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A Nonsubstitution Dynamic Model for Optimal Fertilizer Recommendations

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1. THE PROBLEM

REASONS FOR A REASSESSMENT

This study analyzes yield response to nutrients in order to make optimal fertilizer recommendations. This has been a continuing objective of agronomists and soil scientists since Justus von Liebig's efforts in 1840. While knowledge of the dynamics of soil fertility and crop yields is of paramount importance in providing reliable fertilizer recommendations to farmers, the fertilizer problem is largely an economic one, and from the very beginning it has been recognized as such. Yet, during the last 25 years agronomists and agricultural economists have tended to use quite different methodologies.

The divergence began during the early 1950s when agricultural economists proposed a revision of the experimental designs used to analyze yield response to fertilizers. The production function concept postulated a smooth, concave, differentiable function, possessing a point maximum, and characterized by substitution among all nutrients (Baum, Heady and Blackmore, 1956; Baum, et al., 1957). The functional form was justified by the numerous combinations of fertilizer treatments needed to estimate multinutrient polynomial yield response surfaces with precision. A major modification in the design of the fertilizer experiment was recommended: the use of many treatments but few replications, instead of the customary few treatments repeated a number of times.

The production function approach had enormous effect on the scientific conception and work of soil scientists. In the late 1950s, there were several joint efforts by agronomists and agricultural economists to investigate the advantages of polynomial responses. But more than 20 years after the initial formulation of the new methodology, soil scientists, by and large, have continued conducting fertilizer experiments following the scientific framework and the statistical techniques developed 50 years ago (Fisher and Eden, 1929). That is, the experiments are based on a single nutrient response with a few treatments and several replications, with an exponential form being the frequent choice to represent the response. At the same time, soil scientists have assumed increasing responsibility in the worldwide battle against famine by preparing fertilizer recommendation tables for use in developing countries.

In reevaluating these polynomial models, Anderson and Nelson (1975) concluded that, in many cropping situations, they result in costly upward biases of fertilizer recommendations. This bias may be due to the presence of an extended range within which crops do not respond to nonlimiting nutrients. Thus, the time may be propitious for a critical reevaluation of the methodology proposed by agricultural economists, with the hope of finding reasons for the substantial lack of common language and the scarcity of collaboration with soil scientists.

AGRONOMIC PRINCIPLES

A detailed presentation of soil fertility theory is beyond the scope of this study. For the economic and empirical analysis of the fertilizer problem it is sufficient to consider the following five agronomic principles: (1) the "law of the minimum" of von Liebig; (2) the notion of plateau maximum of the yield response function; (3) the influence of weather and soil type conditions upon the response function; (4) the fertility carryover effect; and (5) the calibration of soil tests. (See Lanzer (1979) for an expanded analysis.)

Shape of Yield Response Curve

Mitscherlich (1909) and Spillman (1923) were the first to attempt to develop an algebraic specification for yield responses to nutrients. Independently, both scientists selected the following relation to represent the yield response to a single nutrient:[1]

$$y = A(1 - e^{-c(x + b)}) \tag{1.1}$$

where y is yield, x is the quantity of the variable nutrient added to the soil as a fertilizer, and A, b, and c are parameters. Parameter b is interpreted as the level of nutrient available in the soil prior to fertilizer application. Equation (1.1) presents diminishing returns for all $x > 0$, and y approaches A asymptotically from below.

Mitscherlich's theory that the c coefficient was the same for all crops and growing conditions, generated considerable controversy among soil scientists. Still, the algebraic form of his yield response function is by far the most frequently used in applied research conducted by soil scientists (Crowther and Yates, 1941; Willcox, 1947; Hanway and Dumenil, 1955; Bray, 1963; Rouse, 1968; Cope and Rouse, 1973).

Equation (1.1) allows neither for an initial state of increasing returns nor for a final stage of yield depression caused by excessive fertilizer use. However, empirical observations led agronomists to believe that the phase of increasing returns, is in general, relatively short, and yield depression usually occurs at quantities far beyond the minimum needed to attain a "yield plateau" (Figure 1). The yield plateau is generally flat and extended for macronutrients such as nitrogen (N), phosphorus (P), and potassium (K). but the response to many micronutrients such as manganese and zinc is sharp and extends over a short range of exposure (Corey and Schulte, 1973, p. 31). The notion that the yield response for macronutrients tends to present a plateau maximum rather than a point maximum seems to be well established among soil scientists.

Other algebraic forms proposed by soil scientists have retained the yield plateau concept introduced in the Spillman-Mitscherlich equation. Examples are Balmukand's (1928) "resistance" function, equation (1.2), and Cate and Nelson's (1971) "linear response and plateau" function, equation (1.3):

$$y^{-1} = A^{-1} + c(b + x)^{-1} \tag{1.2}$$

$$y = \min\ [c(b + x), A] \tag{1.3}$$

where the symbols are as in equation (1.1).

Agricultural economists have tended to disregard yield response models with plateau because they do not allow for yield depression. Heady and Dillon (1961, p. 80), for example, state that "under many circumstances, except for potash or the single variable

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[1]Both assumed that the supply of other nutrients was kept at "adequate" or "nonrestrictive" levels.

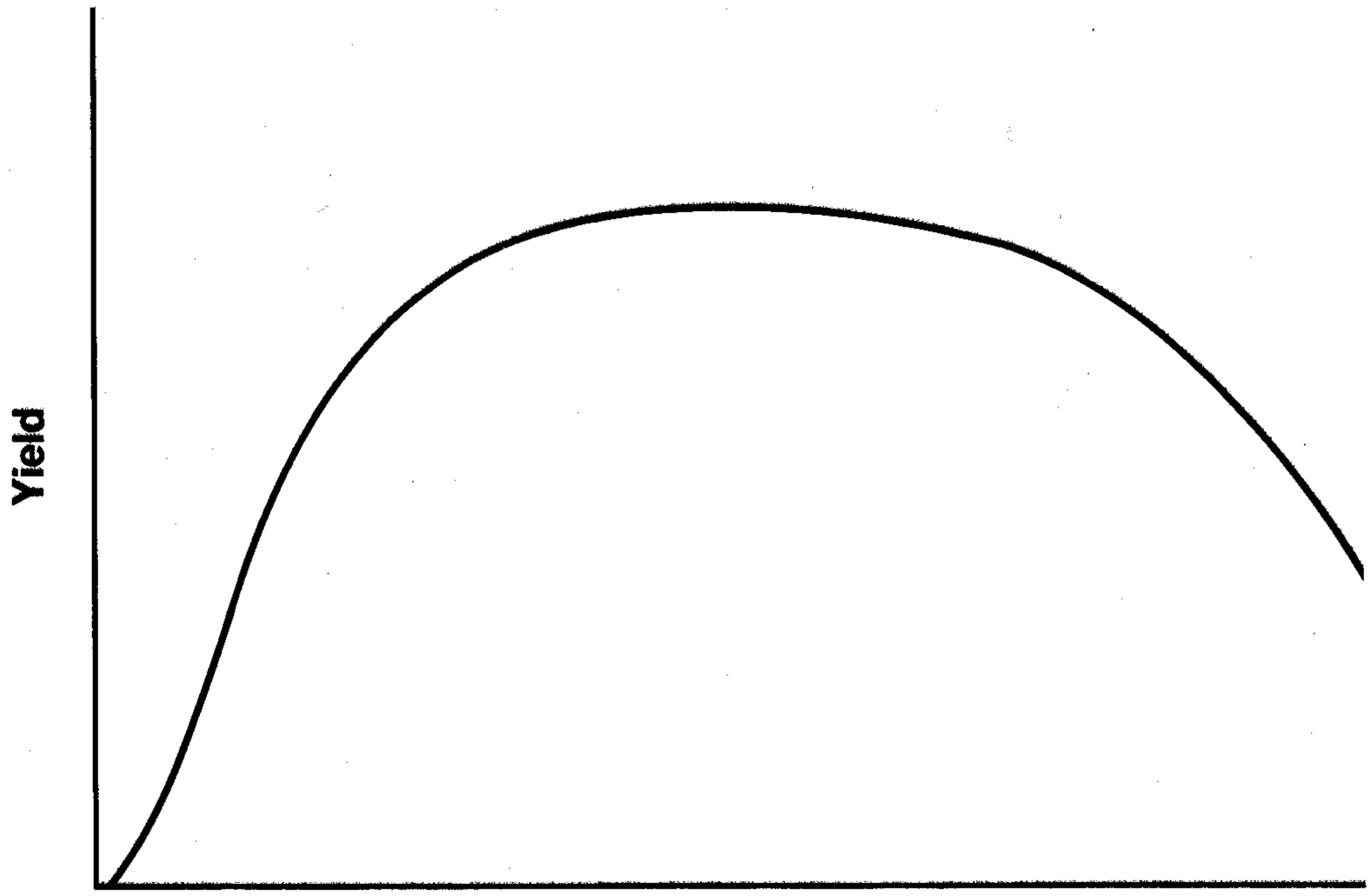

Figure 1. General relation between any particular nutrient or growth factor and the amount of growth made by the plant.

under certain soil climatic conditions, a function allowing negative marginal products would be needed for fertilizer application." However, given the acceptance of a plateau type response by soil scientists and, given that conventional microeconomic equilibrium analysis requires only the existence of diminishing returns, it seems that requiring negative marginal products may be too restrictive for applied research.

The choice of an appropriate algebraic form for yield response has typically been difficult in applied research conducted by agricultural economists. Heady and Dillon (1961, p. 210), who during the 1950s had considerable experience with this problem state:

> Having fitted a number of functions, it may be found that some satisfy some of the criteria better than others and vice versa. The function chosen as best will depend on the weight the researcher attaches to the various criteria, statistical and logical. At such state, the selection of a function is more of an art than a science.

Theil (1971, p. 545) concurs:

> Statistical procedures should not be regarded as the only tool for handling the selection problem. The analyst may be convinced on *a priori* grounds that one specification is more realistic than another, in which case he should feel justified in applying the former even if the latter has a slightly smaller residual variance estimate. The real test is provided by prediction based on an independent set of data.

Perrin (1976) tested point vs. plateau maximum models by fitting both quadratic and linear response and plateau (LRP) functions to the same set of experimental data. With each estimated function, he maximized expected profits subject to the experimental grids associated with an independent set of data. Profits associated with fertilizer recommendations from each respective function were computed from the independent

set of data. The average profit associated with fertilizer recommendations from the LRP function was higher than that associated with recommendations from the quadratic function —through the difference was not statistically significant. Perrin (1976, p. 59) concluded that it might be "surprising to some that the LRP provides recommendations as valuable as those from the quadratic function."

Shape of Yield Isoquants

Liebig (1840, 1863) was among the first agricultural chemists who empirically demonstrated that the growth of plants depended on the uptake of chemical elements, such as phosphorus and potassium. These discoveries led him to propose the use of soil tests and inorganic fertilizers as means to exert control over soil fertility and agricultural production. Liebig's theory on the relationship between yields and nutrients is known as the Law of the Minimum. Redman and Allen (1954, p. 454) define the law:

> This concept holds that the yield of any crop is governed by any change in the quantity of the scarcest factor called the minimum factor, and as the minimum factor is increased the yield will increase in proportion to the supply of that factor until another becomes the minimum. If another factor, not at the minimum, is increased or decreased the yield would not be affected.

The Law of the Minimum introduced *two* distinct concepts. The first was that crops respond in proportion, i.e., linearly, to additions of the limiting nutrient. The second concerned the strong complementarity among plant nutrients, i.e., the notion that various nutrients play different roles in plant physiological processes and cannot substitute for each other. However, apparently, no statistical test of the nonsubstitution hypothesis has ever been performed either by agronomists or by agricultural economists.

Baule (1918) generalized the Mitscherlich equation (1.1), to two and more nutrients. Without loss of generality, consider the two-nutrient specification:

$$y = A\,(1 - e^{-c_1(x_1 + b_1)})\,(1 - e^{-c_2(x_2 + b_2)}) \tag{1.4}$$

where y is crop yield, x_1 and x_2 are respective quantities of two nutrients added to the soil as fertilizer and A, b_1, b_2, c_1, and c_2 are parameters of the model. In particular, the b parameters are associated with the natural supply of the respective nutrient in the soil.

Balmukand (1928) proposed an alternative specification for crop multinutrient relationships:

$$y^{-1} = A^{-1} + c_1(b_1 + x_1)^{-1} + c_2(b_2 + x_2)^{-1} \tag{1.5}$$

where the symbols are as before. (For a detailed account of the mathematical properties of equations (1.4) and (1.5), see Heady and Dillon, 1961, pp. 89 and 97.)

A proper combination of nutrients is essential for a balanced plant growth, and in the absence of an essential nutrient, no growth can take place. Equations (1.4) and (1.5) are attempts to translate this concept of nutrient essentiality into a mathematical equation. With respect to an electrical network, (1.5) states that yields have a theoretical maximum (A) that cannot be attained in general because of the existence of a set of resistances. By analogy, the resistances are deficiencies in the nutrient's availability levels. These resistances (deficiencies) act independently and can individually set an upper bound on the value attained by the dependent variable. Neither the Baule nor the Balmukand equations, however, strictly satisfy the concept of nutrient essentiality. In particular, the requirement of nutrient *specificity* (or nonsubstitution) is not satisfied in either (1.4) or in (1.5). However, both Baule and Balmukand worked when the mathe-

matics of fixed proportions was still intractable and when inequalities were unfamiliar notions to most soil scientists and economists.

Now, under the concept of strict essentiality, a multinutrient-yield specification may be stated:

$$y = \min_{j \in M} [g_j(b_j + x_j)] \tag{1.6}$$

where M is the set of plant nutrients, g_j is the yield response function to nutrient j, given that other nutrients are nonlimiting. For reasons already discussed, g_j is assumed to be quasiconcave and to present, in general, a plateau maximum. Let A be the plateau maximum of g_j. The j-th nutrient is said to be at a nonlimiting level if g_j for this level equals A. On the other hand, it is assumed that $g_j(0) = 0$.

Special cases of the above formulation include the linear response and plateau model of Liebig:

$$y = \min [A, c_1(b_1 + x_1), c_2(b_2 + x_2)]. \tag{1.7}$$

This specification can be traced to Cate and Nelson (1971). Another special case is the Liebig-Mitscherlich model, which might be interpreted as a formal compromise between Liebig's principle of nonsubstitution and Mitscherlich's principle of diminishing returns:

$$y = \min [A(1 - e^{-c_1(b_1 + x_1)}), A(1 - e^{-c_2(b_2 + x_2)})] \tag{1.8}$$

There are, however, other indications that the specification (1.6) closely represents implicit assumptions (or explicit principles) largely accepted by soil scientists. In order to design fertilizer recommendation tables, agronomists usually estimate response functions for a single nutrient at nonlimiting levels; see, for example, Rouse (1968). This is, of course, the most intuitive way of estimating the g_j functions of specification (1.6). And such an estimation procedure is justified by the shape of yield response commonly found for macronutrients, i.e., positive response up to a yield plateau. An experienced soil scientist usually has little difficulty in keeping the levels of all nutrients, but one, under nonlimiting supply conditions.

Effect of Weather and Soil Type Variables and Relative Yield Theory

The study of the several factors affecting plant growth was highly stimulated by Mitscherlich's theory on the constancy of the proportionality factor. Mitscherlich maintained that the coefficient c in equation (1.1) was a constant for each nutrient, irrespective of everything else, including crops. Variations in weather conditions, soil variables, and the kind of crop would be felt only on the parameter A, the asymptotic maximum. In the words of Russell (1973, p. 53):

> Mitscherlich's work was extraordinarily stimulating and caused a veritable flood of controversy when it was first developed. His equation has been of great practical value though it is certainly not exact. Thus ... (coefficient c) ... for a particular nutrient is not a constant, but depends somewhat on the other conditions of growth.

Because y/A equals $1 - e^{c(b + x)}$, a natural outgrowth of Mitscherlich's theory was the use of relative (or percentage) yield as a means to standardize response data obtained under different growing conditions. In spite of the shortcomings attributed to Mitscherlich's theory, Ryan (1972, p. 25) notes that:

> Most soil laboratories recognize that the fitting of Mitscherlich type *relative yield curves* to soil analysis is generally the "best" (emphasis added).

Agricultural economists, on the other hand, generally claimed that effects of weather and soil type conditions (e.g., pH, depth, percent clay, moisture characteristics,

redox potential) should be evaluated through the explicit incorporation of weather and soil type variables into "generalized" yield response functions (Heady, 1956; Munson and Doll, 1959). The estimation of generalized functions was deemed necessary for the design of site specific fertilizer recommendations. In particular, the incorporation of weather variables was required in order to make an assessment of risks associated with fertilizer use (Smith and Parks, 1967; de Janvry, 1972; Ryan, 1972; da Fonseca, 1976).

From an applied research point of view, the difficulties presented by the "generalized yield response function" approach seem to lie mostly on the availability of data for the sites where experimentation took place. This kind of limitation is particularly true in lesser developed countries.

Weather and soil type, in any event, are not under the control of farmers who have already chosen specific locations for their activities. Thus, it is unnecesary to know the *levels* of weather and soil class variables. Only knowledge of their *likely effects* on the yield-soil test-fertilizer relationships is required in order to make fertilizer recommendations. Ideally, the required knowledge would be incorporated into a *single and readily available index* of weather and soil type effects. Such an index can be obtained under a set of relatively mild assumptions, and the result is directly related to Mitscherlich's relative yield theory.

Definition: Given a particition with two subgroups of variables [$w = (w_1, w_2, \ldots w_n)$, $x = (x_1, \ldots x_m)$]. A function $F(w,x)$ is said to be weakly separable if the ratio of first partial derivatives with respect to any two variables in one subgroup is independent of any variable in the other subgroup.

Assumption 1: For predictive purposes, the generalized yield nutrient relationships can be adequately represented by a weakly separable function with respect to the partition (set of nutrients; set of other factors of growth) such as:

$$y = f(w, s)g(b + x) \tag{1.9}$$

where y is yield, w is a random vector of weather variables, s is a vector of soil type variables, b is the vector of the nutrient quantities available in the soil prior to fertilizer application, and x is a vector of nutrient quantities applied as fertilizers.

Assumption 2: There exists a set of weather, soil type and nutrient conditions, w^*, s^*, $(b + x)^*$, such that:

$$f(w^*, s^*)g(b + x)^* = A^* \geq f(w, s)g(b + x) \tag{1.10}$$

where A^* is the maximum attainable yield.

Dividing and multiplying the right-hand side of (1.9) by A^* one obtains:

$$y = A^*h(w, s)g^*(b + x) \tag{1.11}$$

where $h(w, s) = f(w, s)/A^*$. Since, by Assumption 2, $y \leq A^*$, equation (1.11) implies that $0 \leq h(w, s)g(b + x) \leq 1$.

It can be assumed, without any loss of generality, that:

$$0 \leq h(w, s) \leq 1 \tag{1.12}$$

$$0 \leq g(b + x) \leq 1 \tag{1.13}$$

From (1.11), (1.12), (1.13) and the definition of A^*, it is clear that $h \to 1$ as $f(w, s) \to f(w^*, s^*)$. Also, $g \to 1$ as $(b + x) \to (b + x)^*$. In particular, by letting $A_{ws} = A^* h(w, s)$, one can write (1.11) as:

$$y = A_{ws}\, g(b + x). \tag{1.14}$$

Equation (1.14), a generalization of Mitscherlich's theory, explicitly identifies the index to be used for pooling experimental data under different soil class and weather conditions. This location index, A_{ws} represents the yield plateau of each experiment.

