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Introduction to game-theory calculations

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Abstract. Game theory can be defined as the study of mathematical models of conflict and cooperation between intelligent and rational decision makers (Myerson 1991). Game-theory concepts apply in economy, sociology, biology, and health care, and whenever the actions of several agents (individuals, groups, or any combination of these) are interdependent. We present a new command `gamet` to represent the extensive form (game tree) and the strategic form (payoff matrix) of a noncooperative game and to identify the solution of a nonzero and zero-sum game through dominant and dominated strategies, iterated elimination of dominated strategies, and Nash equilibrium in pure and fully mixed strategies. Further, `gamet` can identify the solution of a zero-sum game through maximin criterion and the solution of an extensive form game through backward induction.

Keywords: st0088, Game theory, Nash equilibrium, payoff matrix, zero-sum game, game tree

1 Introduction

Game theory can be defined as the study of mathematical models of conflict and cooperation between intelligent and rational decision makers, also called players (Myerson 1991). *Rational* means that each players' decision-making behavior is consistent with the maximization of subjective expected outcome. *Intelligent* means that each player understands everything about the structure of the situation, including the fact that others are intelligent, rational, decision makers. A game involves at least two players, in which each player might be one individual or a group of individuals. Game theory provides general methods of dealing with interactive optimization problems; its methods and concepts, particularly the notion of strategy and strategic equilibrium, find a vast number of applications throughout the social sciences, including biology and health care (see among many others, von Neumann and Morgenstern 1944; Shubik 1964; Gibbons 1992; Tomlinson 1997; Dowd and Root 2003; Dowd 2004; Riggs 2004; Nowak and Sigmund 2004).

This paper provides a brief introduction of two types of noncooperative models: the strategic game and the extensive game. In strategic or normal-form games, the players choose their actions simultaneously, so when they choose a plan of action, they are not informed of the plan of action chosen by any other player. Instead, in an extensive game, each player can consider his plan of action not only at the beginning of the game but also whenever he has to make a decision.

 st0088

The rest of the article is organized as follows: section 2 provides basic notation and introduces some methods to analyze the game; sections 3 and 4 present an extensive game and a zero-sum game; section 5 describes the syntax; section 6 explains a key feature behind `gamet`; and section 7 shows some examples extracted from cited references.

2 Strategic game

Whenever the strategy spaces of the players are discrete (and finite), the game can be represented compactly as a matrix. In such a game, player 1 has R possible actions, and player 2 has C possible actions; the payoff pairs to any strategy combination can be neatly arranged in an $R \times C$ table; and the game is easily analyzable (table 1). A payoff is a number, also called a utility, that reflects the desirability of an outcome to a player, for whatever reason (Turocy and von Stengel 2002). We denote the set of strategies for player 1 and player 2 with $S_1 = (1, 2, \ldots, r, \ldots, R)$ and $S_2 = (1, 2, \ldots, c, \ldots, C)$. The numbers of S_1 and S_2 may have labels. There are two elements $(u_{1rc}; u_{2rc})$ within each cell of the table. The first subscript takes only two values (1, 2) and simply denotes player 1 or player 2. The subscripts r and c denote, respectively, the strategies played by player 1 and player 2. Thus u_{1rc} is the payoff for player 1 when player 1 chooses strategy r and player 2 chooses strategy c, whereas u_{2rc} is the payoff for player 2 when player 1 chooses strategy r and player 2 chooses strategy c. Many methods are available to seek the solution of the game. We start analyzing the game by collecting the maximum payoffs and their related subscripts for each player given the choice of the other player into lists of numbers.

Table 1: Notation for an $R \times C$ payoff matrix

	S_2					
S_1	1	2	$\ldots$	c	$\ldots$	C
1	$(u_{111}; u_{211})$	$(u_{112}; u_{212})$	$\ldots$	$(u_{11c}; u_{21c})$	$\ldots$	$(u_{11C}; u_{21C})$
2	$(u_{121}; u_{221})$	$(u_{122}; u_{222})$	$\ldots$	$(u_{12c}; u_{22c})$	$\ldots$	$(u_{12C}; u_{22C})$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
r	$(u_{1r1}; u_{2r1})$	$(u_{1r2}; u_{2r2})$	$\ldots$	$(u_{1rc}; u_{2rc})$	$\ldots$	$(u_{1rc}; u_{2rC})$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
R	$(u_{1R1}; u_{2R1})$	$(u_{1R2}; u_{2R2})$	$\ldots$	$(u_{1Rc}; u_{2Rc})$	$\ldots$	$(u_{1RC}; u_{2RC})$

