Measuring Social Welfare:
A Dog's Leg Possibility Postulate

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Abstract:

Our current methods of analysing policies and the distributions of wealth insure that society is on an efficient frontier. This is not the same as a social optimum. To choose the optimal point on the frontier we need a social welfare function. Following the ordinal revolution in demand theory, a large body of research concluded that social welfare functions don't exist. The intensity of people's preferences cannot be observed and hence interpersonal comparisons are essentially impossible. This paper argues that the intensity of people's preferences can be observed and could be incorporated into a social welfare function.

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Introduction

It's interesting when two prejudices collide. It's even more interesting when they are both my own. I was in the basement of the library, quietly reading a most excellent book on demand analysis by Pollack and Wales (1992), who were discussing interpersonal comparisons of utility and social welfare, when the collision occurred.

> Prejudice 1: There is no such thing as a dumb farmer.
>
> Example: Farmers just don't seem to get the concept of probability. Many academic researchers, me included, have tried to explain it to them, to no avail. How can successful managers in such a risky business fail to understand what we believe to be a fundamental concept? Somebody needs to think again, but in my prejudice, it's not the farmers. If farmers make good decisions and survive in a risky business with no knowledge of probability, it must not be a fundamental concept after all.
>
> Prejudice 2: Measuring social welfare is impossible.
>
> Example: People keep making up indexes of sustainability, quality of life, quality of universities and just about every other imaginable thing, as if interpersonal comparisons were possible. We all live sustainably, choose to live in the best place in the world and work in the best university. As professional economists, we all know from the theory of consumer demand that preferences are ordinal, intensities of preferences are unobservable and happiness does not have a number. Yet people keep making up indexes as if the meaning of life really is 42.

My two prejudices collided as I read the following passage from Pollack and Wales:

> "We are agnostic about the possibility of making interpersonal welfare comparisons. We find the most convincing argument that such comparisons are possible is the frequency with which they are made.
>
> "…Economists lack the expertise to analyse the language of welfare comparisons. Despite our inability to provide a satisfactory account of interpersonal comparisons, we are unwilling to dismiss them as *a priori* impossible or clearly meaningless."

Perhaps the world is right and I need to think again. Just, Hueth and Schmitz (2004) summarise the collective wisdom very nicely. I have distilled 3 points from the introductory chapters of their book and one point from the conclusions.

- Preferences are ordinal, or at least utility as a ranking of preferences cannot be observed and the assumption of cardinal preferences has no scientific merit because it cannot be refuted or confirmed.
- Even if preferences were cardinal, interpersonal comparisons are not possible because economists, philosophers and others cannot agree on the form of a social welfare function for weighting individuals in society.
- Even if we agree on a social welfare function, the task of ranking all possible outcomes of all possible policies for all people in society is virtually impossible.

- The first three points are of academic interest but make little difference for welfare analysis. In representative democracies, it is the job of policy-makers to weight individuals in society, rank policy outcomes and make decisions. Our job as economists is to calculate who will win or lose and make the information available.

The collective wisdom dates from the ordinalist revolution of the 1930s (Cooter and Rappoport, 1984) By the time Arrow advanced his impossibility theorem in 1951, there was no need to even mention ordinal preferences as one of the axioms. Subsequent volumes of literature tried, but largely failed, to find a possible social ranking of policies while maintaining ordinal preferences. Social rankings without interpersonal comparisons seem impossible. Interpersonal comparisons without cardinal preferences seem impossible. The bottom line is that we really don't know whether or not our policies will enhance social welfare.

Back in the basement of the library, I was reading Pollack and Wales to get ideas about a demand system that was misbehaving. I was working on a model of lifetime consumption and savings in an economy that depends upon the ecosystem. A pesky parameter was supposed to disappear from the system, but refused. If I normalised as you should, the whole system crashed. Upon reflection, the pesky parameter, and one other, defined a transformation that made lifetime utility cardinal—cardinal in the strict sense that mathematicians define it. Happiness has a number. And even more worrying, the pesky parameter was in the demand and expenditure functions. If this result proved correct, there would scientific merit to cardinal preferences, after all. If preferences are cardinal and the other three points can be addressed, perhaps we can measure social welfare.

