The Australian Journal of Agricultural and Resource Economics, 44:1, pp. 107–127

Imposing regularity conditions on a system of cost and factor share equations

William E. Griffiths, Christopher J. O'Donnell and Agustina Tan Cruz*

Systems of equations comprising cost functions and first-order derivative equations are often used to estimate characteristics of production technologies. Unfortunately, many estimated systems violate the regularity conditions implied by economic theory. Sampling theory methods can be used to impose these conditions globally, but these methods destroy the flexibility properties of most functional forms. We demonstrate how Bayesian methods can be used to maintain flexibility by imposing regularity conditions locally. The Bayesian approach is used to estimate a system of cost and share equations for the merino wool-growing sector. The effect of local imposition of monotonicity and concavity on the signs and magnitudes of elasticities is examined.

1. Introduction

The duality approach to applied production economics often involves the estimation of flexible functional form cost or profit functions. Examples include Bigsby (1994), Mullen and Cox (1996) and O'Donnell and Woodland (1995) using the translog functional form, Fisher and Wall (1990) and Shumway and Alexander (1988) using the normalised quadratic functional form, and Lopez (1980) and Lopez and Tung (1982) using the generalised Leontief functional form. Unfortunately, these estimated functions frequently violate the monotonicity, concavity and convexity conditions implied by economic theory.

A partial solution to this problem involves the imposition of parametric restrictions which ensure that at least some conditions hold at all non-negative prices (i.e. globally). It is possible to impose global curvature restrictions, for example, using eigenvalue decomposition methods and methods involving Cholesky factorisation (e.g. Wiley, Schmidt and Bramble 1973; Talpaz, Alexander and Shumway 1989; Coelli 1996). Unfortunately,

* William E. Griffiths, Christopher J. O'Donnell and Agustina Tan Cruz are all at the Graduate School of Agricultural and Resource Economics, University of New England NSW 2351, Australia.

the global imposition of regularity conditions forces many flexible functional forms to exhibit properties not implied by economic theory. For example, imposing global concavity on a translog cost function may lead to an upward bias in the degree of input substitutability, and imposing global concavity on a generalised Leontief cost function will rule out complementarity between inputs (Diewert and Wales 1987). An alternative approach which can be used to maintain flexibility involves the imposition of regularity conditions locally, that is, at a single point, at several points, or over a region of interest, usually the region over which inferences will be drawn. Methods which can be used to impose curvature restrictions locally include the numerical methods of Lau (1978) and Gallant and Golub (1984). More recently, Chalfant and Wallace (1992) and Terrell (1996) have used a Bayesian approach. The Bayesian approach has the important advantage of allowing us to draw finite sample inferences concerning nonlinear functions of parameters.

Empirical implementation of the Bayesian approach involves the use of Markov Chain Monte Carlo (MCMC) simulation methods. The use of MCMC methods has grown rapidly in recent years with the availability of inexpensive high-speed computers, and with the further development of powerful computer-intensive statistical algorithms. These algorithms, which include the Gibbs sampler and the Metropolis-Hastings algorithm, can be used to draw samples from a marginal probability density indirectly, without having to derive the density itself. Not surprisingly, MCMC methods have revolutionised Bayesian econometrics, where posterior marginal densities can be difficult or impossible to derive analytically.

In this article we illustrate how MCMC methods can be used to estimate a system of cost and factor share equations for a sector of the Australian wool-growing industry. This empirical application of the MCMC methodology is motivated by the large number of curvature violations reported in the study by O'Donnell and Woodland (1995). Although we retain most of the features of the O'Donnell and Woodland model, and we use their data set, we estimate a system of cost and factor share equations which has a less complex stochastic structure, and we focus on only one Australian wool-growing sector (merino wool-growing) instead of three. These simplifications allow us to better illustrate the applicability and usefulness of the MCMC technique, and still allow us to validate the elasticity estimates obtained in O'Donnell and Woodland's earlier work. Two MCMC techniques are employed: the Gibbs sampler is used when inequality restrictions are not imposed, and a Metropolis-Hastings algorithm is used when they are imposed. Using both techniques not only serves the purpose of our application, but it demonstrates alternatives for carrying out Bayesian inference in seemingly unrelated regression equations with equality and inequality constraints on the coefficients. Our choice of a Metropolis-Hastings algorithm for imposing curvature restrictions locally has significant computational advantages over importance sampling and Gibbs sampling algorithms used in other Bayesian studies, including those by Chalfant and Wallace and Terrell. Indeed, the need to generate enormous numbers of Gibbs sampler observations (sometimes hundreds of thousands) to obtain a few legitimate draws is easily overcome, a fact that does not seem to be generally appreciated in the literature.

The outline of the article is as follows. In the second section we translate a standard economic model of producer behaviour into a system of empirical cost and factor share equations. This empirical model takes the form of a seemingly unrelated regression (SUR) model. In the third section we describe two alternative but equivalent iterative procedures for obtaining maximum likelihood estimates of the SUR model parameters. We also describe the Gibbs sampler and Metropolis-Hastings algorithms, and the manner in which monotonicity and curvature restrictions can be imposed. The data are described in the next section and the estimation results are presented in the fifth section. The results present information on parameter estimates, predicted factor shares, estimated eigenvalues and estimated input-price elasticities. Eigenvalues are useful for assessing curvature violations, while input-price elasticities are useful for feeding into studies which examine the welfare implications of policy decisions and technical change (see, for example, Zhao *et al.* 2000). Conclusions are drawn in the final section where we review our work and offer suggestions for further research.

2. Model

Our model is predicated on the assumption that the technological possibilities faced by the firm can be summarised by the cost function

$$C(\mathbf{w}, q) \equiv \min_{\mathbf{x}}\{\mathbf{w}'\mathbf{x}: f(\mathbf{x}) \geq q, \mathbf{x} \geq 0\} \tag{1}$$

where $\mathbf{x}$ is an $I \times 1$ vector of inputs, $\mathbf{w}$ is an $I \times 1$ vector of input prices and q is scalar output. If the production function $f(\mathbf{x})$ satisfies a standard set of relatively weak assumptions then the cost function will be non-negative for all positive prices and output, and linearly homogenous, non-decreasing (i.e. monotonic), concave and continuous in prices (Chambers 1988). Moreover, the Hessian matrix of second-order price derivatives will be symmetric. Our interest lies in the properties of monotonicity and concavity, and the manner in which these properties can be imposed locally on an estimated flexible functional form.

