The Australian Journal of Agricultural and Resource Economics, 50, pp. 313–326

The optimal allocation of water along a system of rivers: a continuous model with sequential bidding*

Alex Coram†

This paper uses a control theory approach to analyse the collectively optimal rate of extraction along a river system and constructs a bidding mechanism that would produce the required prices at each point. It also analyses some characteristics of this mechanism. This approach brings some new perspective to existing work on externalities. It also helps bring to light some aspects of the system as a whole that may be less obvious in a more piecemeal analysis, including the fact that there may not be an optimal solution to the allocation problem. Although the bidding mechanism may be difficult to implement, it may be possible to design various forms of hybrid schemes that have practical value.

Key words: control theory, mechanism design, optimal allocation, rights markets, waters.

1. Introduction

Social planners and policy-makers increasingly have to make decisions about the optimal allocation of water from an interconnected system of rivers in which demand exceeds supply. This raises questions about the calculation of the collectively optimal rate of extraction at each point. It also raises questions about whether the optimal program can be implemented. The problem here is that the information on pay-off functions required by a centralised distribution mechanism may not be available. If so, is it possible to design a decentralised mechanism that produces the correct price for water at every point? This paper deals with some of these questions.

The solution to the problem of allocating water that has received most attention is to provide individuals with rights, or a permit, to a specified amount of water and allow trading in these rights (Burness and Quirk 1979; Kanazawa 1991; Quiggin 2001; Weber 2001; Ambec and Spumont 2002; Anderson 2004). The problem here is that a river is a unidirectional flow and an agent higher in the order may inflict costs on an agent lower in the order by removing or polluting

* An early version of part of this paper was presented in the seminar series in the School of Economics of the University of Queensland and I am grateful to participants for comments. I am also grateful to the reviewers for *The Australian Journal of Agricultural and Resource Economics* for suggesting a number of revisions and extensions to the analysis.

† Alex Coram (email: a.b.t.coram@rgu.ac.uk) is Professor of Political Economy at the Robert Gordon University, Aberdeen, Scotland, UK and Professor of Political Science at the University of Western Australia, Perth, Australia.

doi: 10.1111/j.1467-8489.2006.00357.x

water. This means that the optimal price cannot, in general, be uniform and that wherever there are externalities any trading scheme must support a set of location-specific prices that shadow the collectively optimal price. An appropriate price could always, in principle, be achieved if a sufficiently complete set of rights could be allocated to all users at every point along the river, but this raises serious, and probably insuperable, transaction costs. The alternative is to design some sort of tradeable permits scheme that would generate the optimal price. This problem has received considerable attention in the literature and more recently there have been attempts to design trading mechanisms that takes the flow properties of a river into account (Weber 2001). As will be shown below, however, there is still some way to go in the design of a satisfactory pricing mechanism.

The purpose of this paper is to fill some of these gaps by analysing the collectively optimal rate of extraction along a river system in a control theory framework and constructing a bidding mechanism that would produce the required prices at each point. It also analyses some characteristics of this mechanism.

The control theory approach used in this paper is also intended to provide a simple way of looking at the problem as a whole that differs from that available in much of the literature, and gives some fresh insights into existing results on externalities. The results of the optimal control approach could, in principle, be established in the more familiar discrete framework, such as that used in Weber (2001) and Kilgour and Dinar (2001). Nonetheless it has the advantage of extending the toolkit available for policy-makers and analysts. It may also help bring to light some aspects of the system that may be less obvious in a more piecemeal analysis. One of these is that there may not be an optimal solution to the maximising problem under all conditions. This may impose a serious constraint on optimal allocation. The result is easy to see, although something of a surprise.

The approach in this paper also differs from recent work in cooperative game theory. This takes advantage of the fact that, because water flows continuously, coalitions must be connected and hence the pay-off available to a player depends on its marginal value (Moulin 1995, p. 407; Ambec and Spumont 2002). This does not tell us anything about collective optimal rates of extraction.

Among the main findings are that, although it is possible to design a mechanism that guarantees the optimum price at each point, this may be difficult to implement if there are a large number of extracting agents or environmental interests are endogenised in the pay-off function. On the other hand it may be possible to design various forms of hybrid schemes that take over part of the suggested mechanism and allow partial avoidance of information and optimal pricing problems.