Bray's (1954, 1958, 1963) empirical research sought to establish the conditions under which the relative yield principle seems to hold. His findings strongly support Mitscherlich's relative yield theory for nutrients such as phosphorus and potassium. The relative yield concept seems to hold when several conditions are held constant: (1) the form of the nutrient, (2) the distribution pattern of the nutrient in the soil relative to plant distribution, (3) the kind of plant, and (4) the planting pattern and rate of planting. Condition (1) refers both to the kind of fertilizer employed, such as rock phosphate or superphosphate, and to the form of the nutrient available in the soil. Conditions (2) and (4) refer to agricultural techniques, such as band and broadcast fertilizer application and number of plants per hectare (ha). Condition (3) says that the relative yield response functions differ by crop.

Figures 2A and 2B represent, schematically, yield responses to immobile nutrients (such as P and K) and to mobile nutrients (such as N), respectively. According to Bray, the relative yield concept would not apply to the more mobile nitrogen, but some authors have had considerable success in pooling large sets of nitrogen experiments under the assumption that coefficient c (in Mitscherlich's equation) is a constant (see, for example, Hanway and Dumenil, 1955; Hildreth, 1956). It may be, therefore, that Mitscherlich's relative yield theory has wider applicability than Bray indicated.

In conclusion, an estimate of the maximum yield attained at a given experiment may be used as an index for the weather and soil type conditions associated with that experiment. The shape of yield responses (plateau type maximum) simplifies the task of obtaining an estimate for the maximum yield. Moreover, under the relative yield concept, the maximum yield index may be used as a simple scaling factor. The relative yield principle has sound theoretical support for immobile nutrients such as phosphorus and potassium, but there is evidence that it can serve as a useful simplifying assumption for a mobile nutrient such as nitrogen as well.

Soil Fertility Carryover and Control

Agronomists attribute considerable importance to fertilizer carryover effects. For example, Tisdale and Nelson (1975, p. 538) state:

> As fertilizer is applied in increasing quantities, it becomes apparent that increased attention must be given to the value of carryover. In many cases, the cost of fertilization is charged to the crop treated. However, carryover fertilizer is like money in the bank and is part of fertilizer economics. Hence, it is apparent that if we are to make a critical evaluation of fertilizer use, the carryover value must be considered.

Agronomists make recommendations for correcting the level of soil fertility to keep it at the minimum required to achieve the yield plateau (Rouse, 1968, pp. 16-17). These recommendations are based on observations from soil tests of fertility levels. If all of the fertilizer available in the soil in a given season were used by the plants and/or leached out by the end of the season, there would be no need for dynamic specification and analysis of the yield response problem. In many instances, however, a large part of the applied fertilizer remains in trhe soil for extended periods of time. Thus, following Burt (1972, p. 135):

> The only correct procedure is to approach the problem for what it is — a truly dynamic phenomenon.

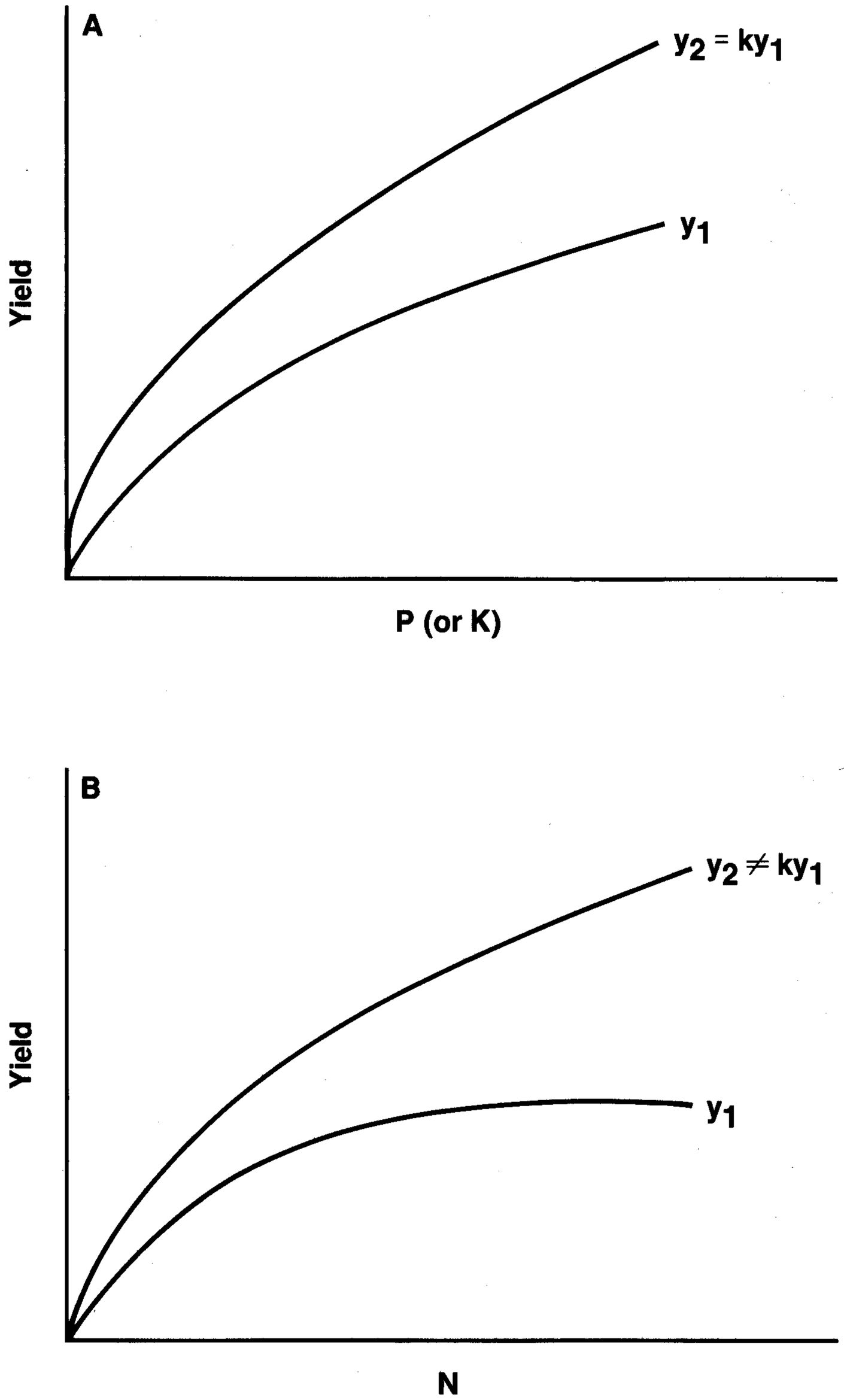

Figure 2. Yield response to immobile nutrients P or K (2A) and to mobile nutrient N (2B) under different weather and soil type conditions.

Fuller (1965) used the following model to analyze nitrogen fertilization for continous planting of corn:

(a) Carryover of nitrogen

$$c_t N_{t-1}^T (k_1 - k_2 e^{k_3 N_{t-1}^T}) \tag{1.15}$$

(b) Total nitrogen supply

$$N_t^T = N_t^A + C_t + a_1 \mu_{t-1} + a_2 [R_{t-1}\mu_{t-1} - E(R_{t-1}\mu_{t-1})] \tag{1.16}$$

(c) Yield response

$$Y_t = \alpha_t + \beta_t e^{\gamma N_t^T} + v_t \tag{1.17}$$

where

C_t = carryover of available nitrogen from t-1,

N_t^T = total nitrogen available for plants in year t,

N_t^A = nitrogen fertilizer applied in year t.,

$\mu_t = y_t - E(y_t)$,

R_t = a rainfall index $[E(R_t) = 0]$,

v_t = a chance variable

and other symbols are parameters of the model.

From the model, Fuller was able to derive mean-variance maps for two fertilization strategies: (1) a fixed rate of nitrogen applied every year and (2) an application of nitrogen at an annual rate to maximize expected profits, including the allowance for carryover for the current year. From equations (1.15) to (1.17) above, it is clear that decision rules for the latter strategy require the knowledge of past levels of nitrogen fertilization, rainfall, and yields.

Kennedy et al., (1973), analyzed nitrogen carryover for multiharvest sorghum in Australia. Dynamic programming was used to derive optimal fertilizer and recycling policies. Nitrogen carryover was specified as follows:

$$C_T = V_T \bullet V_{T-1} \bullet V_{T-2} \bullet\bullet\bullet V_1 \bullet N; \qquad V_t \epsilon\ (0.1) \tag{1.18}$$

where C_T is the amount of nitrogen carryover from an application of N units of fertilizer T periods before. The authors stated that coefficients V_t would depend on variables such as past yields and weather conditions, although they did not formalize these relationships. Instead, based on previous research, they parameterized V_t in the range of 0.2 to 0.4. Then they defined the optimal application of fertilizer for a given period as the amount needed to bring total nutrient supply up to its optimum stock level. Although the optimum stock level of nitrogen was found to be very stable with respect to the choice of V_t, the sequence of rates of nitrogen application was less stable.

Stauber, Burt, and Linse (1975), studied the economics of fertilizing grasses, accounting for carryover. They specified nitrogen carryover:

$$C_t = \alpha_o (A_{t-1} + C_{t-1})^{\alpha_1} W_{t-1}^{\alpha_2} \tag{1.19}$$

where

C_t = nitrogen carryover available for plants in period t,
A_t = applied nitrogen in period t and
W_t = seasonal precipitation in period t.

The parameters of the carryover equation were estimated implicitly in the yield response function. The problem of maximizing discounted expected returns was solved by a stochastic dynamic programming approximation. Optimal fertilizer policies were presented as S,s policies (where S is the desired level of nitrogen and s represents the reorder point). Control of the system required the knowledge of the level of nitrogen in the soil, but this variable was not directly observable in their model.

This review shows the need for casting the analysis of fertilizer recommendations into a dynamic framework. In particular, the (S,s) approach employed by Stauber, Burt, and Linse (1975), reveals the nature of the problem. First, the optimal stock of soil fertility which maximizes the stream of discounted expected net returns must be estimated. Second, the fertility stock must be achieved and maintained at its optimal level by the supply of fertilizer inputs; this is a control problem. Control rules can be developed from the knowledge of past fertilizer applications and carryover rates (Kennedy et al., 1973) or carryover functions (Fuller, 1965, and Stauber, Burt and Linse, 1975). In other words, the design of fertilizer recommendations requires updated estimates of the level of extractable nutrients in the soil. Such updated estimates can be obtained from the chemical analysis of soil samples in the laboratory.

Soil Tests and Calibration

Soil scientists consider that a correct interpretation of soil test measurements is of paramount importance for the design of fertilizer recommendations (see Walsh and Beaton, 1973, particular by chapters 1, 2, 4 and 14), because, in many instances, most of the information concerning a particular farming site comes embodied into a single composite soil sample. It is from the chemical analysis of such small pieces of "information" that fertilizer recommendations are usually derived. In the terminology of optimal control theory, soil test measurements are "sensor measurements signals" of a dynamic-stochastic physical process (Athans, 1972). Soil testing chemical methods, on the other hand, constitute the "sensors." Ideally, the soil sensors should produce measurements expressed in the same units employed to measure fertilizer quantities.

Agronomists usually assume that soil test measurements are *proportional* to the "true" b values of existing nutrients in soil (Hanway and Dumenil, 1955, p. 78). Therefore, by defining b^* as a soil test measurement and λ as a proportionality factor, one can rewrite (1.14) as:

$$y = A_{ws}g(\lambda b^* + x). \tag{1.20}$$

In particular, for the case of the Mitscherlich equation:

$$y = A_{ws}(1 - e^{-c^*b^* - cx}) \tag{1.21}$$

where $c^* = c\lambda$.

Equation (1.21) is the specification suggested by Bray (1954, 1958, 1963). The proportionality coefficient λ is observed to vary somewhat with soil type. The pragmatic approach followed by soil scientists in this case is to work with "homogeneous" groups of soil types in terms of λ coefficients. This procedure ensures that the chemical form of the nutrient in the soil is approximately the same for the soils included in a given group. Under the proportionality assumption, estimates of λ can be obtained from:

$$\lambda = x/b_t^* - b_o^*) \tag{1.22}$$

where b_o^* is a soil test measurement taken prior to fertilizer application and b_t^* is a soil test measurement taken sometime after a given quantity of fertilizer (x) has been incorporated into the soil. The process of classifying the soils according to the magnitude of their λ coefficients is called calibration in soil test terminology. Indeed, the λ coefficient "calibrates," or filters the sensor signals (b*) with respect to the actual signals (b). In practice, however, the calibration process is undertaken via the comparison of c* estimates for different soil types (Rouse, 1968; Cope and Rouse, 1973). In more general terms, the calibration of soil tests is made by comparing the response functions of a given crop or rotation cultivated in different soil types. For these reasons, the λ coefficients are functions of at least some soil type variables, so that the relative yield concept expressed in (1.20) is better understood as:

$$y = A_{ws}g(\lambda_s b^* + x) \tag{1.23}$$

where λ_s means that the λ coefficient depends on some soil type variables (other symbols retain the previous meaning).

In view of (1.23), a truly "generalized response function" should incorporate the λ_s *functions*. This procedure, however, requires that soil *type* measurements be made in addition to conventional soil test measurements. The alternative, of course, is the calibration approach, that is, the aggregation of soil types into groups for which the range of the λ_s function is relatively narrow. The choice of approach depends, obviously, on economic considerations. There are the costs of additional soil analysis and research to develop the λ_s functions, but there are potential gains to realize from a "finer tuning" than that allowed by the calibration procedure. In any event, it is clear that the λ coefficients (or λ_s functions) play a crucial role in the formulation of fertilizer recommendations. To some agricultural economists, however, e.g., Ryan (1972, p. 20):

> There does not seem to be any particular advantage in attempting to measure a λ factor when the aim is to develop a generalized response function which will *accurately* estimate the importance of currently available soil nutrient measurements in yield response (emphasis added).

Dynamic Nonsubstitution and Relative Yield Model

By pooling the conclusions achieved in the discussion of the main agronomic principles, the dynamic extension of the nonsubstitution relative-yield model can be formulated:

$$y_t = A_{w_t s} \min_{j \in M} [g_j(\lambda_{js} b^*_{jt} + x_{jt})] \tag{1.24}$$

$$b^* = h_{js}(y_{t-1}, b^*_{jt-1} + \lambda_{js}^{-1} x_{jt-1}) \tag{1.25}$$

- Y_t = crop yield in period t
- $A_{w_t s}$ = the yield plateau given w_t (weather conditions in period t) and s (soil type)
- g_j = the relative yield response to nutrient j $[0 \leq g_j(\cdot) \leq 1, j \in M]$
- λ_{js} = the proportionality factor which allows for the addition of soil test level and applied fertilizer for nutrient j given soil type s
- b^*_{jt} = the soil test level for nutrient j in period t
- x_{jt} = the quantity of the j-th nutrient incorporated into the soil in period t
- h_{js} = the carryover function for the j-th nutrient given soil type s
- w_t = the vector of weather conditions in period t.

In equation (1.24), the term $(\lambda_{js} b^*_{jt} + x_{jt})$ represents the total supply of nutrient j in period t as measured in fertilizer units (e.g., kg $P_2 0_5$/ha), whereas the term $(b^*_{jt-1} + \lambda^{-1}_{js} x_{jt-1})$ in equation (1.25) represents the total supply of nutrient j in period t-1 as measured in terms of soil test units (e.g., ppm P).

Algebraic specifications for the carryover functions h_{js}, equation (1.25), can be obtained from the work reviewed in previous sections. For example, the algebraic proposition of Stauber, Burt, and Linse (1975) — see equation (1.19) — could be modified to accommodate soil test observations:

$$b^*_{jt} = \alpha_0(b^*_{jt-1}+\lambda^{-1}_{js}\, x_{jt-1})^{\alpha_1} W_{t-1}^{-\alpha_2} \tag{1.26}$$

The incorporation of a soil test variable into Fuller's carryover equation — see (1.15) — would lead to:

$$b^*_{jt} = (b^*_{jt-1} + \lambda^{-1}_{js} x_{jt-1})\,(k_1 - k_2 e^{k_3(b^*_{jt-1} + \lambda^{-1}_{js} x_{jt-1})}) \tag{1.27}$$

Finally, a carryover formulation based on the proposition of Kennedy et al. — See equation (1.18) — and allowing for soil tests, could be written:

$$b^*_{jt} = V_j(b^*_{jt-1} + \lambda^{-1}_{\kappa\tau} x_{jt-1}); \qquad 0 \leq V_j \leq 1 \tag{1.28}$$

where V_j could be interpreted as the expectation of a random parameter V_{jt}.

Recursive substitution of (1.28) leads to:

$$b^*_{jt} = V_j^t\, b^*_{j0} + \sum_{n=1}^{n=t} V_j\, \lambda^{-1}_{js} x_{t-n} \tag{1.29}$$

so that the parameter V_j is the rate of geometric decline of a distributed lag model.

The previous discussion on fertilizer carryover and soil tests strongly suggests that the problem of making fertilizer recommendations can be profitably cast into an optimal control framework. Target trajectories for the relevant variables would be set and a welfare loss function, defined in terms of deviations from the target which are minimized by the appropriate choice of control levels (Chow, 1975, p. 15). Thus, the techniques of optimal control may yield significant improvements in the fertilizer recommendation arena. The practical feasibility of implementing a system to design "prescription" fertilizer recommendations for the farmers, based on optimal control techniques, depends mostly on the existence of reliable feedback information linkages between farmers and the soil laboratory. From a computational point of view, it seems that optimal control algorithms present definite advantages over the dynamic programming approach (Athans, 1972, p. 454). On the other hand, if high speed computers and/or reliable feedback channels are not available, the next best approach seems to be to embed control rules into fertilizer recommendation tables. Such tables can be designed with the help of dynamic programming or multistage mathematical programming. The latter approach to table design will be developed in the next section.

2. THE METHODOLOGY

Multinutrient Yield Response Function

The multistage programming model to be developed in this section is greatly simplified when the multinutrient yield response function (1.24) is respecified equivalently as:

$$\max y_t$$
$$\text{subject to } y_t - A_{w_t s} g_j(\lambda_{js} b^*_{jt} + x_{jt}) \leq 0; \qquad j \epsilon M. \tag{2.1}$$

The constraints of (2.1) represent single nutrient yield responses and indicate that yields are limited by the lowest level of a single nutrient response. The concept of "nutrient imbalance," so frequently employed by soil scientists, now acquires a very clear meaning: If the availability of a nutrient such as phosphorus is consistent with a yield of 1,800 kg/ha (given w_t and s), whereas the availability of another nutrient such as nitrogen allows only for a yield of 1,500 kg/ha, the resulting yield will then be 1,500 kg/ha, *ceteris paribus.* In this case, the supply of nutrients would be considered to be "unbalanced" in the sense that individual nutrient supplies determine different yield potentials. On the other hand, the nutrients would be "balanced" when all the constraints in (2.1) were binding.

The programming framework (2.1) permits the introduction of economic considerations into the physical model. The quantities x_{jt}'s of applied fertilizers should be viewed as decision variables, whereas the b^*_{jt}'s together with the y_t's are state variables. In this case, the one period profit maximization problem can be written as:

$$\max \pi = P_y y - \Sigma P_j x_j$$

subject to:

$$y - A_{ws} g_j(\lambda_{js} b^*_j + x_j) \leq 0 \tag{2.2}$$

$$b^*_j = \text{given for } j \epsilon M$$

$$x_j \geq 0 \text{ for } j \epsilon M$$

and where the P_i's and P_y stand for input and output prices, respectively.

Optimizing the Use of Fertilizers

Although a complete specification of the fertilizer problem requires a stochastic dynamic framework, as argued by Stauber, Burt, and Linse (1975), a useful first approximation can be obtained from a multistate deterministic formulation. Farmers are assumed to follow a fertilizer application strategy that maximizes the expected stream of discounted profits from fertilizer use. Thus, the problem of economizing fertilizer use over time can be written as:

$$\max \pi = \sum_{t \epsilon T} (1 + i)^{-t} (P_{yt} y_t - \sum_{j \epsilon M} P_{jt} x_{jt}) \tag{2.3}$$

subject to

$$y_t - A_{w_t s} g_j(\lambda_{js} b^*_{jt} + x_{jt}) \leq 0$$

$$b^*_{jt} - h_{js}(y_{t-1}, b^*_{jt-1} + \lambda_{js} x_{jt-1}) = 0$$

$$b^*_{j1} = \text{given}$$

$$x_{jt} \geq 0; \qquad j \epsilon M, \; t \epsilon T$$

where

π = stream of discounted profits,
i = interest rate,
T = set of periods for planning purposes,
P_{yt} = output price in period t,
P_{jt} = j-th input price in period t,
and other symbols retain the previous meaning.