Dynamic Welfare of an Individual

First, look at a general description of the model with the pesky parameter. Following Koopmans (1960; see also Heal, 2005), we will add up an individual's utility over their lifetime, subject to changes in wealth.

$$J(W_0) = \max_{Q_1,Q_2} \int_0^T U(t,Q_1,Q_2)dt + V(T,W_T)$$

$$\text{subject to:}$$

$$\dot{W} = F(W,Q_1,Q_2)$$

Our lifetime utility, J, depends upon initial endowments of manufactured capital, W. We behave as if we maximise lifetime utility by choosing commodities Q_1, and Q_2. These determine our current utility, U, at each age in our life, t. Over our lifetime we consume until, at the end of this life, T, we bequeath wealth to future generations and gain utility V. Wealth evolves over time according to a differential equation and increases with net production, F.

Maximizing lifetime utility subject to a constraint is equivalent to maximizing the Hamiltonian as a dynamic measure of utility, accounting for changes in wealth.

$$H(t,W) = \max_{Q_1,Q_2}[U(t,Q_1,Q_2) + \lambda F(W,Q_1,Q_2)]$$

On the right-hand side, the first term in the square brackets is current utility at time t. The second term is total user costs of wealth. Total user costs, like other total costs, are a price times a quantity. The price is the marginal user cost, λ, and the quantity is the net production, F. If wealth is depreciating, the net production is negative and total user costs subtract from current utility to account for costs to the future. If wealth is appreciating, the net production is positive and total user costs should be called total user benefits. Total user benefits add to current utility to account for benefits in the future.

There are two decisions with two optimality conditions for commodities.

$$\frac{\partial U}{\partial Q_1} + \lambda \frac{\partial F}{\partial Q_1} = 0$$
$$\frac{\partial U}{\partial Q_2} + \lambda \frac{\partial F}{\partial Q_2} = 0$$

These conditions generalize the conditions from a static model of consumer demand. The marginal utility of consumption is compared with the value of the marginal product. The marginal product is valued at the marginal user cost of wealth.

Further optimality conditions define the evolution of the marginal user cost and its' terminal value.

$$\dot{\lambda} = -\lambda \frac{\partial F}{\partial M}; \quad \lambda_T = \frac{\partial V_T}{\partial M_T}$$

If the marginal user cost is known, everything necessary for finding optimal consumption is known. Unfortunately, the marginal user cost is very shy and difficult to get to know.

A Dynamic Generalised CES System

So far, there is no hint of cardinal utility. To investigate further, a special case can be solved for an analytical solution. However, the special case must still be realistic. I found by trial and error that the utility function must be complicated enough so that the cardinal utility parameters become distinct. A dynamic version of the generalized constant elasticity function appears to be the simplest possible utility function to give a demand system which depends upon cardinal utility. It also lets us draw nice graphs of isoquants. Of course, more complex utility functions are possible. The solution below suggests that any static demand system that is integrable is also integrable as a dynamic system.

This special case is still relatively complex. The solution contains lifetime utility and expenditure functions, a dynamic demand system and dynamic methods for welfare analysis. Assume the following functional forms.

$$U(t, Q_1, Q_2) = e^{-\rho t}\left[\beta_1 (Q_1 - \gamma_1)^{-\nu} + \beta_2 (Q_2 - \gamma_2)^{-\nu}\right]^{\frac{-\alpha}{\nu}}$$

$$F(W, Q_1, Q_2) = rW + Y - p_1 Q_1 - p_2 Q_2$$

$$p_1 = e^{-(r-g)(T-t)} p_{1T}$$
$$p_2 = e^{-(r-g)(T-t)} p_{2T}$$

Utility of consumption is a generalized constant elasticity of substitution function (Pollak and Wales, 1992). Parameter ν is the substitution parameter. As it varies from -1 to ∞, the elasticity of substitution, $\sigma = 1/(\nu + 1)$, varies from ∞, for perfect substitutes, to 0, for perfect complements. Figure 1 shows threes sets of isoquants for elasticities of substitution of ∞, 1 and 0.

