	3.75	0.3065	0.4955	0.1980	1.1344	0.8972	0.4757
16	4.00	0.3065	0.4955	0.1980	1.1343	0.8971	0.4756
17	4.25	0.3065	0.4955	0.1980	1.1342	0.8970	0.4756
18	4.50	0.3065	0.4955	0.1980	1.1340	0.8969	0.4756
19	4.75	0.3065	0.4955	0.1980	1.1339	0.8969	0.4756
20	5.00	0.3065	0.4955	0.1980	1.1338	0.8968	0.4755

Table 39. Monte Carlo results, LES model, 3 equations, Set 4, $N = 200$, of Table 35

Number	λ ratio	$\beta_1 = 0.30$	$\beta_2 = 0.50$	$\beta_3 = 0.20$	$\gamma_1 = 1.20$	$\gamma_2 = 0.90$	$\gamma_3 = 0.50$
1	0.25	0.3098	0.4963	0.1938	1.1089	0.8882	0.4807
2	0.50	0.3113	0.4957	0.1930	1.0946	0.8780	0.4795
3	0.75	0.3121	0.4953	0.1926	1.0875	0.8735	0.4791
4	1.00	0.3126	0.4951	0.1923	1.0832	0.8710	0.4790
5	1.25	0.3129	0.4949	0.1922	1.0802	0.8696	0.4789
6	1.50	0.3132	0.4948	0.1920	1.0781	0.8688	0.4789
7	1.75	0.3134	0.4946	0.1919	1.0766	0.8684	0.4790
8	2.00	0.3136	0.4946	0.1919	1.0754	0.8681	0.4791
9	2.25	0.3137	0.4945	0.1918	1.0744	0.8678	0.4791
10	2.50	0.3139	0.4944	0.1917	1.0736	0.8676	0.4792
11	2.75	0.3140	0.4944	0.1917	1.0729	0.8676	0.4793
12	3.00	0.3140	0.4943	0.1916	1.0724	0.8676	0.4794
13	3.25	0.3141	0.4943	0.1916	1.0720	0.8676	0.4795
14	3.50	0.3142	0.4942	0.1916	1.0717	0.8677	0.4796
15	3.75	0.3142	0.4942	0.1916	1.0714	0.8678	0.4798
16	4.00	0.3143	0.4942	0.1915	1.0712	0.8679	0.4799
17	4.25	0.3143	0.4942	0.1915	1.0710	0.8681	0.4800
18	4.50	0.3144	0.4941	0.1915	1.0709	0.8682	0.4802
19	4.75	0.3144	0.4941	0.1915	1.0708	0.8684	0.4803
20	5.00	0.3145	0.4941	0.1914	1.0707	0.8686	0.4805

Table 40. Monte Carlo results, LES model, 3 equations, Set 5, $N = 200$, of Table 35

Number	λ ratio	$\beta_1 = 0.30$	$\beta_2 = 0.50$	$\beta_3 = 0.20$	$\gamma_1 = 1.20$	$\gamma_2 = 0.90$	$\gamma_3 = 0.50$
1	0.25	0.3053	0.5108	0.1840	1.3397	1.1554	0.6433
2	0.50	0.3027	0.5102	0.1871	1.3395	1.1524	0.6257
3	0.75	0.3027	0.5099	0.1874	1.3374	1.1498	0.6232
4	1.00	0.3027	0.5097	0.1876	1.3362	1.1481	0.6215
5	1.25	0.3026	0.5096	0.1878	1.3354	1.1469	0.6204
6	1.50	0.3026	0.5096	0.1878	1.3347	1.1459	0.6198
7	1.75	0.3026	0.5096	0.1878	1.3342	1.1452	0.6193
8	2.00	0.3026	0.5095	0.1879	1.3338	1.1446	0.6189
9	2.25	0.3026	0.5095	0.1879	1.3335	1.1442	0.6186
10	2.50	0.3026	0.5095	0.1879	1.3333	1.1439	0.6184
11	2.75	0.3026	0.5095	0.1879	1.3331	1.1436	0.6183
12	3.00	0.3026	0.5095	0.1879	1.3330	1.1435	0.6182
13	3.25	0.3057	0.5093	0.1850	1.3342	1.1552	0.6339
14	3.50	0.3026	0.5095	0.1879	1.3328	1.1433	0.6181
15	3.75	0.3026	0.5095	0.1879	1.3328	1.1433	0.6181
16	4.00	0.3026	0.5095	0.1879	1.3328	1.1433	0.6181
17	4.25	0.3026	0.5095	0.1879	1.3327	1.1433	0.6181
18	4.50	0.3026	0.5095	0.1879	1.3327	1.1433	0.6181
19	4.75	0.3026	0.5095	0.1879	1.3327	1.1434	0.6181
20	5.00	0.3043	0.5078	0.1879	1.3121	1.1249	0.6118

Table 41. Monte Carlo results, LES model, 3 equations, Set 6, $N = 200$, of Table 35

Number	λ ratio	$\beta_1 = 0.30$	$\beta_2 = 0.50$	$\beta_3 = 0.20$	$\gamma_1 = 1.20$	$\gamma_2 = 0.90$	$\gamma_3 = 0.50$
1	0.25	0.3079	0.4847	0.2074	0.8902	0.5729	0.2916
2	0.50	0.3093	0.4845	0.2062	0.8787	0.5668	0.2953
3	0.75	0.3101	0.4844	0.2055	0.8733	0.5649	0.2978
4	1.00	0.3107	0.4842	0.2051	0.8699	0.5644	0.2992
5	1.25	0.3110	0.4840	0.2049	0.8675	0.5644	0.3000
6	1.50	0.3107	0.4842	0.2051	0.8551	0.5445	0.2943
7	1.75	0.3116	0.4838	0.2046	0.8644	0.5646	0.3012
8	2.00	0.3117	0.4837	0.2045	0.8633	0.5647	0.3016
9	2.25	0.3119	0.4837	0.2045	0.8625	0.5649	0.3019
10	2.50	0.3120	0.4836	0.2044	0.8618	0.5650	0.3022
11	2.75	0.3121	0.4835	0.2043	0.8612	0.5651	0.3024
12	3.00	0.3122	0.4835	0.2043	0.8606	0.5652	0.3026
13	3.25	0.3123	0.4835	0.2042	0.8602	0.5653	0.3028
14	3.50	0.3124	0.4834	0.2042	0.8598	0.5653	0.3029
15	3.75	0.3124	0.4834	0.2042	0.8595	0.5654	0.3031
16	4.00	0.3125	0.4834	0.2042	0.8592	0.5655	0.3032
17	4.25	0.3125	0.4833	0.2041	0.8589	0.5656	0.3033
18	4.50	0.3126	0.4833	0.2041	0.8586	0.5656	0.3034
19	4.75	0.3126	0.4833	0.2041	0.8584	0.5657	0.3035
20	5.00	0.3127	0.4833	0.2041	0.8582	0.5657	0.3036

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