This formulation assumes that both the prices (P_{it}, $t \geq 1$) and yield plateaus ($A_{w_t s}$, $t \geq 1$) are known in advance. Because this is, in general, not a reasonable assumption, these parameters are better understood as expected values. Capital constraints which are easily introduced in the programming formulation, can, for example, be useful for examining institutional credit conditions. The b^*_{j1} represent known soil test levels at the beginning of the first planning period.

The solution of the programming problem (2.3) generates an optimal fertilizer application strategy for the planning horizon T. Such a strategy is obviously dependent upon the initial conditions b^*_{j1}. In practice, however, farmers periodically acquire new information on fertility levels by means of soil tests and use this information for updating their fertilization strategy. A conceptual modification of the problem is required to accommodate this situation. The incorporation of updating information into the programming formulation can be achieved by the "moving horizon" concept. Under this notion, "every decision made is then a first-period decision corresponding with a (finite) horizon" (Theil, 1964, p. 155). In other words, at the beginning of every planning period, an optimal fertilizer strategy for the next T cropping periods is computed. However, only the policy for the current cropping period is actually implemented. At the beginning of each cropping period, new information on soil test levels becomes available. New information on input and output prices may be available also. The updated information is substituted for the corresponding information in the problem and a new optimal strategy for the next T cropping periods is found. Again, only the first period policy is actually implemented, since updated information will be available prior to the time for implementing the next period policy. And so on.

Thus, from a parametric evaluation of the initial conditions b^*_{j1} and expected prices, one can derive a set of heuristic control rules for fertilizer use. This set may take the form of a table where fertilizer recommendations are a function of current soil test levels and expected prices. Fertilizer strategies derived from the moving horizon concept maximize (approximately) the expected stream of discounted profits for an infinite planning horizon (see Theil, 1964, pp. 154-157).[2] Therefore, the control rules obtained as presented above are along the lines discussed in Section 1: For a given set of expected prices there will be an optimum stock of soil fertility to be maintained by means of periodic supply of fertilizer inputs. The stock of soil fertility present at any given point in time is measured via soil tests. The optimum quantity of fertilizer (control) to apply in any given period will be the difference between the current soil fertility level and its desired stock level (target).

Finally, the mathematical programming problem stated in (2.3) is likely to have important nonlinearities. Therefore, the computation of exact solutions in an empirical application may be very difficult and approximations may be required.

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[2]As a general rule, the larger the number of planning periods used in the multistage programming formulation, the better the approximation to the infinite horizon that results. However, if the response functions are steep up to a plateau maximum and if fertilizer carryover is high, then it is likely that a relatively small number of cropping periods is required to provide a very good approximation.

Exact Versus Approximate Solutions

Where nonlinearities exist, approximate solutions can be computed under relatively mild assumptions. By defining $a_{jt} = \lambda_{js} b^*_{jt} + x_{jt}$ as the total supply of the j-th nutrient available in the soil at the beginning of period t (measured in terms of fertilizer units) and $a^*_{jt} = b^*_{jt-1} + \lambda_{js} x_{jt}$ as the same variable as measured in terms of soil test units. The programming formulation given in (2.3) may be rewritten as:

$$\max \pi = \sum_{t \epsilon T} (1 + 1)^{-t} (P_{yt} y_t - \sum_{j \epsilon M} P_{jt} x_{jt}) \qquad (2.4)$$

subject to

$$a_{jt} - \lambda_{js} b^*_{jt} - x_{jt} = 0$$

$$a^*_{jt} - b^*_{jt} - \lambda^{-1}_{js} x_{jt} = 0$$

$$y_t - A_{w_t s} g_j(a_{jt}) \leq 0$$

$$b^*_{jt} - h_{js}(y_{t-1}, a^*_{jt-1}) = 0$$

$$b^*_{j1} = \text{given}$$

$$x_{jt} \geq 0;\ j \epsilon T.$$

Assuming that the carryover functions h_{js} ($j \epsilon M$) are additively separable, they can be written as:

$$b_{jt} = h^y_{js}(y_{t-1}) + h^a_{js}(a^*_{jt-1}).$$

Then, problem (2.4) can be solved by well-known approximation techniques of separable programming (Wagner, 1969, pp. 551-557; Hadley, 1964, pp. 104-125; or Beale, 1968, pp. 124-134). Moreover, given that the first two sets of constraints in (2.4) are concave and since the yield response constraints are assumed to be concave by the principle of diminishing returns, the further assumption that h^k_{js} ($k = y,a$) are concave would lead to the conclusion that the set of feasible solutions for the programming model is convex (see Zangwill, 1969, pp. 27, 31 and 32). The objective function, on the other hand, is obviously concave. Under such circumstances, a local optimum is a global optimum for the constrained maximization problem (see Mangasarian, 1969, p. 73). This result has an important pragmatic implication: The separable programming problem can be solved with conventional linear programming procedures because the appropriate convexity and concavity conditions ensure that the rule of restricted basis entry (or adjacent weights) is automatically satisfied by standard simplex rules (see Hadley, 1964, pp. 124-126).

Last, but not least, if the algebraic form (and parameters) of the g_j and h_{js} functions were known exactly, the separable programming approach could provide approximate solutions to any desired degree of accuracy by finer and finer divisions of the grid around optimal solutions obtained in previous steps (Hadley, 1964, p. 123). The limitations imposed by the choice of the separable programming method are, therefore, not too severe. Thus, the actual difficulties in obtaining an exact solution for the economic problem (2.4) do not lie mainly with the choice of techniques for solving the programming problem. Rather, they stem from the more fundamental question of obtaining more accurate estimates of the g_j and h_{js} functions. This subject is discussed in the following section.

Estimation of the Model

Equations (1.24) and (1.25) constitute the structure of a theory that embodies rather basic soil science principles. Yet, as for any theory, they represent an abstraction from a more complex reality. In this regard, the introduction of random noise terms in both

equations appears to be necessary[3] and leads to:

$$y_t = A_{w_t s} \underset{j \epsilon M}{\text{Min}} \; [g_j(\lambda_{js} b^*_{jt} + x^a_{jt})] + v_t \tag{2.5}$$

$$b^*_{jt} = h_{js}(y_{t-1}, b^*_{jt-1} + \lambda^{-1}_{js} x^a_{jt-1}) + u_{jt}; \tag{2.6}$$

where v_t and u_{jt} ($j \epsilon M$) are assumed to be stochastically independent random errors with zero mean, constant variances and zero lagged covariances.

The assumptions regarding the properties of the errors are admittedly restrictive and will be relaxed later. However, the assumption that the response function noise (v_t) and the carryover function noise (u_{jt}) are stochastically independent variables will be retained throughout and can be defended on the following grounds: y_t is the yield obtained at the end of period t. Thus, v_t is a function mostly of excluded variables that occur during cropping period t; examples include the intensity of attack of various pests and weeds and the degree of success of plant establishment. On the other hand, the soil test level b^*_{jt}, is defined at the beginning of period t. Hence, the error u_{jt} is a function mostly of excluded variables that occur during cropping period t-1, such as the effects of weather conditions not embedded in y_{t-1}. Besides the assumption of independence concerning v_t and u_{jt}, equation (2.6) contains only one of the jointly dependent variables (i.e., b^*_{jt}), whereas equation (2.5) contains both jointly dependent variables (i.e., b^*_{jt} and y_t). Hence, model (2.5) - (2.6) is a recursive system. Therefore, little could be gained from applying simultaneous equation techniques rather than single equation regression techniques.[4] Because single equation techniques will be used and because of the particular nature of (2.6), the equations will be discussed separately.

Estimation of the Multinutrient Yield Response Function

The yield response function (2.5) belongs to the class of nonsubstitution production functions and was initially described in Lanzer and Paris (1981). Specification (2.5), is closely related to recent models of markets in disequilibrium (see, for example, Fair and Jaffee, 1972; Maddala and Nelson, 1974; Goldfeld and Quandt, 1975).

Maximum Likelihood Estimator of the Nonsubstitution-Relative Yield Response Function

As a simplification at the outset, assume that relation (2.5) involves only two nutrients and that $v \sim N(0, \sigma^2 I)$, i.e., the disturbance term is normally distributed with zero expectation and variance σ^2. The log-likelihood function, L, corresponding to this specification is therefore:

$$L = -(N/2)\log 2\pi\sigma^2 - (1/2\sigma^2)[y - \min(f_1, f_2)]' \; [y - \min(f_1, f_2)] \tag{2.7}$$

where N is the number of sample points, $f_1(\alpha_1, X_1)$, $f_2(\alpha_2, X_2)$ are the response functions to the two nutrients, α_1, α_2 are vectors of parameters, while X_1, X_2 are the total quantities of nutrients. The maximization of (2.7) with respect to α_1, α_2 and σ will provide maximum likelihood estimates of the relevant parameters. The development of the estimator, however, is greatly facilitated by operating with an equivalent specification of (2.7):

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[3]With regard to the introduction of random error terms, Bellman and Kalaba (1965, p. 24) note: "Whether or not one wishes to believe in deterministic processes as fundamental with stochastic processes solely as a mathematical device introduced to handle unkown or 'hidden' variables, or to believe that nature is basically stochastic with determinism the result of averaging, again a mathematical device, is a matter of personal philosophy which, fortunately has little effect upon the analytic models that arise."

[4]An alternative approach for the case when set M is composed of only one element, is to substitute (2.6) into (2.5). This approach has been employed by Stauber, Burt, and Linse (1975).

$$\max L = -(N/2)\log 2\pi\sigma^2 - (1/2\sigma^2)\,[y - y^*]'[y - y^*] \tag{2.8}$$

subject to

$$y^* = f_1(\alpha_1, X_1) - s_1 = X_1\alpha_1 - s_1$$

$$y^* = f_2(\alpha_2, X_2) - s_2 = X_2\alpha_2 - s_2$$

$$0 = s_1's_2$$

$$s_1 \geq 0,\ s_2 \geq 0,\ \alpha_1,\ \alpha_2,\ y^* \text{ unrestricted}$$

where y^* is the vector of expected values of yield levels. A further assumption is made that the response functions of f_1 and f_2 are linear in the parameters α_1 and α_2, respectively. This assumption is not restrictive, since any concave function can be approximated to any degree of satisfaction by linear segments. The vectors s_1 and s_2 are nonnegative slack vectors. The constraint $s_1's_2 = 0$ ensures that y^* will be equal to either f_1 or f_2, whichever is smaller. More intuitively, the slack vectors s_1 and s_2 can be thought of as dummy variables with unknown location.

The normal inequalities associated with the above formulation are obtained by differentiating the Lagrangean function corresponding to problem (2.8) with respect to α_1, α_2, s_1, s_2, σ^2 and y^*. The Lagrangean function is specified as follows:

$$\max \phi = -(N/2)\log 2\pi\sigma^2 - (1/2\sigma^2)[y'y - 2y'y^* + y^{*\prime}y^*]$$
$$+ (1/\sigma^2)\pi_1'\,[y^* - X_1\alpha_1 + s_1] + (1/\sigma^2)\pi_2'$$
$$[y^* - X_2\alpha_2 + s_2] - (1/\sigma^2)ws_1's_2 \tag{2.9}$$

Since s_1, s_2 are nonnegative, first order conditions for a maximum of (2.9) are

$$\phi_{y^*} = (1/\sigma^2)\,(y - y^* + \pi_1 + \pi_2] = 0 \tag{2.10}$$

$$\phi_{\alpha_1} = (1/\sigma^2)\,(\delta y^*/\delta\alpha_1)\,[y - y^* + \pi_1 + \pi_2] - (1/\sigma^2)X_1'\pi_1 = 0$$

$$(1/\sigma^2)\,(X_1'y - X_1'y^* + X_1'\pi_2) = 0 \tag{2.11}$$

$$\phi_{\alpha_2} = (1/\sigma^2)\,(X_2'y^* - X_2'y + X_2'\pi_2) = 0 \tag{2.12}$$

$$\phi_{s1} = (1/\sigma^2)\,(\delta y^*/\delta s_1)\,(y - y^* + \pi_1 + \pi_2) + (1/\sigma^2)\,(\pi_1 - ws_2) \leq 0$$

$$= (1/\sigma^2)\,(\pi_1 - ws_2) \leq 0 \qquad \text{[using (2.10)]} \tag{2.13}$$

$$\phi_{s2} = (1/\sigma^2)\,(\pi_2 - ws_1) \leq 0 \tag{2.14}$$

$$\phi_{\pi_1} = (1/\sigma^2)\,(y^* - X_1\alpha_1 + s_1) = 0 \tag{2.15}$$

$$\phi_{\pi_2} = (1/\sigma^2)\,(y^* - X_2\alpha_2 + s_2) = 0 \tag{2.16}$$

$$\phi_w = s_1's_2 = 0 \tag{2.17}$$

$$\phi_{\sigma^2} = -(N/2\sigma^2) + (1/2\sigma^4)\,(y - y^*)'\,(y - y^*) = 0 \tag{2.18}$$

Provided $\sigma^2 < \infty$, from (2.10) is derived:

$$y^* = y + \pi_1 + \pi_2 \tag{2.19}$$

so that the vectors of dual variables π_1 and π_2 must also be interpreted as the vector of residuals, when added together as $(\pi_1 + \pi_2)$. Other important restrictions implied by the dual of problem (2.2) are $X_1'\pi_1 = 0$ and $X_2'\pi_2 = 0$. The conditions are easily obtained by substituting (2.19) in (2.11) and (2.12), respectively. From (2.13), (2.14) and (2.17) we

also have that $\pi_1's_1 = 0$, $\pi_2's_2 = 0$. Given the structure of the above problem, conditions (2.11) through (2.17) and the associated complementarity slackness requirements are necessary and sufficient for a global maximum of (2.9).

Combining relations (2.11) through (2.16) with the condition that $s_1's_2 = 0$, we can obtain maximum likelihood estimates of α_1, α_2, y^*, s_1, s_2, π_1, π_2 by solving the following problem:

$$\min ws_1's_2 \tag{2.20}$$

subject to

$$\begin{bmatrix} X_1' & 0 & 0 & 0 & 0 & 0 & -X_1' \\ X_2' & 0 & 0 & 0 & 0 & -X_2' & 0 \\ I & -X_1 & 0 & I & 0 & 0 & 0 \\ I & 0 & -X_2 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & wI & 0 & 0 & -I \\ 0 & 0 & 0 & 0 & wI & -I & 0 \end{bmatrix} \begin{bmatrix} y^* \\ \alpha_1 \\ \alpha_2 \\ s_1 \\ s_2 \\ \pi_1 \\ \pi_2 \end{bmatrix} \begin{matrix} = \\ = \\ = \\ = \\ \geq \\ \geq \end{matrix} \begin{bmatrix} X_1'y \\ X_2'y \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$s_1 \geq 0$, $s_2 \geq 0$, y^*, α_1, π_1, π_2 unrestriced.

By increasing the penalty w to the level $\tilde{w}$ — as necessary — $\tilde{s}_1'\tilde{s}_2$ is driven to zero and the associated solution $\tilde{y}^*$, $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{s}_1$, $\tilde{s}_2$ (if it exists) is a solution of the original maximum likelihood problem.

Alternatively, using conditions $X_1'\pi_1 = 0$ and $X_2'\pi_2 = 0$ and (2.19) as appropriate, it is possible to reduce the dimension of the problem by solving the system

$$\min ws_1's_2 \tag{2.21}$$

subject to

$$\begin{bmatrix} 0 & 0 & X_1' & 0 & 0 & 0 \\ 0 & 0 & 0 & X_2' & 0 & 0 \\ -X_1 & 0 & I & I & I & 0 \\ 0 & -X_2' & I & I & 0 & I \\ 0 & 0 & 0 & -I & wI & 0 \\ 0 & 0 & -I & 0 & 0 & wI \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \pi_1 \\ \pi_2 \\ s_1 \\ s_2 \end{bmatrix} \begin{matrix} = \\ = \\ = \\ = \\ \geq \\ \geq \end{matrix} \begin{bmatrix} 0 \\ 0 \\ -y \\ -y \\ 0 \\ 0 \end{bmatrix}$$

$s_1 \geq 0$, $s_2 \geq 0$, α_1, α_2, π_1, π_2 unrestricted.

System (2.21) is equivalent to (2.20) but has the advantage of a smaller number of variables since $(y + \pi_1 + \pi_2)$ has been substituted for y^*.

Variance of the Estimates

The variance of the estimators of α_1, α_2, s_1 and s_2 can efficiently be obtained as the inverse of the information matrix associated with the Lagrangean function specified in (2.9). In turn, the information matrix is the negative of the expectation of the matrix of second partial derivatives of ϕ with respect to α_1, α_2, s_1, s_2 and σ^2. It can be easily verified that the information matrix associated with the problem discussed above is:

$$R(\alpha_1, \alpha_2, s_1, s_2, \sigma_2) = \frac{1}{\sigma^2}\begin{bmatrix} \bar{X}_1'\bar{X}_1 & 0 & -\bar{X}_1' & 0 & 0 \\ 0 & \bar{X}_2'\bar{X}_2 & 0 & -\bar{X}_2' & 0 \\ -\bar{X}_1 & 0 & I & wI & 0 \\ 0 & -\bar{X}_2 & wI & I & 0 \\ 0 & 0 & 0 & 0 & \frac{N}{2\sigma^2} \end{bmatrix} \tag{2.22}$$

where $\bar{X}_1$ and $\bar{X}_2$ are the matrices corresponding to those values of the explanatory variables which are limiting, as determined by the solution of the normal equations.

If $\bar{X}_1'\bar{X}_1$ and $\bar{X}_2'\bar{X}_2$ are of full rank, the information matrix has an inverse, and the variance of the estimates can be represented as:

$$\text{Var}\begin{bmatrix} \tilde{a} \\ \tilde{s} \end{bmatrix} = \begin{bmatrix} V_{\alpha\alpha} & V_{\alpha s} \\ V_{s\alpha} & V_{ss} \end{bmatrix}$$

where

$$V_{ss} = \sigma^2 [G - \bar{X}(\bar{X}'\bar{X})^{-1}\bar{X}]^{-1} \tag{2.23}$$

$$V_{\alpha s} = V_{ss}\bar{X}(\bar{X}'\bar{X})^{-1}$$

$$V_{\alpha\alpha} = \sigma^2(\bar{X}'\bar{X})^{-1} + \sigma^2(\bar{X}'\bar{X})^{-1}\bar{X}'[G - \bar{X}(\bar{X}'\bar{X})^{-1}\bar{X}]^{-1}\bar{X}(\bar{X}'\bar{X})^{-1} \tag{2.25}$$

and where $\bar{X} = \begin{bmatrix} \bar{X}_1 & 0 \\ 0 & \bar{X}_2 \end{bmatrix}$, $G = \begin{bmatrix} I & wI \\ wI & I \end{bmatrix}$

A Log-Likelihood Ratio Test for the Nonsubstitution Hypothesis

The null hypothesis of nutrient nonsubstitution can now be tested against any alternative in which a tradeoff among nutrients is permitted. In general, agricultural economists have elected to specify this alternative in the form of polynomial functions; the same tradition will be followed here. The objective is to develop a test for the null hypothesis that the crop response function is of the nonsubstitution type or, equivalently.

$$H_0: y = \min[f_1(\alpha_1, X_1), f_2(\alpha_2, X_2)] + v_0$$

versus the alternative,

$$H_1: y = \text{polynomial function in } X_1 \text{ and } X_2 + v_1.$$

The classical theory for hypothesis testing cannot be applied to the above problem because the null and the alternative hypotheses are nonnested.