Let's define the maximum payoff MU_{1rc} of player 1 if player 2 plays strategy c as the highest value on the left side of column c. Formally,

$$\mathrm{MU}_{1rc} = \max \{u_{11c}, u_{12c}, \ldots u_{1rc} \ldots, u_{1Rc}\} \quad \text{with } c = 1, 2, \ldots, C$$

Let's define the maximum payoff MU_{2rc} of player 2 if player 1 plays strategy r as the highest value on the right side of the row r. Formally,

$$\mathrm{MU}_{2rc} = \max\{u_{2r1}, u_{2r2}, \dots u_{2rc} \dots, u_{2rC}\} \quad \text{with } r = 1, 2, \dots, R$$

We create a list, SMU_1, of C elements containing the subscripts r of all maximum payoffs MU_{1rc} for player 1 and a list, SMU_2, of R elements containing the subscripts c of all maximum payoffs MU_{2rc} for player 2. Furthermore, we create a list, SU_1, of subscripts for all possible strategies of player 1 (1, 2, ..., R) and a list, SU_2, of subscripts for all possible strategies of player 2 (1, 2, ..., C). These lists of numbers are useful for seeking the solution of a general R by C payoff matrix.

2.1 Dominant or dominated strategies

Since all players are assumed to be rational, they make choices that result in the payoff they prefer most, given what their opponents do. The examination of which strategies are strictly dominated can lead us to determine the solution of the game (option `domist`). A strategy strictly dominates another strategy of a player if it always provides a greater payoff to that player, regardless of the strategy played by the opposing player. A strategy is strictly dominated by another strategy of a player if it provides a lower payoff, regardless of the strategy played by the opposing player. Using the notation described in section 2, a comparison between SU_1 and SMU_1 gives us the strictly dominated strategies for player 1. The subscripts of the list SU_1 not belonging to the list SMU_1 identify the strictly dominated strategies for player 1. If the subscripts of the list SMU_1 are all the same, say, a particular value r, it means that player 1 has a dominant strategy identified by the subscript r. In a similar manner, we identify strictly dominated and dominant strategies for player 2 by comparing the subscripts of the lists SU_2 and SMU_2. The subscripts of the list SU_2 not belonging to the list SMU_2 identify the strictly dominated strategies for player 2. If the subscripts of the list SMU_2 are all the same, say, a particular value c, it means that player 2 has a dominant strategy identified by the subscript c. See section 6 for a worked example.

2.2 Iterative elimination of dominated strategies

Rational players will never play strictly dominated strategies because such strategies can never be best responses to any strategies of the other player. Eliminating a strictly dominated strategy should not affect the analysis of the game because this fact should be evident to all players in the game (common knowledge assumption). The option `elids` performs an iterative elimination of all strictly dominated strategies. The steps of this algorithm are as follows: 1) determine MU_{1rc}, MU_{2rc}, SMU_1, SMU_2, SU_1, and SU_2; 2) identify the strictly dominated strategies for each player; and 3) remove them. The option `elids` repeats the steps from 1 to 3 until there are no more strictly dominated strategies to eliminate. Although the process is intuitively appealing, each step of elimination requires a further assumption about the other player's rationality.

2.3 Nash's equilibrium in pure and fully mixed strategies

Another way to find the solution of the game is through Nash's equilibrium (Myerson 1991) in pure and fully mixed strategies (options `neps` and `nefms`). A Nash equilibrium, also called a strategic equilibrium, is a list of strategies, one for each player, which has the property that no player has incentive to deviate from his strategy and get a better payoff, given that the other players do not deviate (Turocy and von Stengel 2002). A mixed strategy is a strategy generated at random according to a particular probability distribution that determines the player's decision. As a special case, a mixed strategy can be a deterministic choice of one of the given pure strategies.

A Nash equilibrium in pure strategy specifies a strategy for each player in such a way that each player's strategy yields the player at least as high a payoff as any other strategy of the player, given the strategies of the other player.

Based on our notation, we can say that Nash equilibriums in pure strategies are all pairs of strategies for which MU_{1rc} and MU_{2rc} have the same pairs of subscripts r and c. In other words, we proceed in two steps: first, we determine the best response; and second, we find the strategy profiles where strategies are best responses to each other. See section 6 for a worked example.

A Nash equilibrium in mixed strategy specifies a mixed strategy for each player in such a way that each player's mixed strategy yields the player at least as high an expected payoff as any other mixed strategy, given the mixed strategies of the other player. Fully mixed strategies mean that the probability associated with each strategy cannot be equal to zero or one. The command `gamet` can find Nash equilibrium in fully mixed strategies if $R = 2$ and $C = 2$ (table 2).