I set out the paper as follows. In Section 2 a simple model of water allocation is developed and analysed. In Section 3 I show that markets for rights cannot meet the conditions required for an optimal allocation.

2. The model of a river system

2.1 The general model of optimal extraction

Suppose there is a waterway, thought of as a set of connected rivers that flow continuously over their entire length, and a planner wishes to determine the collectively optimal volume of water to be extracted at every point. This collectively optimal volume is understood as the volume that maximises a collective welfare function. It is assumed that the welfare function is additive and depends on the pay-offs to different users as well as constraints that reflect the concerns of the community over such things as minimum water levels and limitations to rates of extraction. It is assumed that each river collects, or loses, water as the result of rain, evaporation, tributaries, branches and so on along its way. These gains or losses are represented in a continuous manner. Any continuous path that terminates at the sea, or in a lake, or marsh, or wherever, and any path that terminates on another path are thought of as a separate river. Label each river as σ^i for $i = 1, \ldots, n$. Let σ^1 be single path that does not commence from or terminate on another river in the system. It does no harm to think of this as the main river in the system, but this need not necessarily be the case. A river that ends on some other river is called terminal (σ^2, σ^3), and a river that begins on another river but does not end on a river is called commencing (σ^4). Rivers that begin and end on other rivers are not investigated; see Figure 1 for an example.

The rate of flow in a river does not matter for present purposes, provided that the water available at any point on the system partly depends on the water at the previous point. The analysis does not hold for a lake. Nor does it hold for a system where water extracted at some point is compensated for by an inflow from downstream, such as a slow-flowing river opening into the sea. In this case extraction of water at some point may cause an inflow of sea water to downstream points.

For simplicity we are only interested in a situation where the stand-alone demands of each user for water impose negative externalities on other users or the community. Floods and other situations where it might be desirable to reduce the level of the river are ignored.

The depth and width of any river σ^i in the system is given by a continuous function $x^i(t)$ and each river is normalised to give $t \in [0, 1]$. The change in the volume of water depends on natural gains and losses and the amount extracted. Natural gains and losses are represented by the continuous function $a^i(t)$.

The amount of water extracted from a river σ^i in the system at each point is $u^i(t)$. In the real world the points of extraction may be some distance apart and $u^i(t)$ would be treated as constant in the interval u^i for extraction at the point $t \in u^i$ where u^i has some non-zero measure. These intervals can be thought of as giving a partition of σ^i. In order to simplify the presentation it is assumed that the partition is sufficiently fine that the step function representing extraction in each interval can be approximated as closely as we like by a continuous function interpolating the step function at the point t. This

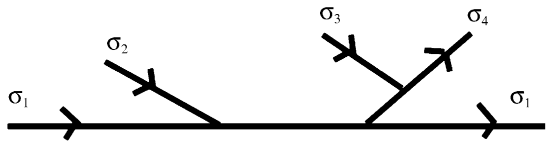

Figure 1 Example of a river system.

gives us the advantage of being able to treat the problem in continuous terms. It involves no loss of generality since the properties of the discrete and the continuous system will be the same (Whittle 1996, p. 132). Now for the details.

The change in the volume of water along a river σ^i is written

$$\dot{x}^i = -u^i(t) + a^i(t) \tag{1}$$

and it is assumed that a river starts at $t = 0$ with an initial amount of water $x^i(0) = 0$ if σ^i is terminating or the main river and $i = 1$ and $x^j(0)$ optimally chosen if σ^j is commencing.

The pay-off to an agent from extracting water depends on quantity extracted as well as the quality and may vary at each point t along the river. The pay-off for extraction along σ^i is written $f^i(t, x^i, u^i)$. Since f varies with t, the extracting agents need not be homogenous. It is assumed that f^i is concave and differentiable in u^i and differentiable in x^i.

For simplicity the volume of water in the river is used as a rough proxy for quality and I do not deal explicitly with pollution discharges and return flows. Although this is only intended as a first approximation it can be justified in a number of ways. Return flows can be eliminated with suitable rescaling, for example, and where these carry back pollution this can be treated as a reduction in the available water. In addition pollution from effluent discharge depends on the amount of water used and on the amount of water in the river, and can also be treated as a function of volume.