A functional form is flexible if it can provide a local 'second-order approximation' to an arbitrary functional form. An excellent discussion of the concept of a second-order approximation can be found in Barnett (1983). The two most commonly used flexible functional forms are the generalised Leontief introduced by Diewert (1971) and the translog introduced by Christensen, Jorgensen and Lau (1971). We follow O'Donnell and Woodland (1995) and assume a constant returns to scale translog functional form, which implies

$$\ln(C/q) = \alpha_0 + \alpha_T T + \sum_{i=1}^{I} \alpha_i \ln(w_i) + .5 \sum_{i=1}^{I} \sum_{j=1}^{I} \alpha_{ij} \ln(w_i) \ln(w_j) \tag{2}$$

where C represents total costs, w_i represents the price of input i, and T is a time trend which is used to capture the effects of exogenous technical change. The factor share equations are obtained using Shephard's lemma:

$$s_i = \alpha_i + \sum_{j=1}^{I} \alpha_{ij} \ln(w_j) \qquad i = 1, \ldots, I \tag{3}$$

where s_i represents the cost share of input i. It is clear from equations 2 and 3 that our assumed form of technical change is Hicks-neutral: factor shares are unaffected by technical change while unit costs decrease at a constant percentage rate.

Some of the theoretical properties of the cost function in equation 1 can be expressed in terms of the parameters appearing in equation 2. Specifically, linear homogeneity and symmetry will be satisfied if

$$\sum_{i=1}^{I} \alpha_i = 1, \quad \sum_{j=1}^{I} \alpha_{ij} = 0 \; (i = 1, \ldots, I) \quad \text{and} \quad \alpha_{ij} = \alpha_{ji} \; (i, j = 1, \ldots, I). \tag{4}$$

Monotonicity will be satisfied if the estimated factor shares are positive, while concavity will be satisfied if the Hessian matrix of second-order derivatives is negative semi-definite. In turn, the Hessian matrix will be negative semi-definite if and only if its eigenvalues are non-positive. In the later section on Bayesian estimation, that part of the parameter space where monotonicity and concavity hold is denoted by Γ_2. The unrestricted parameter space is denoted Γ_1.

Our empirical model is obtained by embedding equations 2 and 3 in a stochastic framework. After incorporating stochastic terms and introducing the firm and time subscripts n and t ($n = 1, \ldots, N$ and $t = 1, \ldots, T$), our empirical model becomes

$$s_{int} = \alpha_i + \sum_{j=1}^{I} \alpha_{ij} \ln(w_{jnt}) + \varepsilon_{int} \qquad i = 1, \ldots, I-1$$

$$\ln(C_{nt}/q_{nt}) = \alpha_0 + \alpha_T T_{nt} + \sum_{i=1}^{I} \alpha_i \ln(w_{int}) + .5 \sum_{i=1}^{I} \sum_{j=1}^{I} \alpha_{ij} \ln(w_{int}) \ln(w_{jnt}) + \varepsilon_{Int} \tag{5}$$

where $\varepsilon_{int}(i = 1, \ldots, I)$ represents statistical noise. We have adopted the usual practice of dropping one share equation to avoid singularity of the error covariance matrix. The share and cost equation errors are assumed to be independently and identically distributed over firms and time with

$$E\{\varepsilon_{int}\} = 0 \tag{6}$$

and

$$E\{\varepsilon_{int}\varepsilon_{mks}\} = \begin{cases} \sigma_{im} & \text{if } n = k \text{ and } t = s \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

The model given by equations 4 to 7 has an identical deterministic structure and a similar stochastic structure to the model of O'Donnell and Woodland. Like O'Donnell and Woodland, our stochastic assumptions allow for within-firm contemporaneous correlation between the disturbances ε_{int}. However, unlike O'Donnell and Woodland, our cost function combines errors that vary over time and firms with any time-specific uncertainty that may exist. As a consequence, our cost function does not have a complicated error components structure. As noted by Pope and Just (1998), this simpler structure is in line with assumptions adopted in most empirical studies.

3. Estimation methods

In this section we describe four methods for estimating the parameters of the model given by equations 4 to 7: two equivalent methods for obtaining maximum likelihood estimates, and two MCMC algorithms (the Gibbs sampler and the Metropolis-Hastings algorithm). The maximum likelihood methods we describe do not allow for the imposition of monotonicity or concavity constraints. Nor does our Gibbs sampler, which is only used in this article to illustrate the MCMC method and to provide a benchmark by which to judge the results of the maximum likelihood and Metropolis-Hastings approaches. Our description of a Metropolis-Hastings algorithm provides details of necessary modifications to the standard approach to ensure that monotonicity and concavity conditions are satisfied.

3.1 Maximum likelihood estimation

For a model consisting of four inputs, the system of equations given by 5 can be more conveniently written:

$$y_{int} = \mathbf{x}'_{int}\beta_i + \varepsilon_{int} \qquad i = 1, \ldots, 4 \tag{8}$$

where

$$y_{int} = s_{int} \qquad i = 1, \ldots, 3$$

$$y_{4nt} = \ln(C_{nt}/q_{nt})$$

$$\beta_i = (\alpha_i, \alpha_{i1}, \ldots, \alpha_{i4})' \qquad i = 1, \ldots, 3 \tag{9}$$

$$\beta_4 = (\alpha_0, \alpha_T, \alpha_1, \ldots, \alpha_4, \alpha_{11}, \alpha_{12}, \ldots, \alpha_{14}, \alpha_{22}, \alpha_{23}, \ldots, \alpha_{44})' \tag{10}$$

and the definitions of the $\mathbf{x}_{int}$ conform to the definitions of the β_i and are obvious. Notice from equations 9 and 10 that the β_i vectors have many elements in common. Indeed, the restrictions given by equation 4 and the restrictions implicit in equations 9 and 10 together mean that only 11 of the 31 parameters in the β_i vectors are 'free'. Those that are redundant or 'not free' can be obtained from the other parameters and the restrictions.

Stacking equation 8 by firm, time period and then by equation we obtain

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \mathbf{y}_4 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1 & & & \\ & \boldsymbol{X}_2 & & \\ & & \boldsymbol{X}_3 & \\ & & & \boldsymbol{X}_4 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix} \tag{11}$$

where $\mathbf{y}_i = (y_{i11}, y_{i21}, \ldots, y_{iN1}, y_{i12}, y_{i22}, \ldots, y_{iN2}, \ldots, y_{iNT})'$ is $NT \times 1$ for all i, and $\boldsymbol{X}_i$ and ε_i are similarly defined, although it is worth noting that $\boldsymbol{X}_i$ is $NT \times 5$ for $i = 1, \ldots, 3$ and $\boldsymbol{X}_4$ is $NT \times 16$. Thus, we can write the empirical model more compactly as:

$$\mathbf{y} = \boldsymbol{X}\beta + \varepsilon \tag{12}$$

where the definitions are obvious. The parametric restrictions implied by equations 4, 9 and 10 and our assumptions concerning the error vector ε can also be written more compactly as:

$$\boldsymbol{R}\beta = \mathbf{r} \tag{13}$$

$$E\{\varepsilon\} = \mathbf{0} \tag{14}$$

and

$$E\{\varepsilon\varepsilon'\} = \Omega = \Sigma \otimes I_{NT} \tag{15}$$

where $\Sigma = [\sigma_{im}]$ and $\boldsymbol{R}$ and $\mathbf{r}$ are known matrices of order 20×31 and 20×1, respectively. The model given by equations 12 to 15 is a standard restricted SUR model (see Judge *et al.* 1985, pp. 469–73).