Table 2: Payoff matrix if each player has two strategies

	S_2	
S_1	1	2
1	$(u_{111}; u_{211})$	$(u_{112}; u_{212})$
2	$(u_{121}; u_{221})$	$(u_{122}; u_{222})$

Player 1 would be willing to randomize between S_1=(1) and S_1=(2) only if these strategies gave him the same expected utility. More formally, we seek the probability p so that both sides of (1) are equal.

$$p \times u_{111} + (1-p) \times u_{112} = p \times u_{121} + (1-p) \times u_{122} \tag{1}$$

Thus player 2's strategy in the equilibrium must be equal to

$$p \times S_2(1) + (1-p) \times S_2(2)$$

to make player 1 willing to randomize between $S_1(1)$ and $S_1(2)$. $S_1(1)$ and $S_1(2)$ indicate the strategies for player 1, while $S_2(1)$ and $S_2(2)$ indicate the strategies for player 2.

Similarly, player 2 would be willing to randomize between $S_2 = (1)$ and $S_2 = (2)$ only if these strategies give him the same expected utility. Again we seek the probability q such that both sides of (2) are equal.

$$q \times u_{211} + (1 - q) \times u_{221} = q \times u_{212} + (1 - q) \times u_{222} \tag{2}$$

Thus player 1's strategy in the equilibrium must be equal to

$$q \times S_1(1) + (1 - q) \times S_1(2)$$

to make player 2 willing to randomize between $S_2(1)$ and $S_2(2)$.

The Nash equilibrium in fully mixed strategies must be equal to (3).

$$\{p \times S_2(1) + (1 - p) \times S_2(2),\ q \times S_1(1) + (1 - q) \times S_1(2)\} \tag{3}$$

We find the solution of (1) and (2) by using explicit formulas.

3 Extensive game

An extensive form game is a dynamic model because it includes a full description of the sequence in which actions and events may occur over time. This section treats games of perfect information. Thus each player is at any point aware of the previous choices of the other player. Only one player moves at a time, so there are no simultaneous moves. Whenever the moves of the game are sequential, we could use a game tree to describe the game. The option `gtree` provides a graphical representation (figure 1) of a sequential game. It is a wrapper for `twoway scatter` that provides information about the players, payoffs, strategies, and the order of the moves. The first move of the game is identified with a distinguished node that is called the root of the tree. A play of the game consists of a connected chain of edges starting at the root of the tree and ending at a terminal node. The nodes in the tree represent the possible moves in the game. We use the rollback or backward induction to find the equilibrium. Suppose that there are two players (figure 1); where player 1 moves first, and player 2 moves second. Start at each of the terminal nodes of the game, and compare the payoffs that player 2 receives at the terminal node; assume that player 2 always chooses the action giving him the maximal payoff. So, player 2 will choose $S_2 = \{$`Buy`$\}$, $2 > 1$, if player 1 chooses $S_1 = \{$`High`$\}$ and $S_2 = \{$`Not_buy`$\}$, $1 > 0$, if player 1 chooses $S_1 = \{$`Low`$\}$. Now treat the next-to-last decision node of the tree as the terminal node. Given player 2's choices compare the payoffs that player 1 receives in the next-to-last node and assume again that player 1 always chooses the action giving him the maximal payoff. Thus, player 1 will choose $S_1 = \{$`High`$\}$ because he will get a better payoff, $2 > 1$. The path indicated by arrows is the equilibrium path (figure 1).

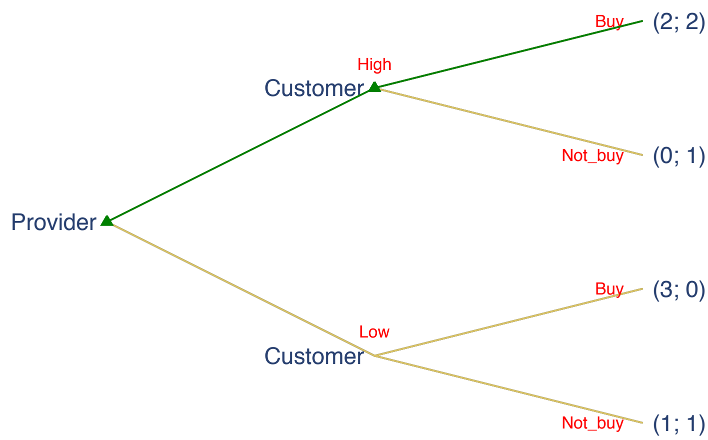

Figure 1: Example of a 2×2 extensive game (Turocy and von Stengel 2002) where player 1 has strategies $S_1 = \{$`High`, `Low`$\}$ and player 2 has strategies $S_2 = \{$`Buy`, `Not_buy`$\}$. The path equilibrium (`High`; `Buy`) with payoffs (2; 2), identified by two arrows, is determined through backward induction (option `gtree`).