It is also reasonable to assume that there are agents that derive a pay-off from the water left in-stream at any point. Examples here would be environmentalist and conservation interests and people who use the river for fishing and boating. It is important to note that environmental interests make a claim to the resource at the same point as the claim made by the extractive agent at every point. The claims of the two types of agents might be thought of as parallel, or simultaneous, rather than being in sequence along a river. This does not create problems for the optimisation problem but will create problems for us in designing a mechanism. Pay-offs to environmental interests are written $g^i(x^i, t, y^i)$ where $y^i(t)$ is the amount of water or permits purchased at a point t. In many cases, of course the environmental interest will not directly purchase permits and the environmental pay-off is thought of as a function of the water in the river.

There may also be conditions on the amount that can be extracted at any point, such as a lower bound on the amount of water that must be left, or an upper bound on the proportion of the available water that can be taken out.

I do not deal with these in this paper, although the framework used here can be extended to take these, and other types of constraint, into account.

In addition to such constraints, a scrap value might be attributed to the water left at the terminal point of a river that drains into the ocean or a lake. Apart from environmental reasons, the water may have commercial value for holiday resorts, or for a fishing industry or for some other users not strictly on the river. Alternatively the river may cross a political border and the scrap value will represent its worth to the next country or political jurisdiction. The set of scrap values is written $\psi = \{\psi^1, \psi^2, \ldots, \psi^n\}$ where ψ^i is the scrap value attached to σ^i. This is simplified by letting $\psi^i = k^i x(1)^i$. I show in Theorem 1 that only one scrap value can be imposed from the outset but leave the problem in general form for the time being.

The planner's problem for the continuous system can now be written

$$\max J = \sum_{i=1}^{n} \int_0^1 (f^i + g^i)dt + \delta^i k^i x(1)^i \tag{2}$$

subject to the dynamics of the system in Equation (1) where $\delta^i = 1$ if the scrap value is imposed and $\delta^i = 0$ otherwise.

2.2 Reduction to a single river with a single specified scrap value

The analysis of the system can be simplified by noting that it can be separated into its components under certain conditions and these components are similar in a formal sense. A river σ^j that terminates at a point t_k on σ^k is similar to σ^k if it can be analysed in the same way as σ^k by assigning a scrap value to the water at its terminal point equivalent to the value of the water in σ^k at t_k. A river σ^j that commences at t_k on σ^k is similar to σ^k if it can be analysed by assigning a value to its initial point equivalent to the value of the water in σ^k at t_k. We might think of analysing σ^1 first, for example, and using this analysis to determine the appropriate scrap values or starting values to provide the correct program for each branching river. Although this is intuitively plausible from the structure of the problem in Equation (2), it needs to be stated precisely and established and I do this in Theorem 1.

An interesting characteristic of the system is that there may not always be a solution to the problem of finding a collectively optimal allocation at each point if the initial specification of the problem attaches a scrap value to more than one river. Call this an original scrap value to distinguish it from the scrap values that need to be assigned as part of the solution. This follows directly from the nature of the optimal control problem. If an initial value and an original scrap value are attached to the same system, the solution must satisfy a differential equation with two end-points. This is, in general, impossible.

Theorem 1. *(a) The problem in Equation (2) has a unique solution and the formal properties of each* σ^i *are similar if only one original scrap value is specified* *for the system.* (*b*) *If more than one original scrap value is specified the optimal solution may fail to exist.*

Proof: See Appendix I.

From a practical point of view part (*b*) is unwelcome news and may impose serious limitations on any attempt to design a policy that gives a collectively optimal result. An analysis of the conditions under which this problem might be avoided is beyond the scope of the present paper.

It follows from part (*a*) that we can concentrate on the single river σ^1 and I do this in what follows. To simplify the notation I refer to this as the river and drop the superscripts.

2.3 Necessary conditions

The collectively optimal amount of water will be extracted at each point only if the system specified in Equations (1) and (2) satisfy the conditions given by the Pontryagin maximum principle (Macki and Strauss 1982). For the single river these are

$$u(t) = \begin{vmatrix} 0 \quad \text{for } \left.\dfrac{\partial f}{\partial v}\right|_{v=0,x(t)} < \alpha(t) \text{ or, otherwise} \\ v(t) : \left.\dfrac{\partial f}{\partial v}\right|_{v(t),x(t)} = \alpha(t) \end{vmatrix} \tag{3}$$

and

$$\dot{\alpha}(t) = -\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right) \tag{4}$$

where v is a candidate path and u is the optimal path. α is the costate variable and can be thought of as giving the value of the water under the optimal program at each instant. This is important for our purposes because it amounts to saying that, in a decentralised decision-making mechanism, every agent at point t should use water up to the point that marginal return equals the marginal value determined by α. Solving Equation 4 gives

$$\dot{\alpha}(t) = k + \int_t^1 \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right) \tag{5}$$

and we can think of the optimal price as $p(t) = \alpha(t)$ where $p(t)$ is the price at t.