The theory of nonested hypotheses is in the early stages of development, in spite of the pioneering work of Cox nearly 20 years ago. Among the various procedures proposed for testing hypotheses belonging to disjointed families of probability density functions, the T statistic elaborated by Cox seems to be the most relevant. Cox (1962) and Pesaran (1974) present a thorough analysis of this test. Thus, the null hypothesis takes on the following structure:

$$H_0: y = y^* + v_0,\ y^* = X_1\alpha_1 - s_1,\ y^* = X_2\alpha_2 - s_2,\ 0 = s_1's_2,$$

$$s_1 \geq 0,\ s_2 \geq 0,\ v_0 \sim N(0, \sigma_0^2 I);$$

while the alternative is stated as:

H_1: $y = Z\beta + v_1$, $v_1 \sim N(0, \sigma_1^2 I)$.

Let the log-likelihood functions of H_0 and H_1 be denoted by $L_0(\theta_0)$ and $L_1(\theta_1)$, respectively, where $\theta_0 = (\alpha_1, \alpha_2, s_1, s_2, \sigma_0^2)$ and $\theta_1 = (\beta, \sigma_1^2)$. Then, Cox's T statistic is defined as:

$$T_0 = [L_0(\tilde{\theta}_1) - L_1(\tilde{\theta}_1)] - E_0[L_0(\tilde{\theta}_0) - L_1(\tilde{\theta}_1)] \tag{2.26}$$

where $\tilde{\theta}_0$ and $\tilde{\theta}_1$ are the maximum likelihood estimators of parameters θ_0 and θ_1, and E_0 is the expectation of the log-likelihood ratio given that H_0 is true. The intuitive idea of the T statistic is to construct a standardized variable which is asymptotically distributed as a normal variate with zero mean and unit variance. Cox and more recently Pesaran have shown that under H_0 and H_1 as specified above:

$$T_0 = (N/2)\log[\tilde{\sigma}_1^2/(\tilde{\sigma}_1^2 + (1/N)e_{10}'e_{10})] \tag{2.27}$$

where $\tilde{\sigma}_0^2 = e_0'e_0/N$, $\tilde{\sigma}_1^2 = e_1'e_1/N$. The vectors e_0 and e_1 are the residual vectors of models H_0 and H_1, respectively. Furthermore, e_{10} is the OLS residual vector of the regression of y^* on the explanatory variables Z of model H_1. The variance of T_0 can be shown to be:

$$\tilde{V}_0\ (T_0) = (\tilde{\sigma}^2 e_{100}'e_{100})/[\tilde{\sigma}^2 + (1/N)e_{10}'e_{10}]^2 \tag{2.28}$$

where e_{100} is the residual vector of the following model:

$$\min wu_1'u_2 = 0$$

subject to

$$\begin{bmatrix} 0 & 0 & X_1' & 0 & 0 & 0 \\ 0 & 0 & 0 & X_2' & 0 & 0 \\ -X_1 & 0 & I & I & I & 0 \\ 0 & -X_2 & I & I & 0 & I \\ 0 & 0 & 0 & -I & wI & 0 \\ 0 & 0 & -I & 0 & 0 & wI \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \pi_1 \\ \pi_2 \\ u_1 \\ u_2 \end{bmatrix} \begin{matrix} = \\ = \\ = \\ = \\ \geq \\ \geq \end{matrix} \begin{bmatrix} 0 \\ 0 \\ -e_{10} \\ -e_{10} \\ 0 \\ 0 \end{bmatrix}$$

$u_1 \geq 0$, $u_2 \geq 0$, γ_1, γ_2, π_1, π_2 unrestricted.

Finally, the relevant statistic is defined as follows:

$$N_0 = T_0/[V_0(T_0)]^{1/2} \tag{2.29}$$

which is asymptotically distributed as a standardized normal variate when H_0 is true. The criterion for action is now rather straightforward. First of all, with nonnested hypotheses it is appropriate to apply one-sided tests since the two parameter spaces are usually disjointed. Second, when $|N_0|$ is less than the value of the standard normal variate corresponding to a specified level of confidence, the null hypothesis is not rejected. If $|N_0|$ is greater than or equal to the chosen level, H_0 is rejected in favor of H_1 if N_0 is negative. Otherwise, the test indicates that both hypotheses should be rejected.

Estimation of a Three- or More-Nutrient Response Function

The generalization of this approach to more than two nutrients is conceptually straightforward, although it may be computationally rather taxing. In principle, the maximum likelihood problem specified in (2.8) can include as many individual nutrient response functions as required. The only modification of the formulation (2.8) regards the nonlinear constraints involving the slacks s_{ij}, $i = 1, 2, \ldots k$, $j = 1, \ldots n$, which, in this

case, become $s_{ij}s_{tj}, \ldots s_{kj} = 0, j = 1, \ldots n, i, t, k = 1, 2, \ldots, K$. The estimation of this more complex specification requires the availability of suitable nonlinear programming subroutines based upon efficient algorithms. For the time being, however, a more basic obstacle, the lack of suitable data, prevents the implementation of the generalized model (2.8). Because agronomists strongly believe in Liebig's nonsubstitution hypothesis, seldom do they design and execute fertilizer experiments involving three or more nutrients. Hence, the lack of suitable data, it must be emphasized, is not exclusively due to the complexity and the cost of carrying out multinutrient experiments but rather to the agronomists' perceived lack of need of proving "the obvious."

Estimation of Individual Nutrient Response Functions by Means of Splines

The implementation of the estimation procedure and test development in the previous sections require data generated from experimental designs with two or more nutrients. However, the yield response curve to the supply of macronutrients — such as N, P and K — tends to present a relatively extensive plateau after a relatively short phase of increasing total product. Under such circumstances, an experienced agronomist has little difficulty in keeping the supply of all nutrients — but one — at nonlimiting levels for a given experiment. Thus, if for a given set of data it is known *a priori* that $f_1(x_{1j})$ is equal to or smaller than $f_2(x_{2j})$ for all j plots, the model of a Liebig-Mitscherlich response functions to be estimated reduces to $y_j = f_1(x_{1j}) + v_j, j = 1, \ldots, m$. In this case, the set of available data from single nutrient experiments can provide estimates for the parameters of f_1 only. Nevertheless, this information can be obtained from conventional simple regression procedures. To estimate the parameters of f_2, a set of data for which it is known that $f_2(x_{2j})$ is equal to or smaller than $f_1(x_{1j})$ for all j observations is required. And so on. For example: The data from an experiment where the supplies of phosphorus and potassium are known to be nonlimiting can be used to estimate the yield response to nitrogen. The individual yield responses estimated in this fashion are combined in the form prescribed by the nonsubstitution model. In this way, an estimate for the multiple nutrient response surface is obtained.

Suppose, therefore, that a set of data from E experiments is available. Suppose also that such experiments were executed at different places and/or different years. Suppose, furthermore, that it is known that in all experiments the only limiting nutrient was, say, potassium. In this case, the model to be estimated from these data is:

$$y_{pe} = A_{w_e s_e} g_k (\lambda_{ks_e} K^*_{pe} + K^a_{pe}) + v_{pe} \qquad (2.30)$$

where

y_{pe} = yield observed on the p-th plot of the e-th experiment, $e \epsilon E$,

$A_{w_e s_e}$ = yield plateau of the e-th experiment,

g_k = relative yield response to potassium,

λ_{ks_e} = proportionality factor between soil test units and fertilizer units for potassium for the soil type of e-th experiment,

K^*_{pe} = observed soil test level for the p-th plot of the e-th experiment,

K^a_{pe} = observed quantity of potash fertilizer applied on the e-th plot of the e-th experiment, and

v_{pe} = white noise error term.

Assume now that unbiased estimates of the λ_{ks_e} factors are available,[5] so that one

[5]Such estimates can be obtained from fitting the carryover equations.

can compute $K_{pe}^{T} = \lambda_{ks_e} K_{pe}^{*} + K_{pe}^{a}$, that is, the total supply of potassium available for the plants grown on the p-th plot of the e-th experiment. Thus, the model to be estimated reduces to:

$$y_{pe} = A_{w_e s_e} g_k(K_{pe}^{T}) + v_{pe}. \tag{2.31}$$

At this stage one generally chooses an (a set of) algebraic formulation(s) to represent the yield response functions. Whatever the final choice of an algebraic form, it will never be more than an *approximation* of the empirical phenomena under study. Awareness of this fact opens the doors to some specific statistical methods such as the fitting of splines — one particularly well-suited to the problem at hand.

Poirier (1973a, p. 516) offers the following definition of a (cubic) spline:

Let the set $\Delta = [x_0 < x_1 < \ldots < x_k]$ of abscissa values be referred to as a mesh of (x_0, x_1) and the $k + 1 > 3$ individual points $x_j (j = 0, 1, \ldots, k)$ as knots. Let $y = [y_0, y_1, \ldots, y_k]$ be an associated set of ordinates. Then a cubic spline on Δ interpolating to y, denoted $S_\Delta(x)$, is a function satisfying:

(i) $S_\Delta(x) \in C^2(x_0, x_k)$,

(ii) $S_\Delta(x)$ coincides with a polynomial of degree at most three on the intervals $[x_{j-1}, x_j] (j = 1, 2, \ldots, K)$, and

(iii) $S_\Delta(x) = y_j$ $(j = 0, 1, \ldots, k)$.

One important special case of the cubic spline is the linear spline. In the linear spline, as the name indicates, all arcs are linear (Figure 3). The main difficulties with linear spline regression appear when the knots are unknown. In this case, not only computational difficulties arise, but also the statistical properties of least squares estimators have not yet been well determined.

The estimation of linear splines with fixed (known) knots, however, can be obtained with conventional multiple linear regression procedures. Moreover, under the classical assumptions concerning the error term, the usual t and F tests apply in a straightforward manner.

The technique employed to estimate fixed-knot-linear splines with conventional regression procedures is described in Poirier (1973b, p. 2):

Suppose y is a linear spline in x consisting of k segments with knots at $x_1 < x_2 < \ldots < x_{k-1}$ (where $x_1 > 0$). Then, for any value of x, y can be written as:

$$y = \beta_0 + \beta_1 Z_1 + \beta_2 Z_2 + \ldots + \beta_k Z_k \tag{2.31}$$

where

$$Z_1 = x$$

$$Z_2 = \text{Max } [x - x_1, 0]$$

$$Z_3 = \text{Max } [x - x_2, 0]$$

$$\bullet$$
$$\bullet$$
$$\bullet$$

$$Z_k = \text{Max } [x - x_{k-1}, 0]$$

> The coefficient β_1 represents the slope of the spline over the first interval, and each of the remaining coefficients β_j (j = 1, 2, . . ., k) represent the change in the slope from interval (j - 1) to interval j, respectively ... The actual slope over the j-th segment is $(\beta_1 + \beta_2 + \ldots + \beta_j)$.

Thus, under the fixed-knot-linear spline assumption, model (2.31) can be written as:

$$y_{pe} = A_{w_e s_e}(\beta_1 Z_{pe} + \beta_2 Z_{2pe} + \ldots + \beta_m Z_{mpe}) + v_{pe} \qquad (2.32)$$

where

$$Z_{1pe} = K_{pe}^T$$

$$Z_{2\text{-}e} = \text{Max}\ (K_{pe}^T - K_1, 0)$$

$$Z_{3pe} = \text{Max}\ (K_{pe}^T - K_2, 0)$$

$$\vdots$$

$$Z_{mpe} = \text{Max}\ (K_{pe}^T - K_{m\text{-}1}, 0)$$

and where the K_j (j = 1, 2, . . ., m-1) are the abscissa knots assumed to be known.

Notice that (2.32) does not include a constant, because it is expected that $g_k = (\Sigma\beta_i Z_i) \to 0$ as $K^T \to 0$ (that is, if the total supply of potassium tends to zero, so does the yield). Moreoever, for large values of K^T — though not so large as to cause yield depression — one should expect that $g_k \to 1$. Convexity of the g_k function, on the other hand, requires that $\beta_1 > 0$ and $\beta_j < 0$ for $j \geq 2$.

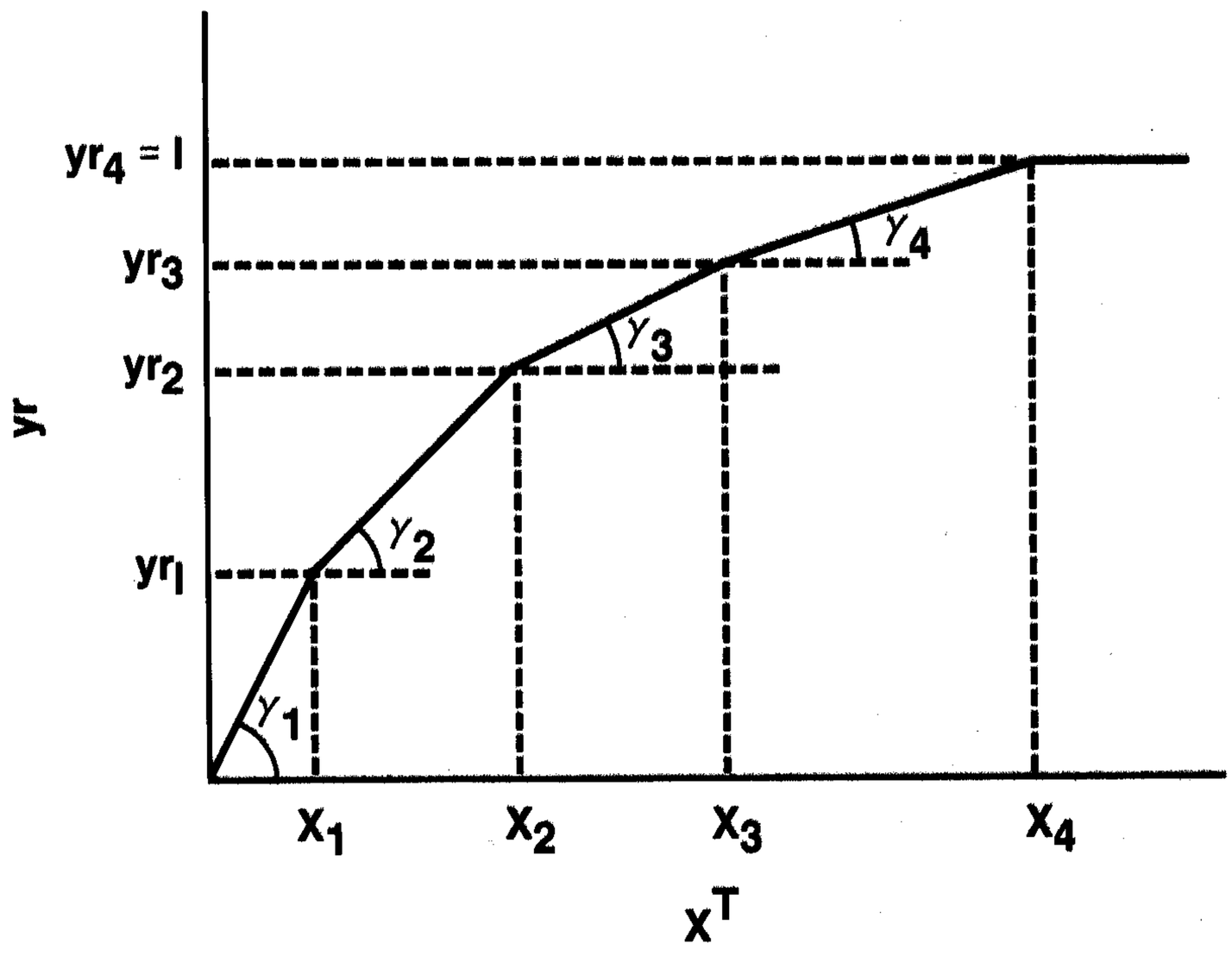

Figure 3. Relative yield (yr) spline response to total supply of nutrient X (X^T).

Finally $A_{w_e s_e}$, the yield plateau parameter of the e-th experiment, can be estimated outside the regression model. An exogenous estimate for $A_{w_e s_e}$ is, clearly, the highest average yield among all treatments that have been tested on the e-th experiment. Let $A^*_{w_e s_e}$ be such an estimate. The β_j coefficients would then be estimated from the regression:

$$y_{pe} = \beta_1 Z^*_{1pe} + \beta_2 Z^*_{2pe} + \ldots + \beta_m Z^*_{mpe} + v_{pe} \qquad (2.33)$$

where

$$Z^*_{jpe} = A^*_{w_e s_e} Z_{jpe}, \qquad j = 1, 2, \ldots, m.$$

There are four main reasons to emphasize fixed-knot-linear splines, rather than, say, unknown knot-cubic splines: First, linear spline regression is the natural counterpart to the separable programming problem presented in previous sections. Second, fixed-know splines can be estimated with conventional — and widely available — multiple linear regression procedures. This is an important consideration for researcher working in developing countries. Third, there is the assumption that yield response functions are concave. Hence, no *a priori* need for either cubic or quadratic splines is detected: Linear splines are sufficient. Fourth, the assumption that the knots are known *a priori* is not as restrictive as it might appear. The help of scatter diagrams may be valuable in determining the knots. Also, depending on the volume and density of the data available, the number of knots can be made considerably large. In this case, the degree of approximation to the true, but unknown, response function is likely to be high.

Finally, and strictly speaking, the Z^*_k regressors of (2.33) are subject to observational errors, because the Z^*_k include estimates for the yield plateau — A^*_{ws} — and for the proportionality constants. However, from the proximity theorem (see Maddala, 1977, p. 153), it appears that the biases caused by this difficulty are likely to be small for the case at hand; since the effects of weather are embedded in the Z^*_k regressors, it is likely that their variance is large with respect to that of the unexplained residuals.

Estimation of the Fertility Carryover Functions

The general specification adopted for the carryover functions is:

$$b^*_{jt} = h_{js}(y_{t-1}, b^*_{jt-1} + \lambda^{-1}_{js} x_{jt-1}) + u_{jt}; \qquad j \epsilon M \qquad (2.34)$$

where

b^*_{jt} = soil test level for the j-th nutrient at the beginning of the t-th cropping period,

y_t = crop yield obtained at the end of the t-th cropping period,

x^a_{jt} = quantity of the j-th fertilizer applied at the beginning of the t-th cropping period.

λ_{js} = proportionality factor between soil test units and applied fertilizer units on soil type s,

u_{jt} = random error with zero mean and constant variance, and

M = set of macronutrients (N, P and K).

A first approximation to the j-th nutrient carryover function that satisfies the separability requirement advanced in a previous section is:

$$b^*_{jt} = \theta_j(b^*_{jt-1} + \lambda^{-1}_{js} x^a_{jt-1}) + \beta_j y_{t-1} + u_{jt}. \qquad (2.35)$$

Notice that (2.35) is the "reduced form" of the following distributed lag model:

$$b^*_{jt} = \sum_{i=1}^{n} \theta_j^{i-1} (\theta_j \lambda_{js}^{-1} x^a_{jt-i-1} + \beta_j y_{t-i-1}) + e_{jt} \qquad (2.36)$$

so that the parameter θ_j is a geometric rate of decline of the availability of the j-th nutrient from one period to another. It is expected that $0 < \theta_j \leq 1$. Notice also that $u_{jt} = e_{jt} - \theta_j e_{jt-1}$. Thus, consistent estimation of (2.35) by ordinary least squares (OLS) requires the autoregressive assumption that $e_{jt} = \theta_j e_{jt-1} + v_{jt}$, where v_{jt} is assumed to be white noise (see, for example, Theil, 1971, p. 261).