4 Zero-sum game

In a zero-sum game, one player wins what the other loses, so their payoffs are diametrically opposed. The sum of the payoffs within each cell of table 1 is zero.

$$u_{1rc} + u_{2rc} = 0 \qquad \forall\ r \text{ and } c$$

The players make moves simultaneously. For a two-person zero-sum game, there is a clear concept of solution. The solution to the game is the maximin criterion (option `maximin`); that is, each player chooses the strategy that maximizes his minimum payoff. The minimal column maximum, MCM_1, of player 1 is

$$\mathrm{MCM}_1 = \min\left\{\mathrm{MU}_{1r1}, \mathrm{MU}_{1r2}, \ldots \mathrm{MU}_{1rc} \ldots, \mathrm{MU}_{1rC}\right\}$$

whereas the maximal row minimum, MRM_2, of player 2 is

$$\mathrm{MRM}_2 = \max\left\{-\{\mathrm{MU}_{21c}, \mathrm{MU}_{22c}, \ldots \mathrm{MU}_{2rc} \ldots, \mathrm{MU}_{2Rc}\}\right\}$$

If $\mathrm{MCM}_1 = \mathrm{MRM}_2$, the solution of the game exists and the pair of strategies associated with that pair of payoffs is called the saddle point.

Furthermore, like for nonzero-sum games, we can use the iterated elimination of strictly dominated strategy and Nash equilibrium in pure strategy and fully mixed strategy described in the previous sections.

5 The gamet command

5.1 Description

We introduce a new command `gamet` to represent the extensive form (game tree) and the strategic form (payoff matrix) of a noncooperative game. `gamet` identifies the equilibrium of a nonzero and zero-sum game through several options: dominant and dominated strategies (`domist`), iterated elimination of strictly dominated strategies (`elids`), Nash equilibrium in pure (`neps`), and fully mixed strategies (`nefms`). Further, `gamet` is able to identify the solution of zero-sum games through maximin criterion (`maximin`) and the solution of extensive form games through backward induction (`gtree`). `gamet` is an immediate command, given that it obtains data not from the data stored in memory, but from numbers typed as arguments (see [U] **19 Immediate commands**).

5.2 Syntax

`gamet,` <u>`pay`</u>`off(`$\#u_{111}$`,` $\#u_{211}$`,`$\ldots\#u_{11c}$`,` $\#u_{21c}\ldots\#u_{11C}$`,` $\#u_{21C}$`\...\`
  $\#u_{1r1}$`,` $\#u_{2r1}\ldots\#u_{1rc}$`,` $\#u_{2rc}\ldots\#u_{1rc}$`,` $\#u_{2rC}$`\...\`
  $\#u_{1R1}$`,` $\#u_{2R1}\ldots\#u_{1Rc}$`,` $\#u_{2Rc}\ldots\#u_{1RC}$`,` $\#u_{2RC}$`)` [`ls1(`*lab_s1*`)` `ls2(`*lab_s2*`)`
  `player1(`*rlab1* ... *rlab_r* ... *rlab_R*`)` `player2(`*clab1* ... *clab_c* ... *clab_C*`)` `domist`
  `elids` `neps` `nefms` `maximin` `gtree` `npath` `aspect(`#`)` `savingpf(`*filename*`)`
  `mlabpls(`*clockpos*`)` `mlabppm(`*clockpos*`)` `mlabpp1(`*clockpos*`)` `mlabpp2(`*clockpos*`)`
  `textpp(`*textsizestyle*`)` `texts(`*textsizestyle*`)` `msizepp(`*relativesize*`)`
  `msizes(`*relativesize*`)` *scatter_options*]

5.3 Options

`payoff(...)` is required and provides a way to input, row after row, a general $R \times C$ payoff matrix by hand; see the `matrix input` command in [P] **matrix define** for more information.

`ls1(`*lab_s1*`)` attaches a label to the set of strategies S_1 for player 1. The default is S_1.

`ls2(`*lab_s2*`)` attaches a label to the set of strategies S_2 for player 2. The default is S_2.

`player1(`*rlab1* ... *rlab_r* ... *rlab_R*`)` attaches a label for each strategy of player 1. The default is A_1, B_2, C_3, and so on.

`player2(`*clab1* ... *clab_c* ... *clab_C*`)` attaches a label for each strategy of player 2. The default is a_1, b_2, c_3, and so on.

`domist` seeks strictly dominated and dominant strategies for each player.

`elids` eliminates iteratively all strictly dominated strategies for each player.

`neps` seeks Nash equilibrium in pure strategies.

`nefms` seeks Nash equilibrium in fully mixed strategies ($0 < p < 1$ and $0 < q < 1$). It works only if R and C are equal to 2.