It should be noted that the environmental interest only enters into the optimal program through its affect on α in Equation 4. This is because the environmental interest does not extract water from the river and does not enter explicitly into the dynamics of the flow. It does, however, influence the amount extracted. As the marginal value of water to the environmental interest increases $\partial g/\partial x$ and hence α increases at each point and less is extracted at the equilibrium price.

As observed in Section 1, this still leaves the optimal allocation undetermined because the values of f and g are not necessarily known to the planner at each t. I now consider the problem of designing a mechanism that will produce the optimal price at each point.

3. The mechanism design problem

The mechanism that induces each agent to pay the correct price at every point on the river needs to capture the fact that damage from extraction affects a smaller number of users as extraction moves downstream. It is intuitively obvious that the price of water in the optimal program should also decrease. This is confirmed by noting that $\dot{\alpha}(t) < 0$ from Equation 4. It also seems obvious that a direct auction, or an open market, for permits, must fail to be optimal as this cannot guarantee that the amount of damage by upstream users to downstream users is reflected in the price at each point. Although this is consistent with what we know about markets and auctions with externalities it cannot simply be asserted. I provide the proofs in Appendix II.

It follows from this that the design problem might be solved if it is possible for an agent at s to prevent an agent at $s - \varepsilon$ buying rights too cheaply where a price is considered too cheap if it does not build in the cost of consumption at $s - \varepsilon$ to the downstream agent. I will sometimes refer to an agent at s as s to avoid repetition.

In what follows I consider variants of a permit trading scheme that has this characteristic. The permit to extract held at the point $t = s$ might be specified in a number of ways. For simplicity it is assumed that what is being bought is the permission to extract some amount of water at a point t and this is rewritten $w(t)$. This means that the extraction plan by any agent in an interval around t would have to satisfy $w(t) \geq u(t)$ in the interval around t.

In order to simplify I ignore environmental interests and concentrate on the pay-off function in Equation (2) rewritten $J = \int_0^1 f + k$. I reintroduce environmental interests near the end of this section.

3.1 Market with extraction and blocking strategies

The possibility of a market with a direct blocking strategy that satisfies the conditions set out above has been considered in some recent work by Weber (2001). Weber claims that such a strategy will be effective and trade will only take place between adjacent users. Is this correct?

The market considered is one in which the planner distributes permits along the river and these can be traded in a Walrasian pure exchange market in which buying and selling can take place between any two points on the river. There is perfect information and offers to buy are made at no cost.

Roughly, the idea behind a direct blocking strategy in this market is that if an agent immediately upstream tries to buy permits, then it pays a downstream agent to match the offer since extraction by the upstream agent will do some damage. This matching will continue until the price is such that the marginal damage to the downstream agent and the marginal benefit to the upstream agent are in equilibrium. This is stated more precisely by defining a blocking strategy for an agent at s as an offer to match any offer by an agent at $s - \varepsilon$ for rights from $t > s$. Remember that the river flows from 0 to 1 and $s - \varepsilon$ is upstream from s. Write this strategy ϑ_s. If this strategy is credible there is an equilibrium in which $s - \varepsilon$ does not attempt to buy from a downstream player. Write this $\vartheta_{s-\varepsilon}$.

For these strategies to work the threat to match an offer must be credible in that if an agent's threat to buy were called they would actually use the threatened strategy. Such a strategy is called subgame perfect in the game theoretical literature. A subgame perfect equilibrium comprises strategies that would be optimal if the subgame were played. I now show that the direct blocking scheme fails this test.

Theorem 2. *There is some f and $s \in (0, 1)$ such that $(\vartheta_{s-\varepsilon}, \vartheta_s)$ is not a subgame perfect equilibrium.*

Proof: Appendix III.

It follows that the direct blocking strategy cannot prevent purchases taking place between non-adjacent users unless they are prohibited. In addition we get the following easy corollary of Theorem 2, which will prove useful later. Define a sequential market as one in which the final price at s depends only on offers made by a single agent at s and the agent directly above or below. The exact nature of the dependence does not matter.