The assumption that the autoregression coefficient is equal to the geometric decline coefficient may be too strong. A less restrictive assumption on the properties of the error term of (2.35) is:

$$u_{jt} = \rho_j u_{jt-1} + v_{jt}, \qquad |\rho_j| < 1$$

where v_{jt} is assumed to be white noise. In this case, consistent estimates of the parameters of (2.35) can be obtained from conditional OLS regressions on:

$$b^*_{jt} - \rho_j b^*_{jt-1} = \theta_j (b^*_{jt-1} - \rho_j b^*_{jt-2}) + \theta_j \lambda_{js}^{-1} (x^a_{jt-1} - \rho_j x^a_{jt-2}) + \beta_j (y_{t-1} - \rho_j y_{t-2}) + v_{jt} \qquad (2.37)$$

where v_{jt} is a white noise term. The coefficient ρ_j is made to vary until the sum of squared residuals is minimized. Under the normality assumption, this procedure produces maximum likelihood estimates of θ_j, λ_{js} and β_j (see Theil, 1971, pp. 414-424).

The estimates of the proportionality factors obtained from the estimation of the carryover functions can be used in the estimation of the yield response equations.

3. EMPIRICAL RESULTS

The Setting

During the 1970s, an intensive program of agronomic research was carried out in southern Brazil. In the state of Rio Grande do Sul, the research centered around the determination of wheat and soybean responses to variable fertilizer application rates. These activities were initially performed under a cooperative contract between the Universidade Federal do Rio Grande do Sul and the University of Wisconsin. The main objective and result of this research program was the development of fertilizer recommendation tables for wheat and soybeans. Today these tables are widely used by the agricultural extension service in southern Brazil.[6]

From the beginning, the great variability of soil fertility and structure posed complex measurement problems, which were simplified by cataloging soils into three broad classes (Table 1). The large percentage of soils testing in the "low" category indicates why soil research has been so important for this region of the country and offers an explanation for the low average yields observed in Rio Grande do Sul. (The average wheat yield in 1970-75 was 844 kg/ha, or 12 bu/acre; average soybeans yield was 1,340 kg/ha, or 20 bu/acre.)

Wheat and soybeans are very important crops for the Brazilian economy and play significant roles in the Brazilian balance of payments (Table 2). Domestic production of both wheat and soybeans is concentrated in southern Brazil: The state of Rio Grande do Sul alone accounted for 59 percent of the wheat production and for 47 percent of the soybean production in Brazil in 1974-75 (Table 3).

The 10-year-old program of fertilizer application practices and incentives has produced dramatic results. The use of fertilizers by wheat-soybean growers is, by now, a common practice in southern Brazil. In fact, FECOTRIGO, the Brazilian federation of wheat and soybean cooperatives, estimates that fertilizer costs of the wheat-soybean enterprise in Rio Grande do Sul amount to nearly one-fourth of the total production expenditures (Fecotrigo, 1976). And this expenditure reflects a government subsidy instituted in 1974, for 40 percent of fertilizer prices. The Brazilian government, however, suspended the subsidy at the beginning of 1977. For most wheat-soybean growers of southern Brazil, this change of agricultural policy meant that decisions on fertilizer use ought to be made with increased care. Helpful guidelines are still needed in this new and more expensive environment. Extension specialists have been trained to formulate their fertilizer recommendations based only on soil tests interpreted by means of the fertilizer recommendation tables. Because principal objective of these tables is to restore soil fertility to a level where it is no longer a binding constraint to the attainment of maximum yields, fertilizer recommendation tables are totally price insensitive. And so extension specialists tend not to take the price changes into account when making their recommendations.

Furthermore, most farmers rely heavily on agricultural credit for their production activities. In many instances, banks require that farmers strictly follow extension specialists' advice on fertilizer use. Therefore, at present, decisions on fertilizer use may not easily be changed, in spite of wide price fluctuations.

The Data

Most of the data used here were provided by Dr. Joa[5]o Mielniczuck of the Department of Soil Science of the Federal University of Rio Grande do Sul (DS/UFRGS) and by Engeniero Agronomo Octa[1]vio Siqueira of the National Wheat Research Center (CNPT).

[6]During the decade the tables were revised twice and extended to cover most crops grown in the state.

Table 1. Soil Test Results of 63,117 Soil Samples Taken by Wheat-Soybean Growers in Rio Grande do Sul, Brazil, 1970-1975.

Soil category[a]	Phosphorus	Potassium	Nitrogen[b]	pH
	percent[c]			
Low	88	31	39	59
Fair	7	10	50	25
Good	5	59	11	16
Total	100	100	100	100

[a]Based on fertilizer recommendation tables presently used in Rio Grande do Sul ("very low" and "low" added together for P and K).
[b]Indirect measure through percentage content of organic matter. Indirect measure through "$CaCO_3$ Requirements for pH = 6.5" (SMP method).
[c]Percentage of sample tests for the respective factors falling in each soil category, e.g., 88 percent of the soils tested were low in phosphorus.
Source: Card records of the soil laboratory of the Universidade Federal do Rio grande do Sul.

Table 2. Main Items in the Brazilian International Trade (1974).

Exports	Value	Imports	Value
	million U.S. dollars		million U.S. dollars
Sugar[a]	1,262	Oil (crude)	2,979
Coffee[b]	980	Machines	1,862
Soybeans[c]	888	Steel sheets	894
Iron Ore	571	Wheat	522
Others	3,701	Others	5,911
Total	7,402	Total	12,168

[a]Crystal and "demerara."
[b]Beans and soluble.
[c]Beans and meal.
Source: FIBGE (1976b)

Table 3. Domestic Production of Wheat and Soybeans in Brazil in 1974/75.

State	Wheat	Soybeans
	1000 metric tons	
Rio Grande do Sul	1,690	4,688
Parana	915	3,625
Others	253	1,579
Total	2,858	9,892

Source: FIBGE (1976a)

Other data were from reports published by the DS/UFRGS, by the CNPT and by the Institute of Agricultural Research (IPAGRO) of the Agricultural Department of the State of Rio Grande do Sul.

Thirty-eight independent experiments carried out in the wheat-soybean producing area of Rio Grande do Sul from 1968 to 1976 were used. The duration of the experiments varied from one single cropping period — either wheat or soybeans — to eight consecutive cropping periods in a double cropping system. In most cases, three to five replications were used. Soil acidity had been corrected through the use of lime in all experiments. The source for P was superphosphate (triple); for K, potassium chloride; and for N, urea. The fertilizers were broadcast, and high yielding plant varieties were used in all experiments. The North Carolina soil test extractant (H_2SO_4 .0025 N + HC1 .005 N) was used to evaluate the levels of P and K in the soil. The soil test method for N was an

indirect measure through the organic matter percent content of the soil (H_2SO_4 + $Na_2Cr_20_7$/oxidation). Some of the experiments had a factorial design with three to five levels of N, P and K. However, since the model allowed for the independent estimation of yield responses to N, P and K. a number of experiments where only one of those elements had been tested was also included. For these experiments, the supply of all but one nutrient had been set at nonlimiting levels. The observations used in the statistical analysis are treatment means.

Results from the Estimation of Carryover Functions

Carryover functions were estimated only for phosphorus and potassium. In southern Brazil, soils have a low organic matter content and rain leaches surplus nitrogen. Hence, soil scientists do not consider that nitrogen has any significant carryover effects in the wheat-soybean double cropping system of southern Brazil. (Nitrogen is applied only for wheat; soybeans are inoculated with Rhizobium bacteria for N fixation.)

The data used to estimate the carryover functions came from a set of seven experiments conducted from 1973 to 1976 by the National Wheat research Center. Five of those experiments were located on soil having a 20 to 40 percent clay, and classified as "type 2." The other two experiments were located on soils having more than 40 percent clay, and classified as "type 1." The soil classification is considered of paramount importance to soil tests for phosphorus. More than 10 years of research led the local soil scientists to conclude that a soil test level of x ppm of P for a soil type 1 reflects the same availability of P as a soil test level of 2x ppm of P for a soil type 2 (see UFRGS, 1973, p. 3). The data had two important limitations for estimating the carryover functions. First, straw residue from the crops was removed rather than incorporated into the soil, as is the customary practice; therefore, fertilizer carryover may have been underestimated. Second, fertilizers were applied only to wheat; soybeans were carried as a "residual" crop making it impossible to estimate separate carryover functions for wheat and for soybeans. Thus, lagged yields were not considered in the carryover functions; rather "average" carryover functions were estimated for the two crops. Some of the results have been reported in Lanzer and Paris (1981) and Lanzer, Paris and Williams (1981).

Phosphorus

The carryover model adopted for phosphorus is:

$$P^s_t = \theta_p \, (P^s_{t-1} + \lambda_p^{-1} P^a_{t-1}) + u_t \qquad (3.1)$$

P^s_t = soil test level for P at the beginning of cropping period t (ppm of P),

P^a_t = quantity of phosphate fertilizer applied on period t (P_20_6 kg per ha),[7]

θ_p = geometric decline parameter,

λ_p = proportionality factor for phosphorus, and

u_t = random error $[u_t = \rho_p u_{t-1} + v_t;\ v_t\ v^5\ N(0, \sigma_p^2 I)]$.

The value of λ_p was known to vary with soil type. Agronomists consider that for southern Brazil:

$$\lambda_{p1} = 2\lambda_{p2} \quad (\text{or } \lambda_{p1}^{-1} = \frac{1}{2} \lambda_{p2}^{-1}).$$

---

[7]Phosphorus is measured in P_20_5 and potassium in K_20 (rather than in P and K) for two reasons. First, these were the original units selected by Brazilian agronomists for carrying out the fertilizer experiments. Second, fertilization tables in southern Brazil are, to this day, formulated in P_20_5 and K_20. The conversion between P_20_5 and P, and between K_20 and K, can be easily performed using: P x 2.29 = P_20_5, K x 1.20 = K_20.

Thus, the phosphorus carryover function for soil type 1 is:

$$P^s_t = \theta_p \, (P^s_{t-1} + \lambda^{-1}_{p1} P^a_{t-1}) + u_t,$$

whereas, for soil type 2 it is:

$$P^s_t = \theta_p \, (P^s_{t-1} + \lambda^{-1}_{p2} P^a_{t-1}) + u_t,$$

Thus, a combined carryover function can be written as:

$$P^s_t = \theta_p \, [P^s_{t-1} + \lambda^{-1}_{p1} \, (P^a_{t-1} + P^a_{p1} \, D)] + u_t \qquad (3.2)$$

where D is a dummy variable which equals zero for soil type 1 and equals 1 for soil type 2.

Lagging and multiplying (3.2) by ρ_p, the autocorrelation coefficient, and subtracting the result from (3.2) yields:

$$P^s_t = \rho_p P^s_{t-1} + \theta_p [P^s_{t-1} + \lambda^{-1}_{p1} \, (P^a_{t-1} + P^a_{t-1} \, D)] - \rho_p \theta_p \, [P^s_{t-2}$$

$$+ \lambda^{-1}_{p1} \, (P^a_{t-2} + P^a_{t-2} \, D)] + v_t \qquad (3.3)$$

where v_t is assumed to be a white noise term. Under this assumption, the least-squares estimates of ρ_p, θ_p and λ_p^{-1} are maximum likelihood estimates (Draper and Smith, 1966, p. 256).

The parameters of model (3.3) were estimated using the Gauss-Newton nonlinear least-squares procedures.[8] Results are presented in Table 4; the fit to the data is satisfactory. The estimates of $\lambda_{p1} = 48.2$ and $\lambda_{p2} = 24.1$ suggest that one ppm of P, for soil type 1, is equivalent to a fertilizer application of 48.2 kg of $P_2 0_5$ per ha and 24.1 kg of $P_2 0_5$ per ha for type 2 soils.[9] The estimated geometric decline coefficient is relatively high, that is, close to unity; thus, the level of soil phosphorus is seen to decrease at a relatively low rate across time, and phosphate fertilizers are estimated to possess a highly significant carryover effect. In summary, the estimated phosphorus carryover functions were:

$$P^s_t = 0.8895(P^s_{t-1} + 0.0207P^a_{t-1}) + u_t, \text{ for soil type 1,} \qquad (3.4)$$

$$P^s_t = 0.8895(P^s_{t-1} + 0.041P^a_{t-1}) + u_t, \text{ for soil type 2.} \qquad (3.5)$$

Table 4. Statistical Results for the Phosphorus Carryover Function.

Parameter	Estimate	Asymptotic Standard Error
ρ_p	-0.6747	0.0416
θ_p	0.8895	0.0209
λ^{-1}_{p1}	0.0207	0.0023

$R^2 = 0.7881$; MSE = 86.45; N = 345 observations

[8]Program BMDO7R (Dixon, 1974).

[9]An unconstrained model was also fitted to the available data. The estimates obtained in this case were $\lambda_{p1} = 51.7$ and $\lambda_{p2} = 22.8$. The MSE for the unconstrained model was 85.96.

Potassium

Agronomists' interpretation of southern Brazilian soil tests for potassium is currently made without allowance for soil classification. Therefore, the model to be estimated is reduced to:

$$K_t^s = \theta_k (K_{t-1}^s + \lambda_k^{-1} K_{t-1}^a) + u_t \qquad (3.6)$$

where

K_t^s = soil test level for potassium at the beginning of cropping period t (ppm of K).

K_t^a = quantity of potassium fertilizer applied on cropping period t ($K_2 0$ kg per ha),

θ_k = rate of geometric decline,

λ_k^{-1} = reciprocal of the proportionality constant for potassium, and

u_t = a random error $[u_t = \rho_k u_{t-1} + v_t; v_t \, v \, \epsilon \, v \sim N(0, \sigma^2 I)]$.

Lagging (3.6), multiplying both sides by ρ_k and subtracting the result from (3.6), yields at:

$$K_t^s = \rho_k K_{t-1}^s + \theta_k (K_{t-1}^s + \lambda_k^{-1} K_{t-1}^a) - \rho_k \theta_k (K_{t-2}^s + \lambda_k^{-1} K_{t-2}^a) + v_t \qquad (3.7)$$

where v_t assumed to be a white noise term. Results from the nonlinear least-squares estimation of (3.7) are presented in Table 5. While R^2 is considerably lower than that for the phosphorus carry over model (3.3), the asymptotic standard errors are small relative to their associated parameter estimates for the potassium carryover function, leading to some confidence in the use of the estimates. The estimate that $\lambda_k = 3.73$ suggests that each ppm of K is equivalent to an application of 3.73 kg of $K_2 0$ per ha. The estimated potassium carryover equation is:

$$K_t^s = 0.8139 (K_{t-1}^s + 0.2682 K_{t-1}^a) + u_t. \qquad (3.8)$$

Table 5. Statistical Results for the Potassium Carryover Function.

Parameter	Estimate	Asymptotic Standard Error
ρ_k	-0.4154	0.0488
θ_k	0.8139	0.0128
λ_k^{-1}	0.2682	0.0365

$R^2 = 0.3542$; MSE = 797.50; N = 420 observations

Results from the Estimation of Yield Response Functions

The first step in estimating the yield response functions was to sort the observations according to the response to a single nutrient. For example, to estimate the wheat response to phosphorus, only data were used for which the levels of both potassium and nitrogen were considered "high," based on the fertilizer recommendation tables currently in use.

The observation row includes: observed yield (y_{ij}), soil test level (x_{ij}^s), applied fertilizer (x_{ij}^a), and maximum observed yield of the experiment (M_j). The subscript i stands for the fertilizer treatment; the subscript j, for the experiment. Note that in contrast to its previous usage, "experiment" now assigns both a geographical and a temporal dimension to data subsets.

The total availability of nutrient was computed:

$$x^T_{ij} = \lambda_x x^s_{ij} + x^a_{ij}$$

where x^T_{ij} is the total availability of nutrient x for the i-th treatment of the j-th location, and λ_x is the proportionality constant estimated from the respective carryover functions. Note that x^T_{ij} is defined in terms of fertilizer units rather than in soil test units. Given the mesh of knot points 0, x_1, x_2, . . ., x_{K-1} and the definitions:

$$Z_{1ij} = x^T_{ij}$$

$$Z_{2ij} = \max (x^T_{ij} - x_1; 0)$$

$$Z_{3ij} = \max (x^T_{ij} - x_2; 0)$$

*

*

$$Z_{kij} = (x^T_{ij} - x_{k-1}; 0),$$

the spline formulation of the yield response to the total availability of x becomes:

$$y_{ij} = A_j \left(\sum_{m-1} \beta_m Z_{mij}\right) + e_{ij} \qquad (3.9)$$

where e_{ij} is assumed to be a white noise term. For this analysis, the knots were located at the middle of the soil test ranges that are used in current fertilizer recommendation tables.

Model (3.9) assumes that A_j, the expected yield plateau of the j-th experiment, is observable. The maximum observed yield of the j-th experiment, (M_j), on the other hand, is an order statistic that is likely to overestimate A_j. Thus, (3.9) was modified to:

$$y_{ij} = \alpha M_j \left(\sum_{m-1}^{m=k} \beta_m Z_{mij}\right) + e_{ij} \qquad (3.10)$$

where parameter α is considered as the expectation of a random coefficient x_j such that $M_j = \alpha_j^{-1} A_j$. In words: The highest order statistic for the yields of a given experiment is assumed to be (stochastically) proportional to the expected maximum of that experiment. Under this random coefficient assumption, e_{ij} must now be viewed as an heteroskedastic error term (see, for example, Theil, 1971, p. 623, but the least-squares estimates of the β_m are unbiased. Thus, since the set of available data for this research was relatively large, and since only point estimates are required by the optimization framework, (3.10) was estimated by a direct least-squares procedure.

The max ($\Sigma\beta_m Z_m$) = 1, that is, the maximum for relative yields equals 1. Figure 3 depicts the relative yield response (yr) to the total supply of nutrient X (X^T). For m = 1, 2, 3, 4:

$$yr_m = \beta_1 X_1 + (\beta_1 + \beta_2)(X_2 - X_1) + \ldots + (\beta_1 + \ldots + \beta_j)(X_m - X_{m-1}) \qquad (3.11)$$

where the X_m are known fixed knots and $\beta_1 = tg\gamma_1$, $(\beta_1 + \beta_2) = tg\gamma_2$, etc.

But concavity of the yield response requires that $\beta_m \leq 0$ for $m \geq 2$. Suppose that it were known that yr equals its maximum for some range of X^T starting at X_m ($m < 4$) and extending at least to X_4. In this case, the spline function must satisfy:

$$\beta_1 X_1 + (\beta_1 + \beta_2)(X_2 - X_1) + \quad + (\beta_1 + \quad + \beta_j)(X_4 - X_3) = 1 \qquad (3.12)$$

Solving for β_1, yields:

$$\beta_1 = \frac{1}{X_4} - \frac{(X_4 - X_1)}{X_4}\beta_2 - \frac{(X_4 - X_2)}{X_4}\beta_3 - \frac{(X_4 - X_3)}{X_4}\beta_4. \quad (3.13)$$

Equation (3.13) is a linear constraint on the parameters of (3.10) that ensures that relative yields attain a maximum which equals 1 at X_m (for $m \leq 4$) and extends at least to X_4 (provided that $\beta_m \leq 0$ for $m \geq 2$). Substituting (3.13) into (3.10), assuming $k = 4$, and rearranging terms yields:

$$y_{ij} = \alpha M_j \left[1/X_4 + \sum_{m=2}^{m=4} \beta_m \left(Z_{mij} - \frac{(X_4 - X_{m-1})}{X_4} Z_{1ij}\right)\right] + e_{ij} \quad (3.14)$$

or

$$y_{ij} = \alpha Z^*_{1ij} + \alpha\beta_2 Z^*_{2ij} + ab_3 Z^*_{3ij} + ab_4 Z^*_{4ij} + e_{ij} \quad (3.15)$$

where

$$Z^*_{1ij} = M_j X_4^{-1}$$

$$Z^*_{2ij} = M_j\left[Z_{2ij} - \frac{(X_4 - X_1)}{X_4} Z_{1ij}\right], \text{ etc.}$$

The parameters of (3.15) were estimated with a nonlinear least-squares algorithm which allowed the β_k estimates ($k > 2$) to be constrained to be nonpositive. The constraint assures that the resulting spline be a quasi-concave function (except for the possibility of an initial range of increasing returns if both β_1 and β_2 are positive).