To obtain maximum likelihood estimates we note that the restricted Generalised Least Squares (GLS) estimator for β is

$$\tilde{\beta} = \hat{\beta} + \boldsymbol{CR}'(\boldsymbol{RCR}')^{-1}(\mathbf{r} - \boldsymbol{R}\hat{\beta}) \tag{16}$$

where $\boldsymbol{C} = [\boldsymbol{X}'(\Sigma^{-1} \otimes \boldsymbol{I}_{NT})\boldsymbol{X}]^{-1}$ and $\hat{\beta} = \boldsymbol{CX}'(\Sigma^{-1} \otimes \boldsymbol{I}_{NT})\mathbf{y}$ is the unrestricted GLS estimator. In practice, restricted Estimated Generalised Least Squares (EGLS) estimates can be obtained by replacing Σ in equation 16 with an estimator, $\hat{\Sigma}$, constructed using restricted or unrestricted OLS residuals. Of course, another estimate of β can then be obtained by replacing Σ with a new estimator based on the restricted EGLS residuals (rather than OLS residuals). In fact, we can continue to update our estimates of β and Σ in an iterative way and, if the disturbances are multivariate normal, this iterative process will yield maximum likelihood estimates.

The iterative process described above can be time-consuming if the number of restrictions to be imposed and parameters to be estimated at each step is large. An alternative but equivalent estimation procedure, which is not only faster but can also be usefully exploited in our Bayesian approach, involves maximum likelihood estimation of the subset of 11 free parameters in β. After convergence, the remaining 20 maximum likelihood estimates are derived using the 20 parametric restrictions $\boldsymbol{R}\beta = \mathbf{r}$. To implement the procedure we rearrange the rows of β and the columns of $\boldsymbol{X}$ and $\boldsymbol{R}$ in such a way that equations 12 and 13 can be written in the following partitioned form:

$$\mathbf{y} = \boldsymbol{X}\beta + \varepsilon = [\boldsymbol{X}_1 \ \boldsymbol{X}_2]\begin{bmatrix}\eta \\ \gamma\end{bmatrix} + \varepsilon \tag{17}$$

$$\boldsymbol{R}\beta = [\boldsymbol{R}_1 \ \boldsymbol{R}_2]\begin{bmatrix}\eta \\ \gamma\end{bmatrix} = \mathbf{r} \tag{18}$$

where $\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{R}_1, \boldsymbol{R}_2, \gamma$ and η are $NT \times 20$, $NT \times 11$, 20×20, 20×11, 11×1 and 20×1, respectively. The vector γ contains the subset of 11 free parameters to be estimated in the first stage, and η contains the 20 remaining parameters in β which will be estimated using estimates of γ and the following equivalent form of equation 18:

$$\eta = \boldsymbol{R}_1^{-1}(\mathbf{r} - \boldsymbol{R}_2\gamma). \tag{19}$$

Recall that the vector γ of free parameters contains parameters which cannot be obtained from other parameters and the restrictions. To estimate γ we use equation 19 to rewrite equation 17 in the form:

$$\mathbf{y}^* = \boldsymbol{X}^*\gamma + \varepsilon \tag{20}$$

where $\mathbf{y}^* = \mathbf{y} - \boldsymbol{X}_1\boldsymbol{R}_1^{-1}\mathbf{r}$ and $\boldsymbol{X}^* = \boldsymbol{X}_2 - \boldsymbol{X}_1\boldsymbol{R}_1^{-1}\boldsymbol{R}_2$. The model given by equations 20, 14 and 15 is an unrestricted SUR model, with (unrestricted) GLS estimator for γ given by

$$\hat{\gamma} = \boldsymbol{C}^*\boldsymbol{X}^{*\prime}(\Sigma^{-1} \otimes \boldsymbol{I}_{NT})\mathbf{y}^* \tag{21}$$

where $\boldsymbol{C}^* = [\boldsymbol{X}^{*\prime}(\Sigma^{-1} \otimes \boldsymbol{I}_{NT})\boldsymbol{X}^*]^{-1}$. Again, in practice, EGLS estimates can be obtained by replacing Σ with an estimator constructed using OLS residuals. Moreover, if the disturbances are multivariate normal, a maximum likelihood estimate for γ can be obtained using the iterative procedure described above.

3.2 Bayesian estimation

The formulation of our empirical model as an unrestricted SUR model (equations 20, 14 and 15) is convenient for Bayesian analysis because a number of relevant results already appear in the mainstream econometrics literature (e.g. Judge *et al.* pp. 478–80). We begin by stating Bayes Theorem:

$$f(\gamma, \Sigma \mid \mathbf{y}^*) \propto L(\mathbf{y}^* \mid \gamma, \Sigma)p(\gamma, \Sigma) \tag{22}$$

where $\propto$ denotes 'proportional to', $f(\gamma, \Sigma \mid \mathbf{y}^*)$ is the posterior joint density function for γ and Σ given $\mathbf{y}^*$ (the posterior density summarises all the information about γ and Σ after the sample $\mathbf{y}^*$ has been observed), $L(\mathbf{y}^* \mid \gamma, \Sigma)$ is the likelihood function (summarising all the sample information), and $p(\gamma, \Sigma)$ is the prior density function for γ and Σ (summarising the nonsample information about γ and Σ). Our interest lies in the posterior density $f(\gamma, \Sigma \mid \mathbf{y}^*)$ and characteristics (e.g. means and variances) of marginal densities which can be derived from it.