`maximin` seeks the saddle point through the minimal column maximum for player 1 and maximal row minimum for player 2. It works for zero-sum games; that is, $\#u_{2rc} + \#u_{1rc} = 0$.

`gtree` seeks the equilibrium path through backward induction (player 1 moves first). It produces a game tree.

`npath` specifies no equilibrium path on the game tree.

`aspect(`*#*`)` modifies the aspect ratio (height/width) of the plot region. By default, `aspect()` is set to 1 (equal height and width), so the plot region is a square.

`savingpf(`*filename*`)` saves the variables obtained by the conversion of the payoff matrix in a file. If the `elids` option is specified, `savingpf()` saves one file for each iteration.

`mlabpls(`*clockpos*`)` specifies the position for label *lab_s1* and *lab_s2* on the game tree. The default is `mlabpls(9)`. See [G] ***clockposstyle***.

`mlabppm(`*clockpos*`)` specifies the position for $\#u_{2rc}$ and $\#u_{1rc}$ on the game tree. The default is `mlabppm(3)`.

`mlabpp1(`*clockpos*`)` specifies the position for strategy labels on the game tree for player 1.

`mlabpp2(`*clockpos*`)` specifies the position for strategy labels on the game tree for player 2.

`textpp(`*textsizestyle*`)` specifies the text size style for *lab_s1*, *lab_s2*, and ($\#u_{2rc}$, $\#u_{1rc}$). The default is `textpp(medium)`. See [G] ***textsizestyle***.

`texts(`*textsizestyle*`)` specifies the text size style for strategy labels for both player. The default is `texts(small)`.

`msizepp(`*relativesize)* specifies the sizes of objects *lab_s1*, *lab_s2*, and ($\#u_{2rc}$, $\#u_{1rc}$). The default is `msizepp(2)`. See [G] ***relativesize***.

`msizes(`*relativesize)* specifies the sizes of strategy labels. The default is `msizes(2)`.

scatter_options are any of the `scatter` options. See [G] **graph twoway scatter**.

5.4 Saved results

`gamet` saves in `r()`:

Scalars			
`r(R)`	# player 1's strategies	`r(C)`	# player 2's strategies
`r(p)`	player 1's probability	`r(q)`	player 2's probability
`r(expep1)`	player 1's expected payoff	`r(expep2)`	player 2's expected payoff
Macros			
`r(cmd)`	`gamet`		
`r(ls1)`	player 1's label	`r(ls2)`	player 2's label
`r(player1)`	player 1's strategies labels	`r(player2)`	player 2's strategies labels
`r(dtp1)`	player 1's dominant strategy	`r(ddsp1)`	player 1's dominated strategy
`r(dtp2)`	player 2's dominant strategy	`r(ddsp2)`	player 2's dominated strategy
`r(mcmp1)`	player 1's min col maximum	`r(mrmp2)`	player 2's max row minimum
`r(neps#)`	Nash equilibrium number #	`r(sp#)`	saddle-point number #
Matrix			
`r(PM)`	payoff matrix		

6 Key features: lists and subscripts

The use of lists and subscripts might be useful for handling a bimatrix. To clarify the notation described in section 2, we consider a simple example of two players with two strategies (table 3).

Table 3: Basic example of a 2 × 2 payoff matrix

	S_2	
S_1	1	2
1	(2; 2)	(4; 0)
2	(1; 0)	(3; 1)

Looking at table 3, we can say that $R = 2$, $C = 2$, $S_1 = \{1, 2\}$, and $S_2 = \{1, 2\}$. S_1 and S_2 may have labels. Let's find the maximum payoffs MU_{1rc} and MU_{2rc} for each possible strategy combination. If player 2 plays strategy $S_2 = \{1\}$, $c = 1$, the highest value on the left side of column 1 is 2 for $S_1 = \{1\}$,

$$\mathrm{MU}_{111} = \max\{2, 1\} = 2$$

We put the subscript r of MU_{111}, equal to 1, in a list called SMU_1. If player 2 plays strategy $S_2 = \{2\}$, $c = 2$, the highest value on the left side of column 2 is 4 for $S_1 = \{1\}$,

$$\mathrm{MU}_{112} = \max\{4, 3\} = 4$$

Again, we put the subscript r of MU_{112}, equal to 1, in the list SMU_1. There are no more strategies for player 2, and $\mathrm{SMU}_1 = \{1, 1\}$.