Soybean Response Functions

Two yield response functions were estimated for soybeans: one for phosphorus and one for potassium.

Phosphorus

The knots chosen for the phosphorus response were:

X_1 = 75 kg P_2O_5/ha (approx. 1.5(3.0) ppm P for soil type 1(2)),

X_2 = 225 kg P_2O_5/ha (approx. 4.5(9.0) ppm P for soil type 1(2)),

X_3 = 375 kg P_2O_5/ha (approx. 7.5(15.0) ppm P for soil type 1(2)),

X_4 = 525 kg P_2O_5/ha (approx. 10.5(21.0) ppm P for soil type 1(2)),

X_4 = 675 kg P_2O_5/ha (approx. 13.5(27.0) ppm P for soil type 1(2)),

Because agronomists consider that a soil test level of 9(18) ppm P for soil type 1(2) is high for Southern Brazil the spline function was constrained to attain a maximum of 1, at 675 kg P_2O_5/ha or below. The model fitted to the empirical data was:

$$y_{ij} = \alpha M_j \left(675^{-1} + \sum_{m=2}^{m=6} \beta_m Z^*_{mij}\right) + e_{ij} \quad (3.16)$$

where

$$Z^*_{2ij} = Z_{2ij} - \frac{(675\text{-}75)}{675} Z_{1ij}$$

$$Z^*_{3ij} = Z_{3ij} - \frac{(675\text{-}225)}{675} Z_{1ij}$$

$$Z^*_{3ij} = Z_{3ij} - \frac{(675\text{-}225)}{675} Z_{1ij}$$

and given that

$$Z_{1ij} \begin{cases} = 48.26\ P^s + P^a \text{ for soil type 1} \\ = 24.13\ P^s + P^a \text{ for soil type 2} \end{cases}$$

$$Z_{2ij} = \max\ (Z_{1ij} - 75;\ 0)$$

$$Z_{3ij} = \max\ (Z_{1ij} - 225;\ 0), \text{ etc.}$$

(Recall that subscripts i, j stand for i-th treatment of the j-th experiment.)

Once the estimates of $\beta_m (m = 2, 3, \ldots, 6)$ are available, the estimate of β_1 is obtained from:

$$\beta_1 = \frac{1}{675} - \frac{600}{675}\beta_2 - \frac{450}{675}\beta_3 - \frac{300}{675}\beta_4 - \frac{150}{675}\beta_5. \qquad (3.17)$$

Constraint (3.17) does not include β_6 because it requires the spline to attain its maximum at or below the level of 675 kg P_2O_5/ha. The constraint, however, does not prevent relative yields from falling after this level. An additional condition that $\beta_6 = 0$ could be imposed to keep yields from falling after this maximum phosphorus application.

Model (3.16) was fitted to the empirical data using nonlinear least-squares. Parameter α was constrained to the interval (0.5, 1.5), and parameters β_m (m = 3,4,5,6) were constrained to be nonpositive (in order to ensure concavity of the response functions for levels of P^T equal to or above 225 kg (P_2O_5/ha).

Results of the estimated soybean response to phosphorus application are summarized in Table 6; the fit is satisfactory. The estimated grid of points for the relative yield (yr) spline function (as a fraction of maximum expected yield) is:

At P^T = O kg P_2O_5/ha yr = 0,

at P^T = 75 kg P_2O_5/ha yr = 0.549,

at P^T = 225 kg P_2O_5/ha yr = 0.868,

at P^T = 525 kg P_2O_5/ha yr = 0.997,

at P^T = 675 kg P_2O_5/ha yr = 1.

No change of slope occurred at the level of 375 kg P_2O_5/ha (the nonpositive constraint of β_4 was binding). The estimate for β_6 indicates that yields tend to decrease very slowly after the level of 675 kg P_2O_5/ha. This result, together with the relative yield estimate of 99.7 percent for P^T = 525 kg P_2O_5/ha, strongly suggests the presence of a yield plateau. Using separable programming, the relative yield response of soybean to total phosphorus can also be expressed as:

$$P^T = 0w_1 + 75w_2 + 225w_3 + 525w_4 + 675w_5$$

$$yr^{sp} = 0w_1 + 0.549w_2 + 0.968w_3 + 0.997w_4 + 1.000w_5 \qquad (3.18)$$

Table 6. Spline Regression Results for Soybean Response to Phosphorus.

Parameter	Estimate	Asymptotic Standard Error
α	0.906100	0.013332
β_1	0.007324[a]	—
β_2	-0.005194	0.000604
β_3	-0.001706	0.000299
β_4	0.000000[b]	—
β_5	-0.000397	0.000248
β_6	0.000060	0.000180

R^2 = 0.7885; MSE = 125.300; N = 340 observations.
[a]Computed according to equation (3.17).
[b]Binding constraint.

$$\sum_{i=1}^{i=4} w_i = 1;\ w_i \geq 0 \text{ for } i = 1,2,3,4$$

at two (adjacent) weights $w_i \neq 0$, where P^T stands for total supply of phosphorus (in kg. P_2O_5/ha) and yr^{sp} stands for relative yield response of soybeans to phosphorus (as a fraction of the expected maximum). P^T is computed as:

$$P^T \begin{cases} = 48.26\ P^s + P^a \text{ for soil type 1} \\ = 24.13\ P^s + P^a \text{ for soil type 2} \end{cases}$$

where P^s stands for soil test (ppm P) and P^a stands for applied phosphate (kg P_2O_5/ha).

Postassium

The knots chosen to estimate the soybean response to potassium were:

X_1 = 40 kg K_20/ha (approx. 10 ppm K),

X_2 = 110 kg K_20/ha (approx. 30 ppm K),

X_3 = 185 kg K_20/ha (approx. 50 ppm K),

X_4 = 260 kg K_20/ha (approx. 70 ppm K),

X_5 = 410 kg K_20/ha (approx. 110 ppm K).

The spline function was constrained to attain a maximum of 1 at or below the level of 410 kg K_20/ha, because agronomists consider a soil test level of 60 ppm K as nonlimiting. The model fitted to the empirical data was:

$$y_{ij} = \alpha M_j(410^{-1} + \sum_{m=2}^{m=6} \beta_m Z^*_{mij}) + e_{ij} \qquad (3.19)$$

where

$$Z^*_{2ij} = Z_{2ij} - \frac{(410 - 40)}{410} Z_{1ij}$$

$$Z^*_{3ij} = Z_{3ij} - \frac{(410 - 40)}{410} Z_{1ij} \text{ etc.}$$

and given that

$$Z_{1ij} = 3.7\ K^s_{ij} + K^a_{ij}$$

$$Z_{2ij} = \max\ (Z_{1ij} - 40;\ 0)$$

$$Z_{3ij} = \max\ (Z_{1ij} - 110;\ 0) \text{ etc.}$$

(Again, recall that subscripts i, j stand for the i-th treatment of the j-th experiment.) The point estimate of β_1, that is, the inclination of the yield response function in the interval K^T = (0;40)kg of K_20/ha, is given by:

$$\beta_1 = \frac{1}{410} - \frac{370}{410}\beta_2 - \frac{30}{410}\beta_3 - \frac{225}{410}\beta_4 - \frac{150}{410}\beta_5. \qquad (3.20)$$

The estimation procedure for model (3.19) was the same as for phosphorus, already described. The regression results are reported in Table 7. The R^2 statistic and the relatively small standard errors associated with the estimated parameters indicate that the fitted equation has a high predictive power. The estimated grid of points for the soybean response to potassium as a fraction of maximum expected yield and computed from the information presented in Table 7 is:

at K^T = 0 kg K_20/ha yr = 0

at K^T = 40 kg K_20/ha yr = 0.534

at K^T = 110 kg K_20/ha yr = 0.721

at K^T = 185 kg K_20/ha yr = 0.915

at K^T = 260 kg K_20/ha yr = 0.970

at K^T = 410 kg K_20/ha yr = 1.

The small numerical estimate for β_6 indicates that relative yields tend to decrease very slowly beyond K^T = 410 kg K_20/ha. Again, the existence of a yield plateau is strongly suggested. The relative yield response of soybean to total potassium can also be expressed as:

$$K^T = 0w_1 + 110w_3 + 185w_4 + 260w_5 + 410w_6$$

$$yr^{sk} = 0w_1 + 0.534\ w_2 + 0.721w_3 + 0.915w_4 + 0.970w_5 + 1.000w_6 \qquad (3.21)$$

$$\sum_{i=1}^{i=6} w_i = 1;\ w_i \geq 0 \text{ for } i = 1, \ldots, 6$$

Table 7. Spline Regression Results for Soybean Response to Potassium.

Parameter	Estimate	Asymptotic Standard Error
α	0.91240	0.010010
β_1	0.01335[a]	—
β_2	-0.01068	0.003481
β_3	-0.00008	0.000003
β_4	-0.00186	0.000975
β_5	-0.00053	0.000394
β_6	-0.00028	0.000198

R^2 = 0.8819; MSE = 50.960; N = 273 observations.
[a]Computed according to equation (3.20).

at two (adjacent) $w_i \neq 0$, where K^T stands for total potassium (kg K_20/ha) and yr^{sk} stands for relative yield response of soybean to potassium (fraction of the expected maximum yield). K^T is defined as:

$$K^T = 3.73\ K^s + K^a$$

where K^s stands for soil test level (ppm K) and K^a stands for applied potassium (kg K_20/ha).

Wheat Response Functions

Yield response functions for phosphorus, potassium, and nitrogen were estimated for wheat. The methods employed for P and K are identical to those estimated for soybeans (including the choice of knots); therefore, the results are presented directly. The nitrogen function will be presented in more detail.

Phosphorus

The results from fitting a spline to the wheat response to phosphorus (Table 8) are statistically satisfactory. The $\beta_2 > 0$ implies increasing returns for relatively low levels of phosphorus — a finding not observed for soybeans. The estimated grid of points for the wheat response to phosphorus (as a fraction of maximum expected yield) is:

at P^T = 0 kg P_20_5/ha yr = 0

at P^T = 75 kg P_20_5/ha yr = 0.197

at P^T = 225 kg P_20_5/ha yr = 0.799

at P^T = 375 kg P_20_5/ha yr = 0.905

at P^T = 675 kg P_20_5/ha yr = 1.

Another way of expressing the estimated spline is:

$$P^T = 0w_1 + 75w_2 + 225w_3 + 375w_4 + 675w_5$$

$$yr^{wp} = 0w_1 + 0.197w_2 + 0.799w_3 + 0.905w_4 + 1.00w_5 \qquad (3.22)$$

$$\sum_{i=1}^{i=5} w_i = 1;\ w_i \geq 0 \text{ for } i = 1, \ldots, 5$$

Table 8. Spline Regression Results for Wheat Response to Phosphorus.

Parameter	Estimate	Asymptotic Standard Error
α	0.883700	0.017720
β_1	0.002630[a]	—
β_2	0.001386	0.001091
β_3	-0.003316	0.000653
β_4	-0.000380	0.000413
β_5	0.000000[b]	—
β_6	-0.000313	0.000146

R^2 = 0.9009; MSE = 59.070; N = 179 observations.
[a]Computed according to equation (3.17).
[b]Binding constraint.

at two (adjacent) $w_i \neq 0$, where P^T stands for total phosphorus (kg $P_2 0_5$/ha) and yr^{wp} stands for relative yield of wheat. Recall of P^T is defined as:

$$P^T \begin{cases} = 48.26P^s + P^a \text{ for soils type 1} \\ = 24.13P^s + P^a \text{ for soils type 2} \end{cases}$$

where P^s stands for soil test level (ppm P) and P^a stands for applied phosphate (kg $P_2 0_5$/ha).

Potassium

The results from fitting a spline for wheat response to potassium (Table 9) are also satisfactory. The estimated grid of points for the wheat response to potassium (as a fraction of maximum expected yield:

at K^T = 0 kg $k_2 0$/ha yr = 0

at K^T = 40 kg $k_2 0$/ha yr = 0.83

at K^T = 110 kg $k_2 0$/ha yr = 0.553

at K^T = 185 kg $k_2 0$/ha yr = 0.885

at K^T = 260 kg $k_2 0$/ha yr = 0.970

at K^T = 410 kg $k_2 0$/ha yr = 1.

The negative estimate, -0.83 at K^T = 40 kg $k_2 0$/ha is meaningless because there were no observations in the range of 0 to 40 kg $K_2 0$/ha. Therefore, the first spline segment "floated" free in order to link the lower point estimate for the second spline segment; for all practical purposes, the knot (40, -0.83) can be excluded from the analysis. Thus, the estimated spline can also be written as:

$$K^T = 0w_1 + 110w_2 + 185w_3 + 260w_4 + 410w_5$$

$$yr^{wk} = 0w_1 + 0.555w_2 + 0.885w_3 + 0.970w_4 + 1.000w_5 \qquad (3.23)$$

$$\sum_{i=1}^{i=5} w_i = 1;\ w_i \geq 0 \text{ for } i = 1, \ldots, 5$$

at two (adjacent) $w_i \neq 0$.

Table 9. Spline Regression Results for Wheat Response to Potassium.

Parameter	Estimate	Asymptotic Standard Error
α	0.88230	0.01916
β_1	-0.02074[a]	—
β_2	0.04048	0.03755
β_3	-0.01531	0.01525
β_4	-0.00331	0.00308
β_5	-0.00092	0.00148
β_6	-0.00032	0.00037

R^2 = 0.9485; MSE = 34,440; N = 125 observations.
[a]Computed according to equation (3.20).

Nitrogen

The estimation method for wheat response to nitrogen (Table 10) differs from that for phosphorus and potassium for two main reasons. First, the agronomists' soil test method used to evaluate the supply of N is not a direct measurement, as it is in the case of P and K. Rather, nitrogen is indirectly evaluated by analyzing the organic matter content of the soil. Second, nitrogen carryover is not important in the wheat-soybean double cropping system, in part because soybeans can produce their own nitrogen via the *Rhizobium* bacteria. Therefore, an explicit carryover function was not needed for N. Instead, the relation between soil nitrogen, applied nitrogen, and yields was estimated directly via the yield response function. The model adopted was based on the work of Mitscherlich and of Bray (recall equation (1.1)):

$$y_{ij} = A_j \left(1 - e^{-c^*N^s_{ij} - cN^a_{ij}}\right) + v_{ij} \tag{3.24}$$

where

y_{ij} = yield obtained on the i-th treatment of the j-th experiment (kg of wheat per ha)

A_j = expected asymptotic yield plateau of the j-th experiment (as before, it is assumed that $M_j = a_j{}^1A_j$ and $E(a_j) = \alpha$; M_j is the maximum yield observed on the j-th experiment)

N^s_{ij}= soil test level for nitrogen (percent organic matter)

N^a_{ij}= quantity of applied nitrogen fertilizer (kg N/ha)

v_{ij} = white Gaussian noise term and

c^* = a parameter of the model such that $c^* = c\lambda_n$, where λ_n is the proportionality factor between percentage organic matter and kg N/ha.

In (3.24), the term in parenthesis is the relative yield response of wheat (yr) to total nitrogen (N^T). Total nitrogen, in turn, is defined as:

$$N^T = \lambda_n N^s + N^a. \tag{3.25}$$

From (3.25) it is clear that N^T will be measured in kg N/ha. Model (3.24) was estimated by nonlinear least-squares with A_j being substituted for αM_j of (3.10). Results (Table 10) indicate a statistically satisfactory fit for the wheat response to nitrogen. The estimated proportionality factor for nitrogen is $\lambda_n = 13.1$ ($=c^*/c$), suggesting that each percentage unit of organic matter is equivalent to an application of 13.1 kg N/ha. Thus, the estimated wheat response to nitrogen can also be written as:

$$Y = A \left(1 - e^{-0.0429N^T}\right) \tag{3.26}$$

$$N^T = 13.1\ N^s + N^a. \tag{3.27}$$

Table 10. Regression Results for Wheat Response to Nitrogen.

Parameter	Estimate	Asymptotic Standard Error
α	0.8507	0.0133
c^*	-0.5634	0.0557
c	-0.0429	0.0299

R^2 = 0.8823; MSE = 60,040; N = 158 observations.

The term in parentheses in (3.26) is the relative yield response of wheat to total nitrogen (yr^{nw}). Because a grid of points approximately this relative yield response was needed for the separable programming model, the following function was specified:

$$N^T = 0w_1 + 30w_2 + 60w_3 + 90w_4 + 120w_5$$

$$yr^{nw} = 0w_1 + 0.724w_2 + 0.924w_3 + 0.978w_4 + 1.000w_5 \qquad (3.28)$$

$$\sum_{i=1}^{i=5} w_i = 1;\ w_i \geq 0 \text{ for } i = 1, \ldots, 5$$

at two (adjacent) $w_i \neq 0$.

This completes the estimation of the physical relationships among yields, soil tests and applied fertilizers. Based on these relationships we turn to the economic optimization of fertilizer use.

Programming Results and Economic Analysis

The moving horizon concept for a planning period of four years (or eight consecutive cropping periods) was used for the economic analysis. Besides the estimates of carryover and relative yield functions, the programming formulation requires estimates for expected yield plateaus of wheat and soybeans which in southern Brazil are 1,800 kg/ha and 2,800 kg/ha, respectively.[10] The prices used in the basic programming model were:[11]

nitrogen: Cr$ 5.61/kg of N

phosphorus: Cr$ 7.06/kg of $P_2 0_5$

potassium: Cr$ 2.49/kg of $K_2 0$

wheat: Cr$ 2.03/kg

soybeans: Cr$ 1.84/kg

These prices, which we will refer to as 1976 prices, do not include the 40 percent subsidy for fertilizers that went into effect in 1974 and was revoked at the beginning of 1977.

The programming formulation to establish the optimum levels of soil fertility for the wheat-soybean double-cropping system was:

(1) **Objective Function**

$$\max PV_p = PVR - PVVC$$

where

PV_p = present value of expected net revenues.

PVR = present value of expected total revenues, and

PVVC = present value of expected fertilizer costs;

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[10]The estimates were derived at a series of seminars held by the National Wheat Research Center; they are based on opinions of researchers, extension specialists, and farmers who attended the seminars (see EMBRAPA, 1975a, and 1975b).

[11]This set of prices reflects average prices in the wheat-soybean producing area of southern Brazil in 1976 (see Stulp, 1977).

(2) **Constraints**

First:

$$PVR = \frac{3654}{1.03} yr_1 + \frac{5152}{1.03^2} yr_2 + \frac{3654}{1.03^3} yr_3 + \frac{5152}{1.03^4} yr_4 + \ldots + \frac{5152}{1.03^8} yr_8$$

where yr_j is relative yield obtained in the j-th cropping period. The index j = 1,3,5,7 indicates wheat, while j = 2,4,6,8 indicates soybeans. The coefficient 3,654 (= 1.800 kg of wheat/ha times Cr$ 2.03 per kg of wheat) represents the expected revenue plateau per ha of wheat (in Cr$/ha). The coefficient 5.152 (= 2,800 kg of soybeans/ha times Cr$ 1.84 per kg of soybeans) represents the expected revenue plateau per ha of soybeans (in Cr$/ha). A subsidized semiannual interest rate of 3 percent is used to discount future revenues — the rate officially adopted for savings accounts ("cadernetas de poupanχa") in Brazil. The cost of fertilizer applications is assumed constant.

Second:

$$PVVC = 5.61N^a_1 + 7.06P^a_1 + 2.49K^a_1 + \frac{7.06}{1.03} P^a_2 + \frac{2.49}{1.03} K^a_2 + \frac{5.61}{1.03^2} N^a_3$$

$$+ \frac{7.06}{1.03^2} P^a_3 + \frac{2.49}{1.03^2} K^a_3 + \ldots + \frac{2.49}{1.03^7} K^a_8$$

where x^a_j is the quantity of nutrient X added to the soil (fertilizer) at the beginning of the j-th cropping period. Recall that no nitrogen is applied for soybeans.