Corollary 1 of Theorem 2. *A market mechanism is optimal only if it is sequential.*

Proof: Suppose a market is not sequential. If the price at s depends on offers by a non-adjacent agent it is not optimal from the proof of Theorem 2. If there is more than one agent at s neither has an incentive to offer a price that takes into account the full marginal cost of consumption at $s - \varepsilon$.

It is possible to put the last part of the proof slightly differently. Where there are two different types of agents competing for permits in the same interval the externalities will not be internalised but the optimal price requires that externalities are taken into account at every point.

I now consider a mechanism which produces the optimal price.

3.2 The sequential bidding game with extraction

The sequential bidding game overcomes problems in the blocking strategy by forcing agents to bid for permits in sequence and allowing for re-contracting by a downstream agent if an upstream agent bids too low. This gives the downstream agent an effective means of making its interest effective in the market. In addition there is a penalty attached to any revealed mis-statement by an upstream agent to prevent an upstream agent exhausting the capacity of a downstream agent to re-contract. This game has a Nash equilibrium in which the optimum price is paid at each point. Since this scheme works for any river with a terminal value for water, from Theorem 1 it works for all terminating rivers. In the case of a commencing river the initial value of the water is specified and the scheme will work in these cases by substituting $s \equiv 1 - t$ and running the sequence backward from the initial point.

Under this mechanism buying and selling by agents after the initial allocation of permits is not allowed since, as already shown, this would violate the optimal price. This leaves the problem of how to handle the fact that pay-off functions might change over time. One way in which we might take care of this is to make the permits time-dependent and re-auction them on a regular basis.

This mechanism is specified as follows and the general idea is illustrated in Figure 2. An agent at s has perfect knowledge of f in some interval around s greater than $B(s; 2\varepsilon)$. The planner can levy a penalty $\pi \geq 0$.

Stage 1. Starting at $s = 1$, s nominates an amount $w(s)$ to be bought at the price $p(s) = \alpha(1) = k$.
Stage 2. $s - \varepsilon$ nominates $p(s - \varepsilon)$ and $w(s - \varepsilon)$.
Stage 3. s can either: (*a*) take no action or (*b*) return a portion $w(s)/b$ for $b > 1$ of its rights and purchase $w(s)/b$ at $(s - \varepsilon) - \int_{s-\varepsilon}^{s} (\partial f/\partial x) + \gamma$ for some arbitrarily small $\gamma > 0$ from $s - \varepsilon$.
Stage 4. If (*a*) $s - 2\varepsilon$ bids and $s - \varepsilon$ can either return $w(s - \varepsilon)/b$ of its rights and purchase from $s - 2\varepsilon$ or take no action.
If (*b*) a penalty π is levied on $s - \varepsilon$ and it is required to return all unsold rights and nominate a new price $\acute{p}(s - \varepsilon)$ and $w(s - \varepsilon)$.
Stage 5. s can either: (*a*) take no action or (*b*) return a portion $(w(s)/b)(1-1/b)$ of its rights and purchase from the player at $s - \varepsilon$ at $\acute{p}(s - \varepsilon) - \int_{s-\varepsilon}^{s} (\partial f/\partial x) + \gamma$
Stage 6. If (*a*) proceed as in Stage 5 (*a*).
If (*b*) proceed as in 4 (*b*).
Stage *n*. $s \in [0, \varepsilon)$ bids and $s = \varepsilon$ holds.

Theorem 3. *The sequential bidding game has an equilibrium at the optimal price for* $\pi \geq 0$.

Proof: See Appendix IV.

3.3 Environmental interests in the sequential bidding game

The sequential bidding game can easily be extended to other constraints such as a lower bound on the amount to be extracted at each point, provided

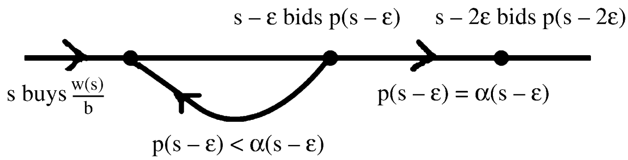

Figure 2 Sequential bidding game where arrows give direction of bid sequence.

we are able to mimic this constraint with a scrap value function. A more difficult question is, what happens if we drop the simplification and restore pay-offs to environmental or recreational interests given by the function g in Equation (2)?