Let's see what happens for player 2, depending on the action of player 1. If player 1 plays S_1={1}, $r = 1$, the highest value on the right side of row 1 is 2. So,

$$\mathrm{MU}_{211} = \max\{2,0\} = 2$$

We put the subscript c of MU_{211}, equal to 1, in a list called SMU_2. Now, if player 1 plays strategy S_1={2}, $r = 2$, the highest value on the right side of row 2 is 1 for S_2={2}:

$$\mathrm{MU}_{222} = \max\{0,1\} = 1$$

We put the subscript c of MU_{222}, equal to 2, in the list SMU_2. There are no more strategies for player 1, and $\mathrm{SMU}_2 = \{1,2\}$.

We use the lists $\mathrm{SU}_1 = \{1,2\}$ and $\mathrm{SU}_2 = \{1,2\}$ to identify the strategies actually available for each player. In this simple example, there is no difference among SU_1, SU_2, and the set of strategies S_1, S_2. However, creating the lists SU_1 and SU_2 is very useful when we take out columns and rows from the payoff table 1 in the iterative elimination of strictly dominated strategies (see section 2.2).

As we said in section 2, the strictly dominated strategies of player 1 are those elements of SU_1 that do not belong to SMU_1. Given that $\mathrm{SU}_1 = \{1,2\}$ and $\mathrm{SMU}_1 = \{1,1\}$, we conclude that $S_1 = \{2\}$ is a strategy strictly dominated for player 1. Again, the strictly dominated strategies of player 2 are those elements of SU_2 not belonging to SMU_2. Given that $\mathrm{SU}_2 = \{1,2\}$ and $\mathrm{SMU}_2 = \{1,2\}$, we conclude that there is no strategy strictly dominated for player 2. Furthermore, since the elements of $\mathrm{SMU}_1 = \{1,1\}$ are all equal to 1, we can say that the strategy $S_1 = \{1\}$ is a dominant strategy for player 1. Instead, player 2 does not have a dominant strategy since the elements in $\mathrm{SMU}_2 = \{1,2\}$ are different.

Based on the definition of Nash equilibrium given in section 2.3, we can easily conclude that $S_1 = \{1\}$ and $S_2 = \{1\}$, with payoff (2, 2), is the Nash equilibrium in pure strategy. In fact, among the maximum payoffs of player 1 (MU_{111}, MU_{112}) and player 2 (MU_{211}, MU_{222}), only the payoffs MU_{111} and MU_{211} have the same pairs of subscripts ($r = 1$ and $c = 1$).

7 Examples

7.1 Dominant and dominated strategies: the prisoner's dilemma

The prisoner's dilemma is often presented in the following manner (Myerson 1991, table 3.4, 97). Two individuals (players) are arrested for a crime they committed together. They are questioned separately and are offered the same deal. They are accused of conspiring in two crimes: one minor crime for which their guilt can be proved without any confession, and one major crime for which they can be convicted only if at least one confesses. If one confesses, he will go free (payoff 6), but the other will go to jail (payoff 0). If both confess, they both go to jail (payoff 1). If neither confesses, they will go to jail for the minor crime (payoff 5). So each prisoner has two possible strategies: $S_1 = \{\texttt{Not\_confess}, \texttt{Confess}\}$ and $S_2\{\texttt{Not\_confess}, \texttt{Confess}\}$.

```
. gamet, payoff(5, 5, 0, 6 \ 6, 0, 1, 1) player1(Not_confess Confess)
> player2(Not_confess Confess) domist

                                S2
           S1   Not_confess    Confess

Not_confess        (5; 5)       (0; 6)
    Confess        (6; 0)       (1; 1)

DOMINATED AND DOMINANT STRATEGIES
    Dominated strategy for S1 is Not_confess
    Dominated strategy for S2 is Not_confess
    Dominant strategy for S1 is Confess
    Dominant strategy for S2 is Confess
```

As we can see from the output of the option `domist`, there is only one solution since $S_1 = \{\texttt{Confess}\}$ and $S_2 = \{\texttt{Confess}\}$ are dominant strategies. Somewhat paradoxically, this outcome is worse than (`Not_confess`; `Not_confess`). Therefore, the individual rational maximum payoff does not lead always to a better payoff to both. Prisoner's-dilemma games arise in several situations, such as environmental pollution, where individual actions at the expense of others lead to overall less-desirable outcomes.

7.2 Iterated elimination of dominated strategies

In this game (Turocy and von Stengel 2002, table 9, 27), there are only two manufacturing firms who can decide to produce a certain quantity of memory chips (high, medium, low, or nonproduction), denoted by $S_1 = \{\texttt{H}, \texttt{M}, \texttt{L}, \texttt{N}\}$ for `Firm_I` and $S_2 = \{\texttt{h}, \texttt{m}, \texttt{l}, \texttt{n}\}$ for `Firm_II`. The market price of the memory chips decreases with the increasing total quantity produced by both companies. In particular, if both choose to produce a high quantity, the price collapses so that profits drop to zero. The payoff matrix is shown below, and this game can be solved by dominance considerations. Using the option `elids`, we can easily find the solution of the game: (`M`, `m`) with payoffs (`16`; `16`).