Third:

$$N^s_1 = \text{given}$$

$$N^T_j = 13.13\, N^s_1 + N^a_j, \qquad j = 1,3,5,7$$

where N^T_j represents the total quantity of nitrogen available for the plants (kg N/ha) in the j-th cropping period, N^s_1 is the percentage of organic matter in the soil (assumed to be constant because of lack of information), and N^a_j is nitrogen applied to the soil as fertilizer in the j-th cropping period (kg N/ha).

Fourth:

$$P^s_1 = \text{given},$$

$$P^s_{j+1} = 48.26\, P^s_j + P^a_j, \qquad j = 1, \ldots, 8,$$

$$P^s_{j+1} = 0.8895(P^s_j + 0.02072\, P^a_j), \qquad j = 1, \ldots, 8.$$

This set of equations describes the phosphorus supply over time. P^T_j is the quantity of phosphorus available for the plants in the beginning of the j-th cropping period (kg P_2O_5/ha). P^s_j is the soil test level for phosphorus in the beginning of the j-th cropping period (in ppm of P). P^a_j is the quantity of phosphorus applied in the beginning of the j-th cropping period (kg P_2O_5 per ha). The coefficients for soil type 1 were used in the analysis; the only difference between soil types 1 and 2 is the "exchange rate" factor λ_p, so the adjustment of results later is straightforward.

Fifth:

$$K^s = \text{given}$$

$$K^T_j = 3.73\, K^s_j + K^a_j$$

$$K^s_{j+1} = 0.8139\, (K^s_j + 0.2692\, K^a_j).$$

This set of equations describes the dynamics of potassium supply. K_j^T is the quantity of potassium available for the plants in the beginning of the j-th cropping period (kg K_20/ha). K_j^s is the soil test level of potassium in the beginning of the j-th cropping period (ppm K). K_j^a is potassium fertilizer applied in the beginning of the j-th cropping period (kg K_20/ha).

Sixth:

$$yr_j \leq 0w_{1nj} + 0.72w_{2nj} + 0.924w_{3nj} + 0.978w_{4nj} + 1.000w_{5nj}$$

$$N_j^T = 0w_{1nj} + 30w_{2nj} + 60w_{3nj} + 90w_{4nj} + 120_{5nj}$$

$$1 = w_{1nj} + w_{2nj} + w_{3nj} + w_{4nj} + w_{5nj}, \qquad j = 1,3,5,7$$

$$yr_j \leq 0w_{1pj} + 0.197w_{2pj} + 0.799w_{3pj} + 0.905w_{4pj} + 1.000w_{5pj}$$

$$P_j^T = 0w_{1pj} + 75w_{2pj} + 225w_{3pj} + 375w_{4pj} + 675w_{5pj}$$

$$1 = w_{1pj} + w_{2pj} + w_{3pj} + w_{4pj} + w_{5pj}, \qquad j= 1,3,5,7$$

$$yr_j \leq 0w_{1kj} + 0.555w_{2kj} + 0.885w_{3kj} + 0.970w_{4kj} + 1.000w_{5kj}$$

$$K_j^T = 0w_{1kj} + 110w_{2kj} + 185w_{3kj} + 260w_{4kj} + 410w_{5kj}$$

$$1 = w_{1kj} + w_{2kj} + w_{3kj} + w_{4kj} + w_{5kj}, \qquad j = 1,3,5,7.$$

This set of restrictions describes the relative yield of wheat in the first, third, fifth and seventh cropping periods as a function of the available supplies of N, P and K; see equations (3.22), (3.23) and (3.28).

Seventh:

$$yr_j \leq 0w_{1pj} + 0.549w_{2pj} + 0.868w_{3pj} + 0.997w_{4pj} + 1.000w_{5pj}$$

$$P_j^T = 0w_{1pj} + 75w_{2pj} + 225w_{3pj} + 525w_{4pj} + 675w_{5pj}$$

$$1 = w_{1pj} + w_{2pj} + w_{3pj} + w_{4pj} + w_{5pj}, \qquad j = 2,4,6,8$$

$$yr_j \leq 0w_{1kj} + 0.534w_{2kj} + 0.721w_{3kj} + 0.915w_{4kj} + 0.970w_{5kj} + 1.000w_{6kj}$$

$$K_j^T = 0w_{1kj} + 40w_{2kj} + 110w_{3kj} + 185w_{4kj} + 260w_{5kj} + 410w_{6kj}$$

$$1 = w_{1kij} + w_{2kj} + w_{3kj} + w_{4kj} + w_{5kj} + w_{6kj}, \qquad j = 2,4,6,8$$

This set of restrictions describes the relative yield of soybean in the second, fourth, sixth, and eighth cropping periods as a function of the available supplies of P and K; see equations (3.18) and (3.21).

The programming model was completed with nonnegativity constraints on all variables. There were 196 variables in all (excluding slacks) and 127 constraints. However, only the optimum levels of nutrients found for the first two cropping periods (N_1^T, P_1^T, K_1^T, P_2^T and K_2^T) are of interest for the analysis, because these levels constitute "soil fertility targets" or optimum nutrient stocks to be maintained for wheat and soybeans. This interpretation derives from the moving horizon concept applied here: It is assumed that the decision maker (farmer) is continuously either on the first (wheat) or second (soybean) period of a total of eight consecutive planning periods.

The computations were carried out by conventional linear programming procedures. No separable programming algorithm was required because the model had the appropriate convexities.

Of special interest in the analysis was an evaluation of the stability of the soil fertility targets (N_1^T, P_1^T, K_1^T, P_2^T and K_2^T) with respect to changes in input prices. First, it was found that parameterization of the initial soil fertility conditions (N_1^s, P_1^s, and K_1^s) did not affect the computed optimum soil fertility targets (N_1^T, P_1^T, K_1^T, T_2^T and K_2^T):

(a) optimum soil fertility targets for wheat (1976 prices)

N_1^T = 57.2 kg N/ha or 4.35 percent organic matter

P_1^T = 375 kg P_20_5/ha or 7.8(15.6) ppm P for soil type 1(2)

K_1^T = 202 kg K_20/ha or 54.3 ppm K.

(b) optimum soil fertility targets for soybean (1976 prices)

P_2^T = 456 kg P_20_5/ha or 9.4(18.8) ppm P for soil type 1(2)

K_2^T = 260 kg K_20/ha or 69.7 ppm K.

Next, the input prices were made to vary within the interval of 0.6 to 1.4 times the 1976 prices (Table 11 and Figure 4). Figures 4A, 4B, and 4C show the stability or the fertility targets for wheat (dashed lines) and for soybeans (solid lines). The optimum fertility targets (stocks) for both wheat and soybeans are relatively stable with respect to changes in fertilizer prices (particularly with respect to increases in fertilizer prices). For example: A decrease of 30 percent in fertilizer prices, *certeris paribus*, increases the nitrogen target for wheat by only 4.9 percent whereas the phosphorus and potassium targets for the same crop are increased by 16.0 and 8.3 percent, respectively. The relative changes in output levels would be even smaller. Such a stability is possibly due to the high carryover effect of both phosphorus and potassium fertilizers. In any event, it seems that the optimum fertility targets computed at current prices can be viewed as solid lower bounds for the purpose of making fertilizer recommendations. In this regard, it appears that these recommendations are relatively well protected against the possibility of small errors in the estimates of the coefficients of the programming model (particularly against a possible overestimation of the expected yield plateaus for wheat and soybeans).

With these results in mind, the analysis turns to a critical evaluation of the fertilizer recommendation tables currently used for the southern Brazilian wheat-soybean double cropping system. The tables do not make a distinction between soil fertility targets for wheat and for soybeans, i.e., for both cases, soil scientists recommend that a level of 9 (18) ppm of P for soil type 1(2) and of 60 ppm of K be maintained in the soil (UFRGS, 1973). Thus, it appears that only minor modifications to the tables are required with respect to target levels for phosphorus and potassium; the relative differences from the table targets to the computed optima are in the range of 5 to 15 percent. In the analysis to follow, the targets in the tables will be assumed to be satisfactory approximations to the optima.[12]

In order *to maintain* the levels of soil fertility for P and K at their desired level, soil scientists in souther Brazil recommend an application of 75 kg P_20_5/ha and 40 kg K_20/ha for each cropping period. Such "maintenance recommendations;; can be evaluated by the carryover equations (3.4), (3.5) and (3.8). By treating (3.4) and (3.5) deterministically, letting $P_{t-1}^s = P_t^s = P_*^s$ (where P_*^T is defined as the desired target level for P in ppm)

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[12]A static (one period) optimization for wheat and for soybeans independently leads to soil fertility targets that are 20 to 50 percent below the targets computed under the moving horizon concept. The static optimization targets for wheat, under current prices, are: N^T = 165.6 kg K_20/ha. The static optimization targets for soybeans, under current prices, are: P^T = 225.0 P_20_5/ha and K^T = 167.2 kg K_2o/ha. These results further illustrate the need for considering fertilizer carryover in the economic analysis of fertilizer data.

Table 11. Stability of Optimum Soil Fertility Stocks (Targets) for Wheat and Soybeans with Respect to Fertilizer Price Changes.

Percentage change in fertilizer prices		Targets for Wheat			Targets for Soybeans		
	N[a]	P[b]	K[c]	yr[d]	P[b]	K[c]	yr[d]
-40%	60.0	435.0	219.4	92.4	525.0	410.0	100.0
	(4.6)	(9.0)	(58.8)		(10.9)	(109.0)	
-30%	60.0	435.0	219.4	92.4	525.0	410.0	100.0
	(4.6)	(9.0)	(58.8)		(10.9)	(109.0)	
-20%	60.0	435.0	219.4	92.4	489.0	331.0	98.4
	(4.6)	(91.0)	(58.8)		(10.1)	(88.7)	
-10%	57.2	375.0	202.6	90.5	489.0	331.0	98.4
	(4.3)	(7.8)	(54.3)		(10.1)	(68.7)	
0%	57.2	375.0	202.6	90.5	456.0	260.0	97.0
	(4.3)	(7.8)	(54.3)		(9.4)	(69.7)	
+10%	57.2	375.0	202.6	90.5	454.0	258.8	96.9
	(4.3)	(7.8)	(54.3)		(9.4)	(69.4)	
+20%	57.2	375.0	202.6	90.5	421.6	239.3	96.9
	(4.3)	(7.8)	(54.3)		(8.7)	(64.2)	
+30%	54.2	346.7	185.0	88.5	421.6	239.3	95.5
	(4.1)	(7.8)	(54.3)		(8.7)	(64.2)	
+40%							

[a]The first entry is in kg N/ha; the entry in parenthesis is in percentage organic matter.
[b]The first entry is in kg P_2O_5/ha; the entry in parenthesis is in ppm P for soil type 1 (for soil type 2: multiply the entry by 2).
[c]The first entry is in kg K_2O/ha; the entry in parenthesis is in ppm K.
[d]Predicted relative yields, in terms of percentage of the expected yield plateau.

and solving for $P^a_{t-1} = P^a_*$ (where P^a_* is defined as the maintenance application of P in kg P_2O_5/ha/cropping period), the estimated fertilizer application to maintain the level of soil phosphorus at any level P^a_* is given by:

$$P^a_* = 5.996\ P^s_* \qquad \text{for soil type 1} \tag{3.29}$$

or,

$$P^a_* = 2.998\ P^s_* \qquad \text{for soil type 2} \tag{3.30}$$

Similarly, from equation (3.8), by letting $K^s_{t-1} = K^s_t = K^s_*$ (where K^s_* is defined as the desired target for K in ppm) and solving for $K^a_{t-1} = K^a_*$ (where K^a_* is defined as the maintenance application of K in kg K_2O/ha/cropping period), the fertilizer application required to maintain the level of soil K at its desired level is given by:

$$K^a_* = 0.853\ K^s \qquad \text{for all soil types.} \tag{3.31}$$

The soil fertility targets associated with current recommendations are $P^s_* = 9$ (18) ppm of P for soil type 1(2) and $K^s_* = 60$ ppm of K for all soil types. By substituting these values into (3.29) to (3.31), the maintenance application levels for P and K should be, approximately, $P^a_* = 54$ kg P_2O_5/ha/cropping period and $K^a_* = 51.2$ kg K_2O/ha/cropping period. These levels contrast with the maintenance levels recommended in southern Brazil of 75 kg P_2O_5/ha/cropping period and 40 kg K_2O/ha/cropping period. Therefore, the recommendations for "maintenance P" are overestimated by about 48 percent, whereas the recommendations for "maintenance K" are underestimated by about 28 percent. If these recommendations were followed for a long period of time, it seems likely that soil phosphorus would be built up to a level above the 9(18) ppm target for soil type 1(2). For potassium, the maintenance recommendation of the tables seem to be insufficient for keeping the level of soil K at its desired target (60 ppm of K of all soil

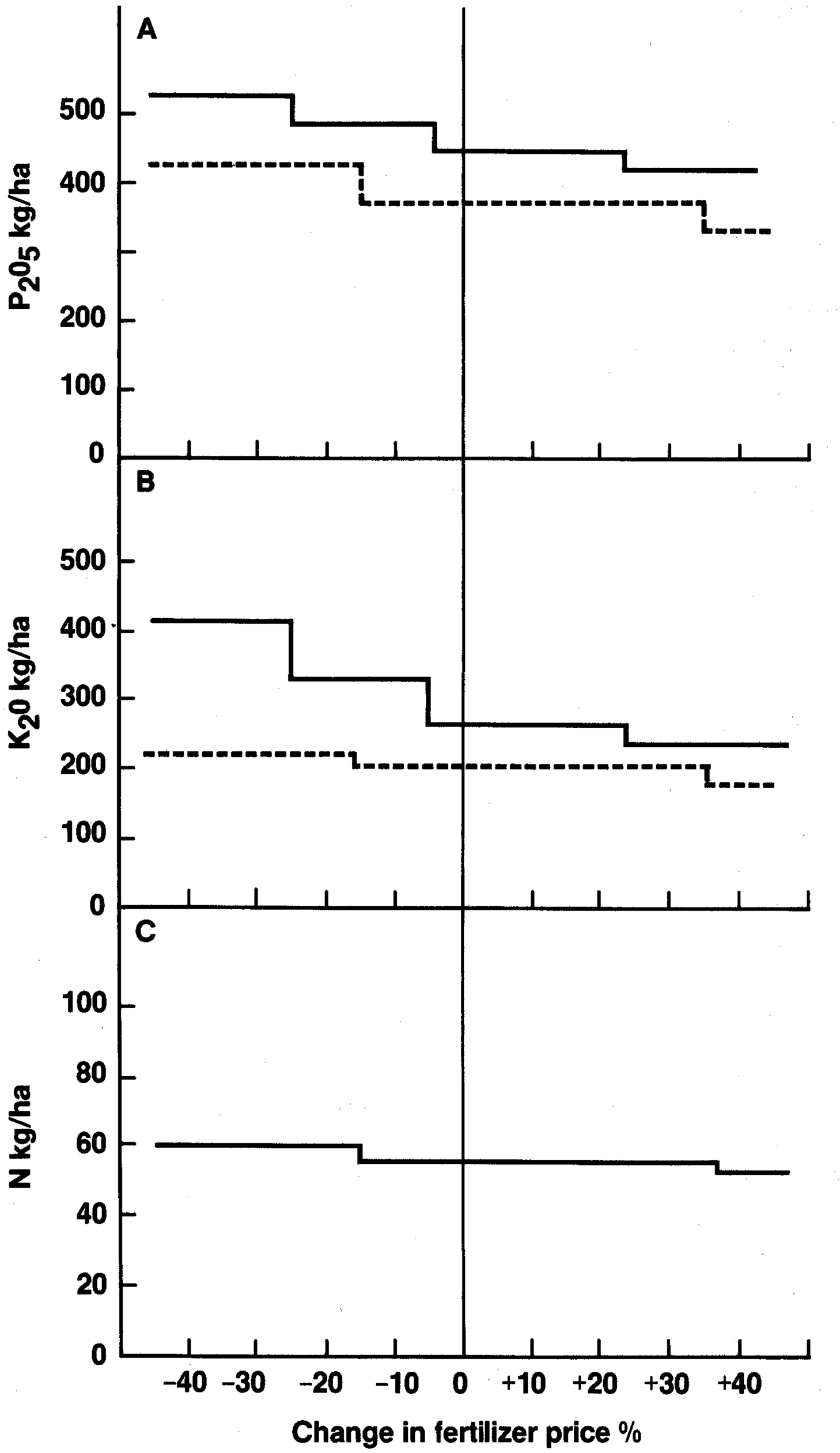

Figure 4. Stability of phosphorus (4A), potassium (4B) and nitrogen (4C) targets for wheat (dashed lines) and soybeans (solid lines) with respect to changes in fertilizer prices.

types).[13] In short: Even though the target levels for P and K are close to the computed optimum (the relative differences were in the range of 5 to 15 percent), it seems that major modifications of the tables are required as far as maintenance P and maintenance K are concerned, for the relative differences found here are in the range of 30 to 50 percent (Table 12).

Table 12. Target Levels and Maintenance Applications Recommendations for Phosphorus and Potassium.

	Current Recommendations		Adjusted Recommendations	
Nutrient	Target[a]	Maintenance[b,c]	Target[a]	Maintenance[b,c]
Phosphorus	9.0	75.0	9.0	54.0
Potassium	60.0	40.0	60.0	51.2
	Optima for Wheat		Optima for Soybeans	
Nutrient	Target[a]	Maintenance[b,c]	Target[a]	Maintenance[b,c]
Phosphorus	7.8	46.8	9.4	56.4
Potassium	54.3	47.4	69.7	60.9

[a]Target levels in ppm units (soil type 1 units for P).
[b]Maintenance levels in kg $P_2 0_5$/ha/cropping period for P and kg $K_2 0$/ha/cropping period for K.
[c]Computed from equations (4.29) and (4.31).

The carryover equations can also provide the information on how much fertilizer is required to change the soil test level found at the beginning of cropping period t to the desired target at the beginning of cropping period t+1. The fertilizer used for the purpose of this change is generally called "corrective fertilizer" (in contrast to the "maintenance fertilizer," the amount required to keep the soil fertility at the desired target through time). To attain soil fertility targets of 9 ppm of P (for soil type 1) and 60 ppm of K in the beginning of cropping period t, given that the soil test levels of period t-1 were P^s_{t-1} and K^s_{t-1} the corrective fertilizer requirements are given by:

$$P^a_{corr} = 488.8 - 48.31\ P^s_{t-1} \tag{3.32}$$

or,

$$K^a_{corr} = 274.9 - 3.73\ K^s_{t-1}. \tag{3.33}$$

(Equations (3.32) and (3.33) were computed from equations (3.4) and (3.8) by setting P^{sT}_t and K^{sT}_t at their target levels and solving for the applied fertilizer quantity.) However, since a maintenance recommendation is also made for each period (54.0 kg $P_2 0_5$/ha/cropping period and 51.2 kg $K_2 0$/ha/cropping period), the corrective recommendations should be diminished by that amount. Thus, the corrective recommendation for soil targets of $P^s_* = 9.0$ ppm of P (soil type 1) and $K^s_* = 60$ ppm of K should be modified to:

$$P^a_{corr} = 434.8 - 48.31\ P^s_{t-1} \tag{3.34}$$

and

$$K^a_{corr} = 223.7 - 3.73\ K^s_{t-1} \tag{3.35}$$

[13]We note that the fertilizer recommendation tables have been recently revised in order to increase the maintenance K recommendation from 40 to 50 kg $K_2 0$/ha/cropping period.