Our Bayesian treatment of the unrestricted SUR model begins with the assumption that ε is multivariate normal. Under this assumption the likelihood function is (Judge *et al.* 1995, p. 478)

$$L(\mathbf{y}^* \mid \gamma, \Sigma) \propto |\Sigma|^{-NT/2}\exp[-.5tr(\boldsymbol{A}\Sigma^{-1})] \tag{23}$$

where $\boldsymbol{A}$ is the 4×4 symmetric matrix with $(i, j)^{\text{th}}$ element $a_{ij} = (\mathbf{y}_i^* - \boldsymbol{X}_i^*\gamma)'(\mathbf{y}_j^* - \boldsymbol{X}_j^*\gamma)$, and $\mathbf{y}_i^*$ and $\boldsymbol{X}_i^*$ are obviously defined sub-vectors and matrices of $\mathbf{y}$ and $\boldsymbol{X}$. In addition, we use a non-informative joint prior:

$$p(\gamma, \Sigma) = p(\gamma)p(\Sigma)I(\gamma \in \Gamma_s) \quad s = 1, 2 \tag{24}$$

where $p(\gamma) \propto$ constant, $p(\Sigma) \propto |\Sigma|^{-(I+1)/2}$ is the limiting form of an Inverted Wishart density, the Γ_s are the sets of permissible parameter values when monotonicity and concavity information is $(s = 2)$ and is not $(s = 1)$

available, and $I(.)$ is an indicator function which takes the value 1 if the argument is true. Thus, $I(\gamma \in \Gamma_1) = 1$ for all values of γ. We choose a noninformative prior because it allows us to better compare our maximum likelihood results with our Bayesian results, whether or not monotonicity and concavity information is available. Note that the algebraic form of the prior $p(\gamma, \Sigma)$ is unchanged by the availability of monotonicity and concavity information, even though the region over which it is defined is different. The same is true of the joint posterior density (Judge *et al.* 1995, p. 479):

$$f(\gamma, \Sigma \mid \mathbf{y}^*) \propto |\Sigma|^{-(NT+I+1)/2} \exp[-.5(\mathbf{y}^* - \boldsymbol{X}^*\gamma)'(\Sigma^{-1} \otimes \boldsymbol{I}_{NT})(\mathbf{y}^* - \boldsymbol{X}^*\gamma)] I(\gamma \in \Gamma_s) \quad s = 1, 2 \tag{25}$$

$$\propto |\Sigma|^{-(NT+I+1)/2} \exp[-.5\mathrm{tr}(\boldsymbol{A}\Sigma^{-1})] I(\gamma \in \Gamma_s) \qquad s = 1, 2. \tag{26}$$

We are particularly interested in the posterior marginal densities of the elements of γ, and the means and standard deviations of these posterior densities. Unfortunately, these results cannot be obtained from equations 25 and 26 analytically. Instead, we must use MCMC methods to draw a sample from the posterior joint density $f(\gamma \mid \mathbf{y}^*)$. We then use these sample observations to estimate the moments of the marginal densities of the elements of γ. The two MCMC algorithms we use to generate these samples are the Gibbs sampler and Metropolis-Hastings algorithms.

The Gibbs sampler

The Gibbs sampler was used for Bayesian estimation without monontonicity and concavity imposed. That is, the parameter space for γ was the unrestricted space Γ_1. Useful introductions to the Gibbs sampler can be found in Casella and George (1992) and Chib and Greenberg (1996). In the present context, the Gibbs sampler is an algorithm which effectively samples from $f(\gamma \mid \mathbf{y}^*)$ by iterating as follows:

Step 1: Specify starting values γ^0, Σ^0. Set $i = 0$.

Step 2: Generate γ^{i+1} from $f(\gamma \mid \Sigma^i, \mathbf{y}^*)$

Step 3: Generate Σ^{i+1} from $f(\Sigma \mid \gamma^{i+1}, \mathbf{y}^*)$

Step 4: Set $i = i + 1$ and go to Step 2.

This iteration scheme produces a chain, $\gamma^1, \Sigma^1, \gamma^2, \Sigma^2, \ldots,$ with the property that, for large k, γ^{k+1} is effectively a sample point from $f(\gamma \mid \mathbf{y}^*)$ (in this case the chain is said to have 'converged'). Thus, in practice, $\gamma^{k+1}, \ldots, \gamma^{k+m}$ can be regarded as a sample from $f(\gamma \mid \mathbf{y}^*)$. In this article we set $k = 25\,000$ (the 'burn-in' period) and draw a sample of size $m = 50\,000$. These values of k and m were determined using the Z-diagnostic of Geweke (1992) and the stationarity and interval halfwidth tests of Heidelberger and Welch (1983). The large values of k and m are partly due to the fact that the observations generated by the Gibbs sampler are correlated. A smaller value of m could have been used if the sample was constructed using only the last observation in m independent Gibbs chains.

Notice from Steps 2 and 3 that in order to make the Gibbs sampler operational we need the conditional probability density functions (pdfs) $f(\gamma \mid \Sigma, \mathbf{y}^*)$ and $f(\Sigma \mid \gamma, \mathbf{y}^*)$. To obtain (the kernel of) the conditional posterior pdf $f(\gamma \mid \Sigma, \mathbf{y}^*)$ we use equation 25 and view Σ as a constant, yielding

$$f(\gamma \mid \Sigma, \mathbf{y}^*) \propto \exp[-.5(\gamma - \hat{\gamma})' \boldsymbol{X}^{*\prime}(\Sigma^{-1} \otimes \boldsymbol{I}_{NT})\boldsymbol{X}^*(\gamma - \hat{\gamma})]I(\gamma \in \Gamma_1) \qquad (27)$$

where $\hat{\gamma}$ is the GLS estimator given by equation 21. Thus, $f(\gamma \mid \Sigma, \mathbf{y}^*)$ is the density function of a multivariate normal random variable with mean vector $\hat{\gamma}$ and covariance matrix $[\boldsymbol{X}^{*\prime}(\Sigma^{-1} \otimes \boldsymbol{I}_{NT})\boldsymbol{X}^*]^{-1}$. Finally, to obtain the kernel of the conditional posterior pdf $f(\Sigma \mid \gamma, \mathbf{y}^*)$ we use equation 26 and view γ as a constant, yielding

$$f(\Sigma \mid \gamma, \mathbf{y}^*) \propto \frac{1}{|\Sigma|^{(NT+I+1)/2}} \exp[-.5tr(\boldsymbol{A}\Sigma^{-1})] \qquad (28)$$

Thus, $f(\Sigma \mid \gamma, \mathbf{y}^*)$ is an Inverted Wishart density function with parameters $\boldsymbol{A}$, NT and I (see Zellner 1971, p. 395).

Terrell has modified this algorithm to impose monotonicity and concavity constraints over a specified grid of prices. For each parameter vector generated by the Gibbs sampler (i.e. for each γ^k), monotonicity and concavity constraints are evaluated at each price point in the grid. The parameter vector is included in the sample if the constraints hold and rejected otherwise. This modification has the effect of changing the conditional density in equation 27 to a truncated multivariate normal density that is only positive in the region Γ_2. Unfortunately, it is often necessary to generate extremely large numbers of parameter vectors before obtaining just one vector that can be included in the sample, and this limits the practical usefulness of the approach. The Metropolis-Hastings algorithm is an alternative MCMC algorithm which does not suffer this disadvantage.