(*Continued on next page*)

```
. gamet, payoff(0,0,12,8,18,9,36,0\8,12,16,16,20,15,32,0\9,18,15,20,18,18,27,0\
> 0,36,0,32,0,27,0,0) player1(H M L N) player2(h m l n)
> ls1(Firm_I) ls2(Firm_II) elids

                              Firm_II
   Firm_I        h          m          l           n

        H    (0; 0)     (12; 8)    (18; 9)    (36; 0)
        M    (8; 12)   (16; 16)   (20; 15)    (32; 0)
        L    (9; 18)   (15; 20)   (18; 18)    (27; 0)
        N    (0; 36)    (0; 32)    (0; 27)    (0; 0)

Iteration 1
DOMINATED AND DOMINANT STRATEGIES
    Dominated strategy for Firm_I is N
    Dominated strategy for Firm_II is n
    No dominant strategy for Firm_I
    No dominant strategy for Firm_II

                         Firm_II
   Firm_I        h          m          l

        H    (0; 0)     (12; 8)    (18; 9)
        M    (8; 12)   (16; 16)   (20; 15)
        L    (9; 18)   (15; 20)   (18; 18)

Iteration 2
DOMINATED AND DOMINANT STRATEGIES
    Dominated strategy for Firm_I is H
    Dominated strategy for Firm_II is h
    No dominant strategy for Firm_I
    No dominant strategy for Firm_II

                    Firm_II
   Firm_I        m          l

        M   (16; 16)   (20; 15)
        L   (15; 20)   (18; 18)

Iteration 3
DOMINATED AND DOMINANT STRATEGIES
    Dominated strategy for Firm_I is L
    Dominated strategy for Firm_II is l
    Dominant strategy for Firm_I is M
    Dominant strategy for Firm_II is m

              Firm_II
   Firm_I        m

        M   (16; 16)
```

```
Iteration 4
DOMINATED AND DOMINANT STRATEGIES
    No dominated strategy for Firm_I
    No dominated strategy for Firm_II
    Dominant strategy for Firm_I is M
    Dominant strategy for Firm_II is m
```

It is always better for each firm to produce a medium quantity of memory chips.

7.3 Nash equilibrium in pure strategies

For an example of a game with two Nash equilibriums in pure strategies, consider the game below (Myerson 1991, table 3.5, 98), known as the *Battle of the Sexes*. The story behind this game is that players 1 and 2 are husband and wife, respectively, and that they are deciding where to go on a Saturday afternoon. Each spouse has two possible strategies: $S_1 = \{\texttt{Football}, \texttt{Shopping}\}$ and $S_2 = \{\texttt{Football}, \texttt{Shopping}\}$. Neither spouse would derive any pleasure from being without the other (payoff 0), but the husband would prefer going to the football match (payoff 3), whereas the wife would prefer going to the shopping center (payoff 3). In this game, there are no dominant strategies. We can seek Nash equilibriums in pure strategies through the option `neps`.

```
. gamet , payoff(3,1,0,0\0,0,1,3) player1(Football Shopping)
> player2(Football Shopping) ls1(Husband) ls2(Wife) domist neps
```

	Wife	
Husband	Football	Shopping
Football	(3; 1)	(0; 0)
Shopping	(0; 0)	(1; 3)

```
DOMINATED AND DOMINANT STRATEGIES
    No dominated strategy for Husband
    No dominated strategy for Wife
    No dominant strategy for Husband
    No dominant strategy for Wife
NASH EQUILIBRIUM IN PURE STRATEGIES
    1. Football Football (3; 1)
    2. Shopping Shopping (1; 3)
```

In the first equilibrium, the players both go to the football match (Football; Football), which is the husband's favorite outcome (but payoff 1 for the wife). In the second equilibrium, the players both go to the shopping center (Shopping; Shopping), which is the wife's favorite outcome (but payoff 1 for the husband). Thus, each player prefers a different equilibrium. There is also a third equilibrium, where the players behave in a random and uncoordinated manner, that could be found using the option `nefms`.