The sequential bidding game cannot, in general, produce an optimal result in this case. Where environmental interests are not represented in the market this is obvious because they will not be taken into account by extractive agents. To see this, note that the optimal price is now given by $\alpha(t) = k + \int_t^1 (\partial f/\partial x + \partial g/\partial x)$ from Equation (5) whereas only the $\partial f/\partial x$ term determines the purchase price.

Where some representative of these interests stands in the market α will be the same as in the previous paragraph and the environmental agent will want to maximise a function of the form $g(x, y, t)$.

The reason for the failure in this case was foreshadowed in Section 3. Environmental interests can only be modelled by assuming there are two pay-off functions at every point but according to Corollary 1 of Theorem 2 this means that the price cannot be optimal, in general. What is happening, roughly, is that because each type of agent optimises separately there is no guarantee that identity $p = \alpha$ is maintained.

In order to show the details I demonstrate this formally in Appendix V. This analysis could be extended to cover other cases where the pay-off function for environmental interests is endogenous to the optimisation problem.

3.4 Remarks on environmental interests

It is not at all clear how these problems might be overcome in any scheme in which the pay-off function of the environmental interest is endogenous to the function to be maximised. It would not be possible to avoid them by altering the bidding mechanism because the sequence of purchases required to attain the correct price is lost.

4. Conclusion

The purpose of this paper has been to use a control theory approach to develop a model of the optimal allocation of water along a system of rivers and to explore the implications of this model for the design of permit bidding systems. Although the results of the continuous approach can also be obtained in a discrete model, the continuous model may have some general value in adding to the tools available for looking at externalities in systems of rivers. It is also capable of extension in a number of directions such as adding constraints on the amount, or proportion, of water that can be extracted at each point.

The use of a continuous flow approach to an entire river system also revealed some interesting features of the optimisation problem. Perhaps the most important of these was that this problem may not have a solution if a value is placed on water at more than one terminal point. It was also shown that auctions cannot give the optimal allocation of water and that markets for rights have an empty core. This is consistent with the more familiar literature on externalities.

It was also shown that it is possible to solve the problem of an optimal price at each point with a sequential bidding mechanism. This mechanism is likely to fail, however, if an attempt is made to optimise a pay-off function that endogenises environmental considerations.

One way in which the environmental problem might be avoided is to treat environmental interests as side constraints. In this case it would be possible to calculate the optimal amount of water to be withheld before bidding, or to lump all environmental interests into a terminal value on a single river. This would give some sort of hybrid scheme in which collective interests in the environment are estimated by a public body and then water permits are marketed.

It should also be noted that the sequential bidding mechanism could be difficult to implement if there is a large number of players. If the number of players is small, however, or individual extractors could be aggregated and represented by a manager, the scheme may be workable.

References

Ambec, S. and Spumont, Y. (2002). Sharing a river, *Journal of Economic Theory* 107, 453–462.

Anderson, T. (2004). Donning Coase coloured glasses: a property rights view of natural resource economics, *The Australian Journal of Agricultural and Resource Economics* 48, 445–462.

Burness, H. and Quirk, J. (1979). Appropriate water rights and the efficient allocation of resoures, *American Economic Review* 69, 25–37.

Kanazawa, M. (1991). Water quality and the economic efficiency of appropriate water rights, in Dinar, A. and Zilberman, D. (eds), *The Economics and Management of Water and Drainage in Agriculture*. Kluwer Academic, Boston.

Kilgour, D. and Dinar, A. (2001). Flexible water sharing within an international water basin, *Environmental and Resource Economics* 18, 43–60.

Macki, J. and Strauss, A. (1982). *Introduction to Optimal Control Theory*. Springer-Verlag, New York.

Moulin, H. (1995). *Cooperative Microeconomics*. Princeton University Press, Princeton, NJ.

Quiggin, J. (2001). Environmental economics and the Murray-Darling river system, *Australian Journal of Agricultural and Resource Economics* 45, 67–94.

Weber. M. (2001). Markets for water rights under environmental constraints, *Journal of Environmental Economics and Management* 42, 53–56.

Whittle, P. (1996). *Optimal Control Basics and Beyond*. John Wiley & Sons, New York.

Appendix I. Proof of Theorem 1

(*a*) From the necessary conditions α^1 is uniquely defined for any river.