The Metropolis-Hastings algorithm

A description of the Metropolis-Hastings algorithm can be found in Chib and Greenberg (1996). In the present context, a Metropolis-Hastings algorithm which allows us to impose monotonicity and concavity at a particular set of prices proceeds iteratively as follows:

Step 1: Specify an arbitrary starting value γ^0 which satisfies the constraints. Set $i = 0$.

Step 2: Given the current value γ^i, use a symmetric transition density $q(\gamma^i, \gamma^c)$ to generate a candidate for the next value in the sequence, γ^c.

Step 3: Use the candidate value γ^c to evaluate the monotonicity and concavity constraints at the specified prices. If any constraints are violated set $\alpha(\gamma^i, \gamma^c) = 0$ and go to Step 5.

Step 4: Calculate $\alpha(\gamma^i, \gamma^c) = \min(g(\gamma^c)/g(\gamma^i)), 1)$ where $g(\gamma)$ is the kernel of $f(\gamma \mid \mathbf{y}^*)$.

Step 5: Generate an independent uniform random variable U from the interval [0, 1].

Step 6: Set $\gamma^{i+1} = \begin{cases} \gamma^c \text{ if } U < \alpha(\gamma^i, \gamma^c) \\ \gamma^i \text{ if } U \geq \alpha(\gamma^i, \gamma^c) \end{cases}$

Step 7: Set $i = i + 1$ and go to Step 2.

Again, this iteration scheme produces a chain, $\gamma^1, \gamma^2, \ldots,$ with the property that, for large k, γ^{k+1} is effectively a sample point from $f(\gamma \mid \mathbf{y}^*)$. Thus, the sequence $\gamma^{k+1}, \ldots, \gamma^{k+m}$ can once again be regarded as a sample from $f(\gamma \mid \mathbf{y}^*)$. Importantly, this sequence satisfies monotonicity and concavity at the specified prices. In this article monotonicity and concavity constraints are imposed at 23 price points: the quantity-weighted averages of observed input prices in each time period $t = 1, \ldots, 11, 13, \ldots, 24$. In Step 3 the monotonicity constraint is evaluated using the signs of the predicted factor shares, while the concavity constraint is evaluated using the maximum eigenvalue of the estimated Hessian matrix. We use a burn-in period of $k = 100\,000$ and draw a sample of size $m = 200\,000$. Again, these values of k and m were set using the convergence diagnostics of Geweke (1992) and Heidelberger and Welch (1983).

Notice from Steps 1, 2 and 4 that in order to make the Metropolis-Hastings algorithm operational we need an arbitrary starting value γ^0 which satisfies the constraints. We also need the transition density $q(\gamma^i, \gamma^c)$ and the kernel $g(\gamma)$.

For starting values we used $\alpha_i = 0.25$ $(i = 1, \ldots, 4)$ and $\alpha_{ij} = 0$ for all $i \neq j$. All other parameters were set equal to their maximum likelihood estimates. These starting values satisfy monotonicity and concavity but may be some distance from the mean of $f(\gamma \mid \mathbf{y}^*)$ (so a reasonably long burn-in period is needed to ensure the convergence of the MCMC chain).

The transition density $q(\gamma^i, \gamma^c)$ is taken to be multivariate normal with mean γ^i and covariance matrix $[X^{*\prime}(\hat{\Sigma}^{-1} \otimes I_{NT})X^*]^{-1}$ (the estimated covariance matrix of the restricted SUR estimator $\hat{\gamma}$). In practice, it is commonplace to multiply the (arbitrarily chosen) covariance matrix by a constant h in order to manipulate the rate at which the candidate γ^c is accepted as the next value in the sequence. In this article we set $h = 0.05$ in order to obtain an acceptance rate of approximately 0.4. This constant was chosen by trial and error.

Finally, the kernel $g(\gamma)$ of the marginal density $f(\gamma \mid \mathbf{y}^*)$ can be obtained by integrating Σ out of the joint posterior (26) (see Judge *et al.* 1985, p. 479):

$$f(\gamma \mid \mathbf{y}^*) \propto |A|^{-NT/2} I(\gamma \in \Gamma_2) = g(\gamma). \tag{29}$$

4. Data

The data were originally collected by the Australian Bureau of Agricultural Economics as part of its Australian Sheep Industry Surveys. Our sample consists of 310 time-series and cross-section observations on Australian merino woolgrowers, covering the periods 1952–53 to 1962–63 ($t = 1, \ldots, 11$) and 1964–65 to 1975–76 ($t = 13, \ldots, 24$). Each observation in the original data set is a record of the average financial and physical characteristics of a group of firms. These observations were used to construct observations on output (q), total cost (C), input prices ($\mathbf{w}$) and input quantities. Inputs were grouped into one of four broad categories: land, capital, livestock and other inputs (including labour, equipment, materials and services). A more complete description of the data can be found in O'Donnell and Woodland (1995).

5. Results

The results were generated using SHAZAM (White 1978) and the R version of CODA (Best, Cowles and Vines 1995). In this section the results are examined in terms of estimates of the unknown parameters, predicted factor shares, eigenvalues of the estimated Hessian matrix of second-order derivatives of the cost function, and estimates of the own- and cross-price elasticities of input demand.

5.1 Parameter estimates

Maximum likelihood estimates of the structural parameters β are presented in table 1, along with the means of the Bayesian samples obtained with and without the inequality constraints imposed. These samples were generated using the Metropolis-Hastings and Gibbs algorithms, respectively. The

Table 1 Parameter estimates

		Maximum [a] likelihood	Gibbs [b] (no inequality constraints)	Metropolis-Hastings [b] (inequality constraints imposed)
Constant		–0.595 (0.058)	–0.597 (0.062)	–0.840 (0.050)
α_1	Land	0.250 (0.005)	0.250 (0.006)	0.251 (0.006)
α_2	Capital	0.674 (0.019)	0.674 (0.020)	0.664 (0.017)
α_3	Livestock	0.440 (0.013)	0.440 (0.013)	0.344 (0.008)
α_{11}	Land/Land	0.023 (0.001)	0.023 (0.001)	0.023 (0.001)
α_{12}	Land/Capital	0.018 (0.001)	0.018 (0.001)	0.018 (0.001)
α_{13}	Land/Livest.	–0.006 (0.001)	–0.006 (0.001)	–0.006 (0.001)
α_{22}	Capital/Capital	0.115 (0.006)	0.115 (0.006)	0.110 (0.006)
α_{23}	Capital/Livest.	–0.007 (0.002)	–0.007 (0.003)	–0.006 (0.002)
α_{33}	Livest./Livest.	0.076 (0.002)	0.076 (0.002)	0.057 (0.001)
α_T	Time	–0.032 (0.002)	–0.032 (0.003)	–0.033 (0.002)

Notes: [a] Numbers in parentheses are estimated standard errors.
[b] Numbers in parentheses are standard deviations of the MCMC samples.

numbers in parentheses are either the estimated standard errors of the maximum likelihood estimates or the standard deviations of our MCMC samples.