Equations (3.34) and (3.35) are graphed in Figures 5A and 5B. These figures also include the corrective recommendations from the tables currently employed by southern Brazilian agronomists (dashed lines). Figures 5A and 5B show that corrective recommendations are quite conservative for soils of every low fertility levels. In other words: It seems unlikely that the corrective fertilizer recommendations of P and K currently adopted in southern Brazil can actually drive soil test levels to their desired targets within only one cropping period. If these recommendations are followed sequentially, farmers may need two to three "corrective" applications (plus "maintenance") in order to hit the desired target levels. On the other hand, this fractioning of the corrective fertilization may be an optimal strategy under capital rationing conditions. Therefore, this point is less of a problem than it might at first appear.

The optimum target level computed for nitrogen was 57.2 kg N/ha/cropping period of wheat. The same level of N is equivalent to a soil test level of 4.5 percent organic matter (using the proportionality factor for N of 13.1). Current nitrogen recommendations for wheat are overestimated with respect to the computed optimum (Figure 5C). The differences accentuate as the level of organic matter increases from 0 to 5 percent. It also seems advisable to make a finer division of classes of soil organic matter percentage content in the tables: Only three classes are currently adopted. At very low levels of soil organic matter, the computed optimum recommendation is close to the current recommendation. The larger difference between the recommendations for higher levels of soil organic matter is caused by a difference in the estimates of nitrogen supply capacity from soil organic matter; in this study it has been estimated that each percentage point of soil organic matter can supply 13.1 kg N/ha/cropping period of wheat. The estimate adopted in the tables, however, is more conservative: approximately 8 kg N/ha/cropping period of wheat for each percentage point of soil organic matter. The difference between there estimates is probably due to the fact that soil pH correction (through limestone applications) has been made for all observations included in this study, and when soil pH is corrected, the supply of nitrogen produced by any given amount of organic matter is significantly increased. The recommendations made by the agronomists, on the other hand, do not necessarily assume that soil pH has been corrected through the use of lime. The result is the agronomists' more conservative estimate for the equivalence factor between soil organic matter and applied nitrogen. In any event, since pH correction itself is also recommended in the tables, consistency requires that the pH factor should be taken into account in making fertilizer recommendations.

In conclusion, it seems that some adjustments on fertilizer recommendation tables for the wheat-soybean system would be highly worthwhile. The maintenance levels recommended for P are too high; a reduction from 75 to 55 kg P_2O_5/ha/cropping period is strongly suggested. For potassium, on the contrary, it appears that the maintenance recommendation should be increased from 40 to 50 kg K_2O/ha/cropping period (and perhaps even a little more for soybeans). For nitrogen it was found that the recommendations could be somewhat reduced, particularly for soils where lime has been applied. It is also suggested that the number of classes of soil organic matter be increased from the current number of three (Figure 5C). The above suggestions are based on the assumption that soil fertility targets are kept at the levels adopted in the tables, as these levels were found to be relatively close to the computed optima (see Table 12). Nevertheless, it is also suggested that the tables could be improved by making a distinction between fertility targets for wheat and for soybeans. Moreover, as the optimum target levels of P and K for soybeans were found to be higher than those computed for wheat, it seems reasonable to recommend higher maintenance levels for soybeans and no P and K at all for the sequential wheat crop.

Finally, it seems important to evaluate the probable gains from the recommended changes. First, since the suggestions assume that current fertility targets will be main-

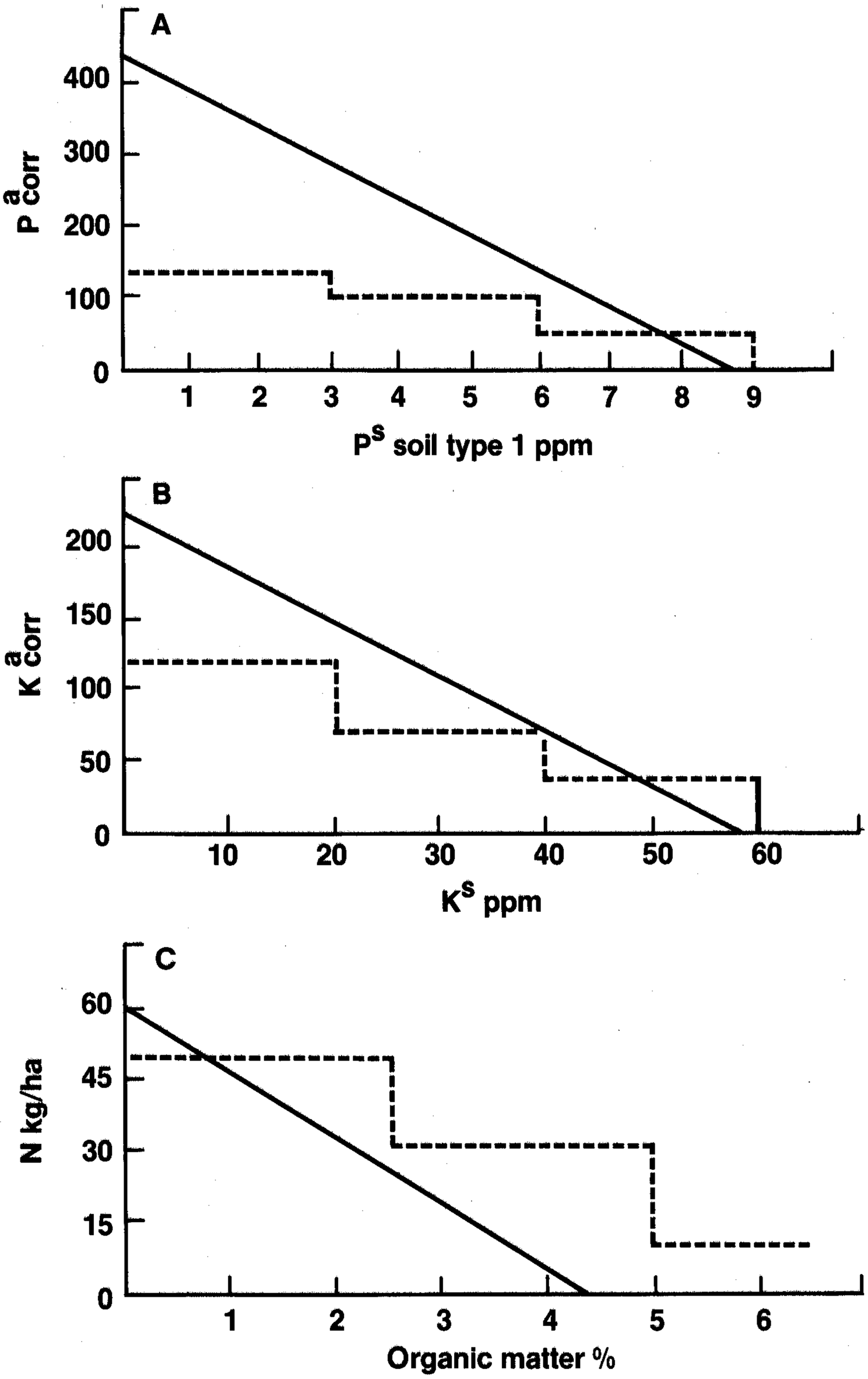

Figure 5. Optimal (solid line) and recommended (dashed line) quantities of P (5A), K (5B), and N (5C) required to change soil tests to desired levels.

tained, no change in output levels would be expected. The gains from the changes would come instead from reduced fertilizer costs. Table 13 summarizes the likely results of the changes in recommended levels of fertilizer use.

Under the table's recommendations, the yearly fertilizer costs of one ha of wheat-soybeans are approximately Cr $1,426. If the suggested changes are implemented, this cost can be reduced to Cr $1,137. The reduction in costs for fertilizer is Cr $288/ha/year or US $27.30/ha/year, at July 1976 exchange rates. The relative decrease in yearly fertilizer costs would be around 20 percent, a significant amount. The costs of changing the tables, on the other hand, are small and, in practice, can be assumed to be insignificant.

4. CONCLUSIONS AND THE ROAD AHEAD

The explicit adoption of agronomists' conception of the fertilizer application problem has produced a series of analytical results of interest to both agricultural economists and soil scientists, albeit for different reasons. For agricultural economists, the significance may lie in the possibility of better understanding the fertilizer problem for what it really is: a dynamic phenomenon subject to specific agronomic principles. Soil scientists, who had correctly perceived and treated the fertilizer problems as a dynamic phenomenon, may benefit from its formalization here. For both groups of scientists, a rebirth of the idea of common goals and scientific collaboration based upon shared conceptions and language seems now a distinct possibility.

These conclusions, however, are preliminary and will require additional verification. The hypothesis of nutrient non-substitution needs to undergo a rigorous statistical test. One main result of this study is the development of such a test. Appropriate experiments can now be designed and executed, and the relevant information collected in a form suitable for performing the test suggested here. The resolution of this question for agronomists and agricultural economists seems of paramount importance for future collaboration between the two groups of scientists.

This study also suggests that more attention be paid to the analytical formulation and solution of the fertility carryover problem. It would appear that in this area agronomists might benefit from a closer association with agricultural economists who are already rather familiar with analogous capital investment issues cast in the form of optimal control problems.

Table 13. Yearly Fertilizer Costs Required to Maintain the Soil Fertility at Desired Targets for the Wheat-Soybean Double Cropping System, 1976 Prices.

Item	Current Recommendation		Suggested Recommendation	
	Quantity[a]	Cost[b]	Quantity[a]	Cost[b]
Maintenance N for Wheat[c]	30	168	20	112
Maintenance P for Wheat	75	529	55	388
Maintenance K for Wheat	40	100	50	125
Maintenance P for Soybean	75	529	55	388
Maintenance K for Soybean	40	100	50	124
Total Cost	—	1,426	—	1,137

[a]In kg N/ha, kg $P_2 0_5$/ha and kg $K_2 0$/ha for N, P and K, respectively.
[b]In Cr$/ha.
[c]Assuming a 3 percent organic matter.

LITERATURE CITED

ANDERSON, R. L. AND L. A. NELSON
1975. A family of models involving intersecting straight lines and concomitant experimental designs useful in evaluating response to fertilizer nutrients. Biometrics, 31:303-318.

ATHANS, M.
1972. The discrete time linear-quadratic-Gaussian stochastic control problem. Annals of Economic and Social Measurements, 1:449-491.

BALMUKAND, B. H.
1928. Studies in crop variation - V. The relatin between yield and soil nutrients. Journal of Agricultural Sciences, 18:602-627.

BAULE, B.
1918. Zu Mitscherlichs Gesetz der physiologischen Beziehungen. Landwirtschaftliche Jahrbücher, 51:363-385.

BAUM, E. L., E. O. HEADY, and J. BLACKMORE, eds.
1956. Methodological procedures in the economic analysis of fertilizer use data. Ames: Iowa State College Press.

BAUM, E. L., E. O. HEADY, J.T. PESEK AND C. G. HILDRETH, eds.
1957. Economic and technical analysis of fertilizer innovations and resource use. Ames: Iowa State College Press.

BEALE, E.M.L.
1968. Mathematical programming in practice. London: Sir Isaac Pitman and Sons, Ltd.

BELLMAN, R. E. AND R. KALABA
1965. Dynamic programming and modern control theory. New York: Academic Press.

BRAY, R. H..
1954. A nutrient mobility concept of soil-plant relationships. Soil Science, 78:9-23.

1958. The correlation of a phosphorus soil test with the response of wheat through a modified Mitscherlich equation. Soil Science Society of America Proceedings, 322:314-317.

1963. Confirmation of the nutrient mobility concept of soil-plant relationships. Soil Science, 95:124-131.

BURT, O. R.
1972. More sophisticated tools for less important problems: the history of range improvement research: reply. American Journal of Agricultural Economics, 54:134-135.

CATE, R. B., JR. AND L. A. NELSON
1971. A simple statistical procedure for partitioning soil test correlation data into two classes. Soil Science Society of America Proceedings, 35:658-660.

CHOW, G. C.
1975. Analysis and control of dynamic economic systems. New York: John Wiley & Sons.

COPE, J. T. AND R. D. ROUSE
1973. Interpretation of soil test results. Soil Testing and Plant Analysis. Walsh and Beaton, eds. Revised edition.. Madison, Wisconsin: Soil Science Society of America, Inc.

COREY, R. G. AND E. E. SCHULTE
1973. Factors affecting the availability of nutrients to plants. Soil Testing and Plant Analysis. Walsh and Beaton, eds. Revised edition. Madison, Wisconsin: Soil Science Society of America, Inc.

COX, D. R.
1962. Further results on tests of separate families of hypotheses. Journal of the Royal Statistical Society, 24, Series B: 406-424.

CROWTHER, E. M. AND F. YATES
1941. Fertilizer policy in wartime: the fertilizer requirements of arable crops. Empire Journal of Experimental Agriculture, 9:77-97.

da FONESCA, V. O.
1976. Análise econômica de doses e fontes de nitrogegno na cultura do trigo, sob condicões de risco. IEPE/UFRGS, Tese No. 47, Porto Alegre, Brazil. (mimeo)

de JANVRY, A.
1972. Optimal levels of fertilization under risk: the potential for corn and wheat fertilization under alternative price policies in Argentina. American Journal of Agricultural Economics, 54:1-10.

DIXON, W. J., ed.
1975. BMDP: biomedical computer programs. Berkeley: University of California Press.

DRAPER, N. R and H. SMITH
1966. Applied regression analysis. New York: John Wiley & Sons.

EMBRAPA
1975a. Sistemas de producão para a cultura da soja. Circular No. 51, Empresa Brasileira de Pesquisa Agropecuária, Pelotas, Brazil.

1975b. Pacotes tecnólogicos para o trigo. Circular No. 71, Empresa Brasileira de Pesquisa Agropecuária. Pelotas, Brazil.

FAIR, R. C. and D.M. JAFFEE
1972. Methods of estimation for markets in disequilibrium. Econometrica, 40:497-514.

FECOTRIGO
1976. Custo de producão — trigo (revisão safra 1976) — soja (estimativa safra 1976/77) — lavouras em sucessão. Federacaío das Cooperativas Brasileiras de Trigo e Soja Ltda., Porto Alegre.

FIBGE
1976a. Anuário estatístico do Brasil, 1976. Fundacão Brasileira de Geografia e Estatística, Rio de Janeiro.

1976b. Boletim estatístico do Brasil, 1976. Fundacão Brasileira de Geografia e Estatística, Rio de Janeiro.

FISHER, R. A. and T. EDEN
1929. Studies in crop variation-VI. Experiments on the response on the potato to potash and nitrogen. Journal of Agricultural Sciences, 19:201-213.

FULLER, W. A.
1965. Stochastic fertilizer production functions for continuous corn. Journal of Farm Economics, 47:105-119.

GOLDFELD, S. M. and R. E. QUANDT
1975. Estimation in a disequilibrium model and the value of information. Journal of Econometrics, 3:325-348.

HADLEY, G.
1964. Nonlinear and dynamic programming. New York: Addision-Wesley Publishing Company, Inc.

HANWAY, J. AND L. DUMENIL
1955. Predicting nitrogen fertilizer needs of Iowa soils: III. Use of nitrate production together with other information as a basis for making nitrogen fertilizer recommendation for corn in Iowa. Soil Science Society of American Proceedings, 19:77-80.

HEADY, E. O.
1956. Methodological problems in fertilizer use. In: Methodological Procedures in the Economic Analysis of Fertilizer Use Data. Baum, Heady and Blackmore, eds. Ames: Iowa State College Press.

HEADY, E. O. and J. L. DILLON
1961. Agricultural production functions. 3rd printing. Ames: Iowa State University Press.

HILDRETH, C. G.
1956. Discrete models with qualitative restrictions. In: Methodological Procedures in the Economic Analysis of Fertilizer Use Data. Baum, Heady and Blackmore, eds. Ames: Iowa State College Press.

KENNEDY, J. O. S., I. F. WHAN, R. JACKSON, and J. L. DILLON
1973. Optimal fertilizer carry-over and crop recycling policies for a tropical grain crop. Australian Journal of Agricultural Economics, 17:104-113.

LANZER, E. A.
1977. Fertilizer recommendations from the dynamic Liebig-Mitscherlich model: the case of wheat-soybeans in southern Brazil. Ph.D. dissertation. University of California, Davis.

LANZER, E. A. and Q. PARIS
1981. A new analytical framework for the fertilization problem. American Journal of Agricultural Economics, 63:93-103.

LANZER, E. A., Q. PARIS and W. A. WILLIAMS
1981. A dynamic model for technical and economic analysis of fertilizer recommendations. Agronomy Journal, 73:733-737.

LIEBIG, J. von
1840. Organic chemistry in its application to agriculture and physiology. London: Playfair.

1863. The natural laws of husbandry. London: Appleton and Co.

MADDALA, G. S.
1977. Econometrics. New York: McGraw-Hill, Inc.

MADDALA, G. S. and F. D. NELSON
1974. Maximum likelihood methods for models of markets in a disequilibrium. Econometrica, 42:1013-1030.

MANGASARIAN, OLVI L.
1969. Nonlinear programming. New York: McGraw-Hill.

MITSCHERLICH, E. A.
1909. Das gesetz des minimums und das gesetz des abnehmenden bodenertrages. Landwirtschaftliche Jahrbücher, 38:537-552.

MUNSON, L. D. and J. P. DOLL
1959. The economics of fertilizer use in crop production. Advances in Agronomy, Vol. XI. New York: Academic Press.

PERRIN, R. K.
1976. The value of information and the value of theoretical models in crop response research. American Journal of Agricultural Economics, 58:54-61.

PESARAN, M. H.
1974. On the general problem of model selection. Review of Economic Studies, 41:153-172.

POIRIER, D.J.
1973a. Piecewise regression using cubic splines. Journal of the American Statistical Association, 68:515-524.

1973b. A note on the use of linear splines in regression analysis. Unpublished manuscript, University of Wisconsin, Madison.

REDMAN, J. C. and S. Q. ALLEN
1954. Interrelationships of economic and agronomic concepts. Journal of Farm Economics, 36:453-465.

ROUSE, R. D.
1968. Soil test theory and calibration for cotton, corn, soybeans and coastal bermudagrass. Auburn University Agricultural Experiment Station, Bulletin No. 375. Alabama.

RUSSELL, E. J.
1973. Soil conditions and plant growth. 10th edition. London: Longmans.

RYAN, J. G.
1972. A generalized crop fertilizer production function. Ph.D. dissertation. North Carolina State University, Raleigh.

SMITH, W. G. and W. L. PARKS
1967. A method for incorporating probability into fertilizer recommendations. Journal of Farm Economics, 49:1511-1515.

SPILLMAN, W. J.
1923. Application of the law of diminishing returns to some fertilizer and feed data. Journal of Farm Economics, 5:36-52.

STAUBER, M. S., O. R. BURT and F. LINSE
1975. Economic evaluation of nitrogen fertilization of grasses when carry-over is significant. American Journal of Agricultural Economics, 57:463-471.

STULP, W. J.
1977. Effects of fertilizer prices under risk on the production of wheat and soybeans in Brazil. Ph.D. dissertation. University of Kentucky.

THEIL, HENRI
1964. Optimal decision rules for government and industry. Amsterdam: North-Holland Co.

1971. Principles of economics. New York: John Wiley & Sons

TISDALE, S. L. and W. L. NELSON
1975. Soil fertility and fertilizers. 3rd edition. New York: Macmillan.

UFRGS
1973. Tabelas de adubacão correctiva e adubacão de manutencão para os solos e culturas dos estados do Rio Grande do Sul e Santa Catarina. Universidade Federal do Rio Grande do Sul, Departmento de Solos, Porto Alegre.

WAGNER, H. M.
1969. Principles of operations research with applications to managerial decisions. New York: Prentice-Hall, Inc.

WALSH, L. M. and J. D. BEATON, eds.
1973. Soil testing and plant analysis. Revised edition. Madison, Wisconsin: Soil Science Society of America, Inc.

WILLCOX, O. W.
1947. How to make a standard yield diagram. Agronomy Journal, 39:74-77.

ZANGWILL, WILLARD I.
1969. Nonlinear programming — a unified approach. New York: Prentice-Hall.