(*i*). σ^j terminating. Let t^1 be the point of termination on σ^1 and write $i \neq 1$ as $i = 2$ to ease the notation. In an optimal program we need $\partial J^1/(\partial x^1(t_1)) = \partial J^2/(\partial x^2(1))$ to ensure that the value of the water at the termination point of σ^2 is the same as the value at the point of termination in σ^1. Write

$$J^i = \int (f^i + g^i + \alpha^i h^i + \dot{\alpha}_i x_i) + \alpha^i(0)x^i(0) - \alpha^i(1)x^i(1)$$

where h^i is the right hand side of (1). Differentiating and cancelling gives $\partial J^1/(\partial x^1(t_1)) = -\alpha_1(t_1)$ and $\partial J^2/(\partial x^2(1)) = -\alpha_2(1)$. From the transversality conditions $\alpha_2(1) = \partial\psi^2/\partial x_2$ where ψ^2 is the scrap value assigned to σ^2. This means that the condition is met by choosing an appropriate ψ^2.

(*b*) Consider, for example, the case where $J = \int (u + x)$ and $\dot{x} = a - u$ with $\alpha(0) = 1$ and $\alpha(1) = 3$. In this case $\dot{\alpha} = 1$ and cannot satisfy the end states. It might be thought that this depends on that fact that $\dot{\alpha}$ is not a function of u or x. Consider $J = \int 2\sqrt{ux}$. In this case $\alpha = \sqrt{u/x}$ to give $\alpha = \alpha(0)\sqrt{2t}$. This will not satisfy an arbitrary $\alpha(1)$.

Appendix II. Direct auctions and markets for rights

Direct auction

The planner puts some fixed amount of permits on sale, equivalent to the total volume of water in the river, and these are sold in public bidding tournament in which a permit goes to the highest bidder.

Proposition 1. *The direct auction does not, in general, produce a price function that solves the allocation problem.*

Proof: The volume of water available at $t = s$ depends on the amount used in $t < s$ and hence $x = x(p(t))$. In this case the problem at $t = s$ is to choose $u(s)$ and $w(s)$ to maximise

$$F(s, x(p), u) + \gamma(w(p) - u) - pw(p) \tag{6}$$

where γ is the multiplier on the inequality constraint. This is done by setting $u = w$ and

$$\frac{\partial f}{\partial u} = \gamma \quad \text{and} \quad \gamma = p \quad \text{with price such that } \frac{\partial f}{\partial x}\frac{\partial x}{\partial p} = w$$

In addition it must be the case that $p = \alpha$ in order to get the optimal level of extraction. Since this gives a system of three equations with two unknowns it is not, in general, consistent.

Unrestricted market for rights

The planner distributes all rights along the river and these can be traded in a Walrasian pure exchange market in which buying and selling can take place between any two points on the river. There is perfect information, offers to buy are made at no cost and are for rights to be used at the point making the offer. It is a direct corollary of Proposition 1 that a market cannot provide a solution that satisfies the optimisation program since the problem is essentially that of Equation (6). In addition it is easy to prove the following stronger theorem.

Proposition 2. *The unrestricted market for rights has an empty core.*

Proof: Suppose $p = p(t)$ is an imputation in the core that offers the same price at all t. Then an individual in an interval around $t = s$ can do better by offering a price $p < q < p + \int_{s-\varepsilon}^{s} (\partial f/\partial x)$ to the individual in the interval around $s - \varepsilon$. This is because $\int_{s-\varepsilon}^{s} (\partial f/\partial x)$ represents the damage done at s by consumption in the interval $s - \varepsilon$ where the integral form comes from the continuity of f. Now consider an imputation where $p(t) \neq p(s)$ for all s, $t \in [0, 1]$. This gives two cases.

(*i*). $p(t)$ is not decreasing as t increases. An individual at s can do better by buying from $s - \delta$ at $p(s - \delta)$.

(*ii*). $p(t)$ is decreasing as t increases as required in the optimisation program. It follows that, if the price between points $(s, s + \varepsilon)$ does not decrease by $\int_{s}^{s+\varepsilon} (\partial f/\partial x)$ there is an incentive to trade. If the price decreases by $\int_{s}^{s+\varepsilon} (\partial f/\partial x)$ then s does better by buying rights from $s + \delta$ for some $\delta > \varepsilon$ at a price $p(s + \delta) + \int_{s+\delta-\varepsilon}^{s+\delta} (\partial f/\partial x) < p(s + \delta) + \int_{s}^{s+\delta} (\partial f/\partial x) = p(s + \varepsilon) + \int_{s}^{s+\varepsilon} (\partial f/\partial x) = p(s)$.