Our maximum likelihood estimates are similar to the estimates obtained by O'Donnell and Woodland (1995). Thus, it appears that our specification of a less complex stochastic structure, and our focus on only one wool-growing sector instead of three, has had little or no effect on the signs or magnitudes of the slope coefficients or their standard errors. Note that all coefficients are statistically different from zero at usual levels of significance. Also note, from the estimated coefficient of the time variable in the cost function, that the annual proportional reduction in unit costs as a result of technical change is estimated to be 3.2 per cent, only slightly higher than the estimate of 2.9 per cent reported by O'Donnell and Woodland.

The strong similarity between the maximum likelihood and Gibbs estimates presented in table 1 reflects our use of a noninformative prior. The location and shape of the likelihood function $L(\mathbf{y}^* \mid \gamma, \Sigma)$ govern the location and shape of the posterior density $f(\gamma, \Sigma \mid \mathbf{y}^*)$ and, of course, our maximum likelihood and Gibbs results have been obtained using these two functions. The standard deviations of the Gibbs samples are slightly higher than the estimated standard errors of the maximum likelihood estimates. These differences arise because, unlike the standard deviations of the Gibbs samples, the estimated standard errors of the maximum likelihood estimates do not account for the uncertainty associated with the estimation of the variance-covariance matrix Σ. For this reason, and because the maximum likelihood and Gibbs estimates are very similar, we shall ignore the maximum likelihood estimates in the remainder of this article.

Finally, there is a reasonable similarity between the Gibbs and Metropolis-Hastings estimates presented in table 1. In fact, only the coefficient of the constant term and the first- and second-order coefficients associated with the livestock input (α_3 and α_{33}) appear to change significantly with the imposition of the monotonicity and concavity constraints. Violations of these constraints are assessed below in terms of predicted factor shares and the eigenvalues of the estimated Hessian matrix.

5.2 Predicted factor shares

Monotonicity requires that the predicted cost shares be positive. The observations in our Gibbs sample were used to check this requirement at our 23 sets of quantity-weighted average input prices. The distributions of the predicted factor shares were uniformly found to lie between zero and one, indicating that monotonicity was satisfied without the imposition of constraints.

5.3 Eigenvalues

For the estimated cost function to be consistent with economic theory it must be concave, requiring that the estimated Hessian matrix of second-order derivatives be negative semi-definite. Since the Hessian matrix is singular, a necessary and sufficient condition for negative semi-definiteness is that the maximum eigenvalue is exactly zero (singularity implies that at least one eigenvalue must be zero).

Each observation in our Gibbs sample was used to construct an observation on the maximum eigenvalue of the Hessian matrix evaluated at a particular point (i.e. a particular set of average input prices). The process was repeated for each of our 23 sets of average input prices, and the means and standard deviations of these 23 samples are presented in table 2. When rounded to three decimal places, the means of the sample distributions of 8 out of 23 maximum eigenvalues are non-zero, implying there is positive probability that concavity is violated for at least 35 per cent of the price vectors, somewhat lower than the proportion of concavity violations reported by O'Donnell and Woodland (1995).

The estimated probability of a concavity violation for each of the price vectors is also reported in table 2. These estimated probabilities are calculated as the proportion of times the maximum eigenvalue exceeded zero when rounded to three decimal places. Note that there are several price

Table 2 Unconstrained maximum eigenvalues[a]

Year	t	Estimated maximum eigenvalue	Estimated prob. of concavity violation	Year	t	Estimated maximum eigenvalue	Estimated prob. of concavity violation
1952–53	1	0.019 (0.004)	1.000	1965–66	14	0.000 (0.000)	0.000
1953–54	2	0.000 (0.000)	0.000	1966–67	15	0.000 (0.000)	0.000
1954–55	3	0.000 (0.000)	0.000	1967–68	16	0.000 (0.000)	0.000
1955–56	4	0.000 (0.000)	0.000	1968–69	17	0.000 (0.000)	0.000
1956–57	5	0.000 (0.000)	0.000	1969–70	18	0.000 (0.000)	0.000
1957–58	6	0.000 (0.000)	0.000	1970–71	19	0.000 (0.000)	0.000
1958–59	7	0.002 (0.001)	0.023	1971–72	20	0.015 (0.003)	0.999
1959–60	8	0.000 (0.000)	0.000	1972–73	21	0.001 (0.001)	0.000[b]
1960–61	9	0.002 (0.001)	0.006	1973–74	22	0.000 (0.000)	0.000
1961–62	10	0.004 (0.002)	0.286	1974–75	23	0.000 (0.000)	0.000
1962–63	11	0.008 (0.002)	0.925	1975–76	24	0.015 (0.002)	1.000
1964–65	13	0.000 (0.000)	0.000				

Notes: [a] Numbers in parentheses are standard deviations of the MCMC samples.
[b] 0.00014 before rounding.

vectors where the probability of violating concavity is approximately 1.00. This result suggests that an attempt to impose concavity via a Gibbs sampling procedure with a truncated multivariate normal density would be doomed to failure, and, hence, our suggested Metropolis-Hastings algorithm is particularly useful.

5.4 Elasticities

The imposition of regularity conditions on our estimated cost function leads to noticeable changes in the posterior distributions of a number of own- and cross-price elasticities. To briefly illustrate, table 3 reports the means and standard deviations of the estimated probability density functions of input price elasticities calculated at the quantity-weighted average of all input prices in the sample.

Three features of table 3 are of particular interest. First, (the means of) all own-price elasticities are correctly signed and indicate that all input demands are inelastic with respect to their own prices. Moreover, the only own-price elasticity which seems to be affected by the imposition of the constraints is the own-price elasticity for livestock. The mean of this own-

Table 3 Estimated input-price elasticities evaluated at average prices [a]

	Price of land	Price of capital	Price of livestock	Price of other inputs
Gibbs (unconstrained)				
Qty of Land	−0.647	0.493	0.027	0.127
	(0.011)	(0.010)	(0.007)	(0.018)
Qty of Capital	0.148	−0.314	0.072	0.094
	(0.003)	(0.022)	(0.009)	(0.022)
Qty of Livestock	0.025	0.218	−0.126	−0.117
	(0.007)	(0.027)	(0.024)	(0.034)
Qty of Other Inputs	0.022	0.053	−0.022	−0.053
	(0.003)	(0.012)	(0.006)	(0.015)
Metropolis-Hastings (constrained)				
Qty of Land	−0.643	0.496	0.030	0.118
	(0.011)	(0.011)	(0.008)	(0.018)
Qty of Capital	0.148	−0.333	0.077	0.108
	(0.003)	(0.021)	(0.008)	(0.020)
Qty of Livestock	0.027	0.229	−0.326	0.070
	(0.007)	(0.023)	(0.005)	(0.023)
Qty of Other Inputs	0.020	0.061	0.013	−0.094
	(0.003)	(0.011)	(0.004)	(0.012)

Note: [a] Numbers in parentheses are standard deviations of the MCMC samples.