7.4 Nash equilibrium in fully mixed strategies

Consider the case of a compliance inspection (Turocy and von Stengel 2002, table 6, 18). Suppose that a consumer (player `II`) purchases a license for a software package, agreeing to certain restrictions on its use. Say also that the consumer has an incentive to violate these rules. The vendor (player `I`) would like to verify that the consumer is abiding by the agreement, but doing so requires inspections that are costly. If the vendor does an inspection and catches the consumer cheating, the vendor can demand a large penalty payment for noncompliance. In this game, there are no dominant strategies and no Nash equilibriums in pure strategies. So, to find a solution to the game, we choose a mixed strategy, where $S_1 = \{\texttt{Not\_inspect}, \texttt{Inspect}\}$ and $S_2 = \{\texttt{Comply}, \texttt{Cheat}\}$. Through the option `nefms`, we find the probabilities p and q and the expected equilibrium payoff.

```
. gamet, pay(0, 0, -10, 10 \ -1, 0, -6, -90) player1(Not_inspect Inspect)
> player2(Comply Cheat) ls1(I) ls2(II) nefms
```

I	II Comply	Cheat
Not_inspect	(0; 0)	(-10; 10)
Inspect	(-1; 0)	(-6; -90)

```
NASH EQUILIBRIUM IN FULLY MIXED STRATEGIES
     p = 0.90
     q = 0.80
    (0.90*[Not_inspect]+0.10*[Inspect], 0.80*[Comply]+0.20*[Cheat])
    Expected equilibrium payoff for I
    0.80*0+(1-0.80)*-10 = -2
    0.80*-1+(1-0.80)*-6 = -2
    Expected equilibrium payoff for II
    0.90*0+(1-0.90)*0 = 0
    0.90*10+(1-0.90)*-90 = 0
```

It is not hard to see that player `I` is indifferent if player `II` complies with probability $q = 0.8$ since the expected payoffs for $S_1 = \{\texttt{Not\_inspect}\}$ and $S_1 = \{\texttt{Inspect}\}$ are the same.

On the other hand, player `II` is indifferent if player `I` does not inspect with probability $p = 0.9$ (or inspect with probability 0.1) since the expected payoffs for $S_2 = \{\texttt{Cheat}\}$ and $S_2 = \{\texttt{Comply}\}$ are the same.

This defines the unique Nash equilibrium of the game. It uses fully mixed strategies and the resulting expected payoffs are (`-2`; `0`).

7.5 Extensive game

Consider again the example from section 7.2 of two chip manufacturers. Suppose now that `Firm_I` moves first and `Firm_II` moves second (option `gtree`).

```
. gamet, payoff(0, 0, 12, 8, 18, 9, 36, 0\ 8, 12, 16, 16, 20, 15, 32, 0 \ 9, 18
> , 15, 20, 18, 18, 27, 0\ 0, 36, 0, 32, 0, 27, 0, 0)
> player1(H M L N) player2(h m l n) ls1(Firm_I) ls2(Firm_II) gtree
```

	Firm_II			
Firm_I	h	m	l	n
H	(0; 0)	(12; 8)	(18; 9)	(36; 0)
M	(8; 12)	(16; 16)	(20; 15)	(32; 0)
L	(9; 18)	(15; 20)	(18; 18)	(27; 0)
N	(0; 36)	(0; 32)	(0; 27)	(0; 0)

```
BACKWARD INDUCTION
    Equilibrium path: H l
    Payoffs pair: (18; 9)
```

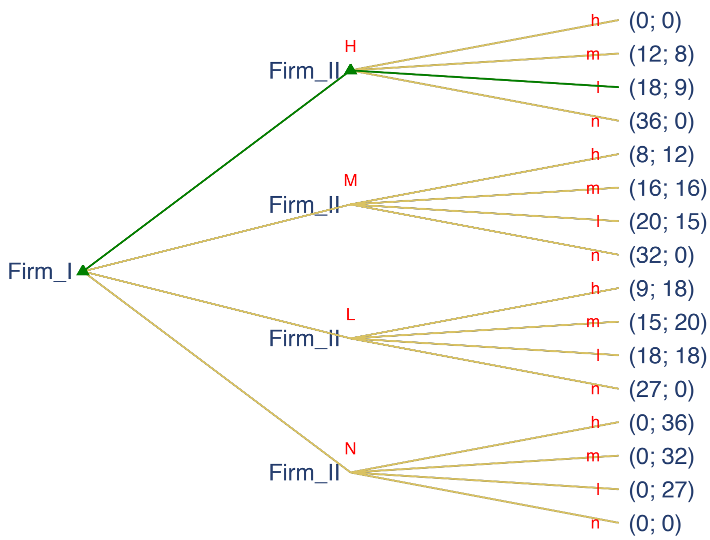

Figure 2: Example of a 4×4 extensive game, option `gtree`.

The equilibrium (figure 2) of the extensive game (`H`, `l`) with payoffs (`18`; `9`) is different from the strategic game (`M`, `m`) with payoffs (`16`; `16`). This is called the first-mover advantage.

8 References

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