Appendix III. Proof of Theorem 2

The only case of interest is where $p(t)$ decreases as t increases. Consider adjacent points $s - \varepsilon$ and s and suppose that ϑ_s is subgame perfect. Because it pays $s - \varepsilon$ to offer any amount in the range $[0, p(s - \varepsilon))$ to $s + \delta$ for $\delta > 0$, s can only implement ϑ_s by buying the right at $p(s - \varepsilon)$. The fact that $(s - \varepsilon)$ can repeat this for all $t > s$ means that s must buy all rights held by $t > s$. Let the cost to s of allowing the purchase by $(s - \varepsilon)$ be c_1, of holding $\bar{\omega}(s)\int_{s}^{1} \bar{x}$ rights be c_2, and selling on to $t > s$ at $p(t) < p(s - \varepsilon)$ be c_3. Then there is some f and s such that $c_1 < \min\{c_2, c_3\}$. Contradiction.

Appendix IV. Proof of Theorem 3

This theorem is proven with the help of the following Lemma which also throws some light on the information problem. Write the optimal allocation as w^*.

Lemma 1. *If* $w = w^*$ *and trades can only take place between adjacent agents* $p(t) = \alpha(t)$.

Proof of Lemma 1. Consider three agents at $s - \varepsilon$, s and $s + \varepsilon$ written a, b and c for convenience. Write the purchase price of a right by z from a as p_a^z. The marginal value of a purchase by b from a is $v_a^b = \partial f(b)/u(b) + \int_b^a (\partial f/\partial x) - p_a^b$. The marginal value of a purchase from c is $v_c^b = [\partial f(b)/u(b)] - p_c^b$. If the market is in equilibrium there are no gains from trade and hence $\partial f(b)/u(b) + \varepsilon(\partial f/\partial x) - p_a^b = [\partial f(b)/u(b)] - p_c^b$. This means $p_a^b = p_c^b + \varepsilon(\partial f/\partial x)$. The price paid by the agent at $s \in (1 - \varepsilon, 1)$ is $p(s) = k = \alpha(1)$. It follows that the price to $s - \varepsilon$ is $p_s^{s-\varepsilon} = k + \int_{1-\varepsilon}^1 (\partial f/x) = \alpha(s - \varepsilon)$. This gives $p(t) = \alpha(t)$ by induction.

Note that, because $\alpha(1)$ is known, $s \in (1 - \varepsilon, 1]$ and $s - \varepsilon$ have sufficient information to bid $p(t) = \alpha(t)$.

Proof of Theorem 3. Suppose $s \in (1 - \varepsilon, 1]$ and the agent at $s - \varepsilon$ bids $p(s - \varepsilon) < \alpha(s - \varepsilon)$. It is optimal for s to buy $w(s)/b$ at $p(s - \varepsilon) - \int_{1-\varepsilon}^1 (\partial f/\partial x) + \gamma = p(s - \varepsilon) - \alpha(s - \varepsilon) + k + \gamma < k$ for γ sufficiently small. For some $\pi \geq 0$ we have $v_{s-\varepsilon}\, p(1 - \varepsilon) < v_{s-\varepsilon}\alpha(s - \varepsilon)$ and $s - \varepsilon$ bids $\alpha(s - \varepsilon)$. The rest of the proof follows by induction.

Appendix V. Demonstration for environmental agents

In the case where both types of agents purchase, each type solves its optimisation problem separately. Consider the agent around $t = 1$. Since the extractive agent only takes account of the payoff to f we know from Equation (5) that $p = k + \int_{1-\varepsilon}^1 f$. For the environmental agent the solution is $p = -(\partial g/\partial x)/(\partial g/\partial y)$ since $p = \partial y/\partial x$. From the fact that there can only be one price at a point the solution is given by the pair $(p(t), w(t))$: $\partial f/\partial x = -(\partial g/\partial x)/(\partial g/\partial y) \neq (\partial f/\partial x + \partial g/\partial x)$ in general. From Equation (5) this means that $p \neq \alpha$, in general, as required.