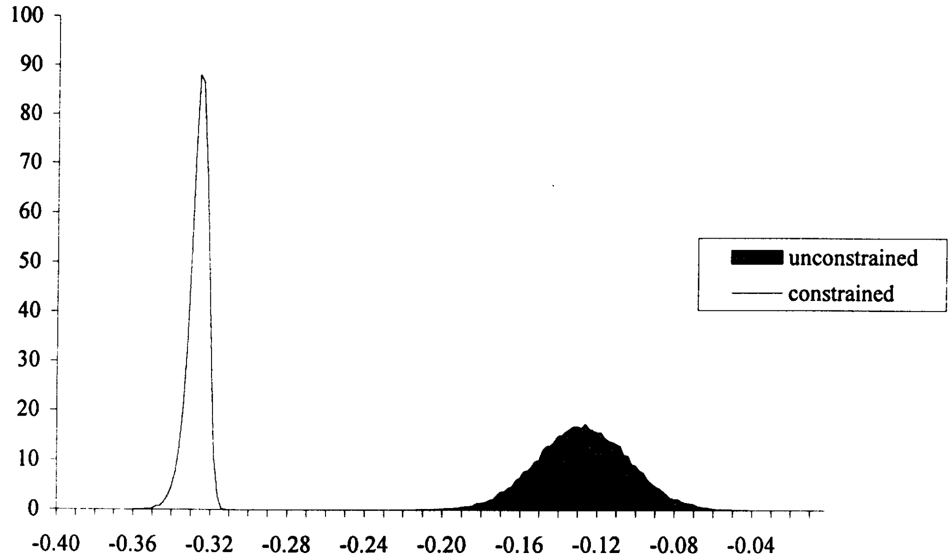

Figure 1 Estimated distribution of the own-price elasticity for livestock evaluated at average prices

price elasticity decreases from −0.13 to −0.33, to become slightly more elastic than the estimate of McKay, Lawrence and Vlastuin (1980) (−0.2). Second, the standard deviations of the constrained and unconstrained probability density functions are generally similar. Again, the only notable exception is the standard deviation of the own-price elasticity for livestock, which falls dramatically with the imposition of the constraints. Finally, the two cross-price elasticities which measure the relationships between the prices and quantities of livestock and other inputs undergo a sign reversal with the imposition of the constraints. Thus, livestock and other inputs appear to be substitutes in production, a result which is consistent with the findings of Watts and Quiggin (1984).

A final illustration of the effects of imposing regularity constraints is provided in figures 1 and 2 where we present the unconstrained and constrained probability density functions of the own-price elasticity for livestock and the cross-price elasticity between livestock and the group of other inputs. Note from figure 2 that the unconstrained (constrained) cross-price elasticity is positive (negative) with estimated probability zero.

From a statistical standpoint, it is interesting that the imposition of concavity changed the coefficient estimates very little despite the fact that the unconstrained estimates led to concavity violations at several price vectors. Furthermore, small differences in the coefficient estimates have led to much greater differences in a few of the elasticities.

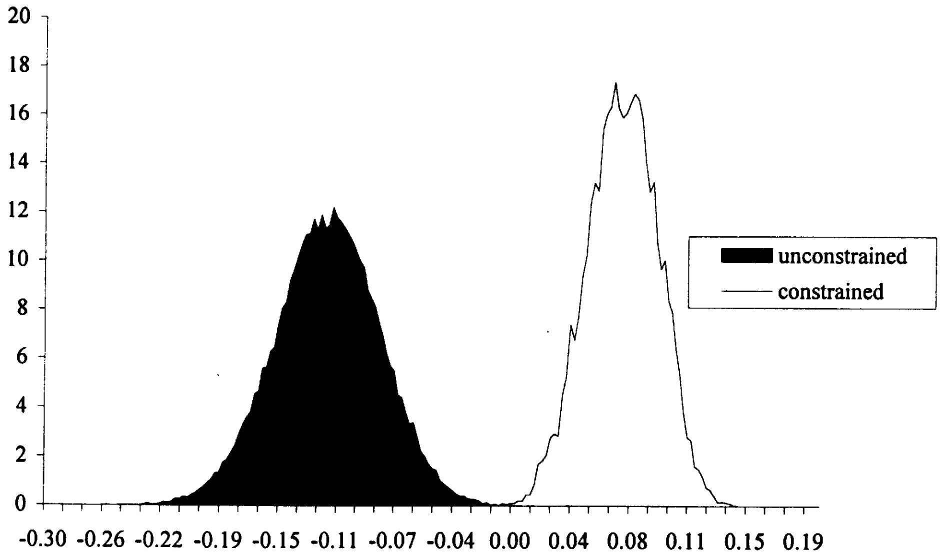

Figure 2 Estimated distribution of the cross-price elasticity between livestock and labour and other inputs evaluated at average prices

6. Conclusion

This article uses Bayesian methods to impose regularity conditions on a system of cost and factor share equations. The Bayesian methodology represents an alternative to conventional sampling theory techniques which can impose regularity, but typically destroy the flexibility properties of many of the more popular functional forms. The Bayesian approach has previously been used by Terrell to estimate the parameters of a cost function using the well-known Berndt and Wood (1975) data set. However, in contrast to Terrell, we use a Metropolis-Hastings algorithm, rather than the Gibbs sampler, to estimate posterior quantities which satisfy the regularity conditions. For problems like ours, with a large number of inequality constraints, the Metropolis-Hastings algorithm is an MCMC technique with much greater practical usefulness.

Our empirical application has been motivated by the large number of regularity violations reported in the study by O'Donnell and Woodland (1995). Thus, our empirical model is based on the translog model of O'Donnell and Woodland and estimated using (a part of) their data set. The empirical results we present include parameter estimates, eigenvalue estimates and estimates of input price elasticities for models with and without regularity constraints imposed. Our unconstrained MCMC estimates are almost identical to our maximum likelihood estimates and the maximum likelihood estimates of O'Donnell and Woodland. Our constrained MCMC estimates differ from our unconstrained estimates in several respects: all maximum eigenvalues become exactly zero in accordance with economic theory, coefficient standard deviations become smaller, and the signs and magnitudes of coefficients and elasticities associated with the livestock input undergo noticeable change. This last result is consistent with the finding of O'Donnell and Woodland that a large number of their regularity violations were associated with the livestock input. Empirically, as far as we are aware, our elasticity estimates are the only ones available for the Australian wool industry which are consistent with the regularity conditions of economic theory. As such, they are a useful input into studies which assign probability distributions to key parameters for the assessment of welfare effects. See, for example, Zhao *et al*. (2000).

Finally, our study offers a number of opportunities for further research. Perhaps the most interesting of these involve the specification of a more complex error structure: one possibility is the heteroskedastic error components structure of O'Donnell and Woodland; another possibility is the truncation of one or more of these error components in line with the stochastic specifications popular in the frontier literature. Other obvious extensions include the use of alternative functional forms and relaxation of the assumption of constant returns to scale.

References

Barnett, W.A. 1983, 'New indices of money supply and the flexible Laurent demand system', *Journal of Business and Economic Statistics*, vol. 1, pp. 7–23.

Berndt, E.R. and Wood, D.O. 1975, 'Technology, prices and the derived demand for energy', *Review of Economics and Statistics*, vol. 57, pp. 259–68.

Best, N.G., Cowles, M.K. and Vines, S.K. 1995, *CODA Manual Version 3.0.*, MRC Biostatistics Unit, Cambridge.

Bigsby, H.R. 1994, 'Production structure and the Australian sawmilling industry', *Australian Journal of Agricultural Economics*, vol. 38, no. 3, pp. 271–88.

Casella, G. and George, E.I. 1992, 'Explaining the Gibbs sampler', *The American Statistician*, vol. 46, no. 3, pp. 167–73.

Chalfant, J.A. and Wallace, N.E. 1992, 'Bayesian analysis and regularity conditions on flexible functional forms: application to the US motor carrier industry', in Griffiths, W.E., Lutkepohl, H. and Bock, M.E. (eds), *Readings in Econometric Theory and Practice: A Volume in Honor of George Judge*, North-Holland, Amsterdam.

Chambers, R.G. 1988, *Applied Production Analysis*, Cambridge University Press, Cambridge.

Chib, S. and Greenberg, E. 1996, 'Markov Chain Monte Carlo Methods in econometrics', *Econometric Theory,* vol. 12, pp. 409–31.

Christensen, L.R., Jorgensen, D.W. and Lau, L.J. 1971, 'Conjugate duality and the transcendental logarithmic production function', *Econometrica*, vol. 39, pp. 255–6.

Coelli, T.J. 1996, 'Measurement of total factor productivity growth and biases in technological change in Western Australian agriculture', *Journal of Applied Econometrics*, vol. 11, pp. 77–91.

Diewert, W.E. 1971, 'An application of the Shephard duality theorem: a generalised Leontief production function', *Journal of Political Economy*, vol. 79, pp. 481–507.

Diewert, W.E. and Wales, T.J. 1987, 'Flexible functional forms and global curvature conditions', *Econometrica*, vol. 55, no. 1, pp. 43–68.

Fisher, B.S. and Wall, C.A. 1990, 'Supply response in the Australian sheep industry: a profit function approach', *Australian Journal of Agricultural Economics*, vol. 34, no. 2, pp. 147–66.

Gallant, A.R. and Golub, E.H. 1984, 'Imposing curvature restrictions on flexible functional forms', *Journal of Econometrics*, vol. 26, pp. 295–321.

Geweke, J. 1992, 'Evaluating the accuracy of sampling-based approaches to calculating posterior moments', in Bernardo, J.M., Berger, J.O., Dawid, A.P. and Smith, A.F.M. (eds), *Bayesian Statistics 4*, Clarendon Press, Oxford.

Heidelberger, P. and Welch, P. 1983, 'Simulation run length control in the presence of an initial transient', *Operations Research,* vol. 31, pp. 1109–44.

Judge, G.G., Griffiths, W.E., Hill, R.C., Lutkepohl, H. and Lee, T.C. 1985, *The Theory and Practice of Econometrics*, 2nd edn, John Wiley, New York.

Lau, L.J. 1978, 'Testing and imposing monotonicity, convexity and quasiconvexity constraints', in Fuss, M. and McFadden, D. (eds), *Production Economics: A Dual Approach to Theory and Applications, Vol. 1,* North-Holland, Amsterdam.

Lopez, R.E. 1980, 'The structure of production and the derived demand for inputs in Canadian agriculture', *American Journal of Agricultural Economics*, vol. 62, no. 1, pp. 38–45.

Lopez, R.E. and Tung, F. 1982, 'Energy and non-energy input substitution possibilities and output scale effects in Canadian agriculture', *Canadian Journal of Agricultural Economics*, vol. 30, July, pp. 115–32.

McKay, L., Lawrence, D. and Vlastuin, C. 1980, 'Input demand and substitution in the Australian sheep industry', *Review of Marketing and Agricultural Economics*, vol. 48, no. 2, pp. 57–70.

Mullen, J.D. and Cox, T.L. 1996, 'Measuring productivity growth in Australian broadacre agriculture', *Australian Journal of Agricultural Economics*, vol. 40, no. 3, pp. 189–210.

O'Donnell, C.J. and Woodland, A.D. 1995, 'Estimation of Australian wool and lamb production technologies under uncertainty: an error-components approach', *American Journal of Agricultural Economics*, vol. 77, August, pp. 552–65.

Pope, R.D. and Just, R.E. 1998, 'Cost function estimation under risk aversion', *American Journal of Agricultural Economics*, vol. 80, May, pp. 288–95.

Shumway, C.R. and Alexander, W.P. 1988, 'Agricultural product supplies and input demands: regional comparisons', *American Journal of Agricultural Economics*, vol. 70, no. 1, pp. 153–61.

Talpaz, H., Alexander, W.P. and Shumway, C.R. 1989, 'Estimation of systems of equations subject to curvature constraints', *Journal of Statistical Computation and Simulation*, vol. 32, July, pp. 201–14.

Terrell, D. 1996, 'Incorporating monotonicity and concavity conditions in flexible functional forms', *Journal of Applied Econometrics*, vol. 11, no. 2, pp. 179–94.

Watts, G. and Quiggin, J. 1984, 'A note on the use of a logarithmic time trend', *Review of Marketing and Agricultural Economics*, vol. 52, no. 2, pp. 91–9.

White, K.J. 1978, 'A general computer program for econometric methods — SHAZAM', *Econometrica*, January, pp. 239–40.

Wiley, D.E., Schmidt, W.H. and Bramble, W.J. 1973, 'Studies of a class of covariance structure models', *Journal of the American Statistical Association*, vol. 68, pp. 317–23.

Zellner, A. 1971, *An Introduction to Bayesian Inference in Econometrics*, John Wiley, New York.

Zhao, X., Griffiths, W.E., Griffith, G.R. and Mullen, J.D. 2000, 'Probability distributions for economic surplus changes: the case of technical change in the Australian wool industry', *Australian Journal of Agricultural and Resource Economics*, vol. 44, no. 1, pp. 83–106.