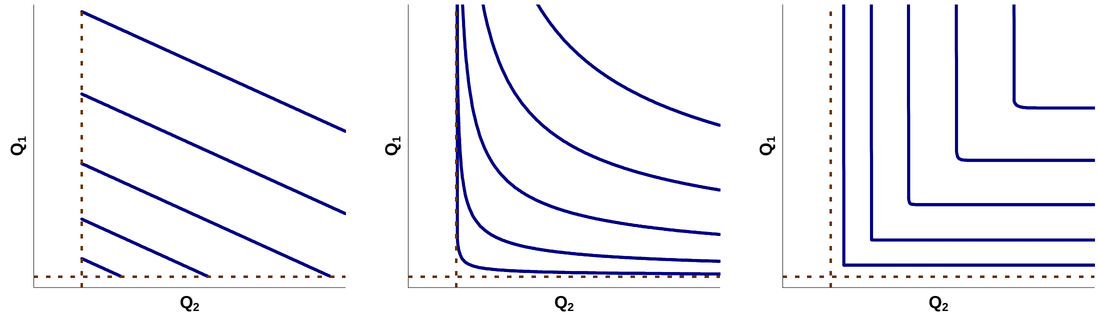

Figure 1. Perfect Substitutes, Substitutes and Perfect Complements

When the elasticity of substitution is infinite, utility is linear. When the elasticity of substitution is one, utility is of the Stone-Geary type. When the elasticity of substitution is 0, utility is of the Leontief type. Subsistence quantities of consumption are γ_1 and γ_2. These are shown in Figure 1 as the dotted lines which effectively shift the origin away from zero. Elasticities of commodities are β_1 and β_2. These change the slopes and curvature of the isoquants. Two additional parameters are the nominal rate of time preference, ρ, and the elasticity of current utility, α. These two parameters define a monotonic transformation of utility and change the spacing among the isoquants. Figure 2 illustrates.

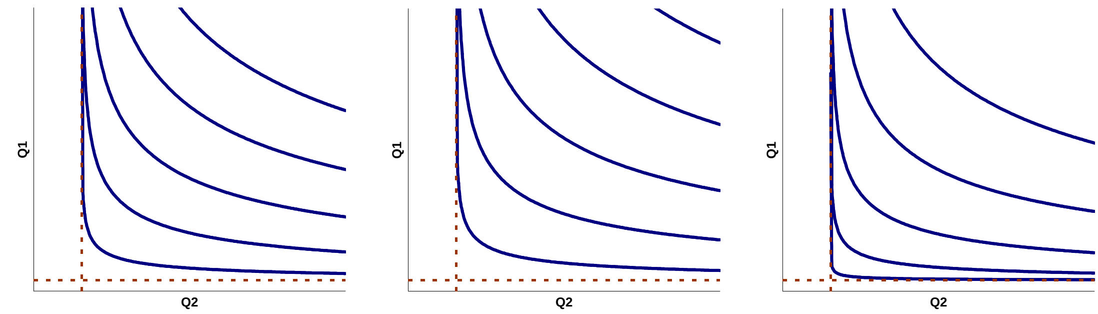

Figure 2. Spacing of Isoquants with a Monotonic Transformation

The first panel is the same as the middle panel of Figure 1. The second panel shows an increase in the nominal rate of time preference. The isoquants are uniformly spaced but further apart. The third panel shows a decrease in the elasticity of current utility. Isoquants near the origin are closer together. Isoquants away from the origin are further apart. In a static demand system, the spacing of the isoquants is unimportant. In the dynamic system the spacing will prove to be crucial for decisions about saving for the future.

Net production is now very simple with investment income, earned income and expenditures. Investment of wealth accrues interest at the rate r. Earned income is Y. Expenditures on commodities are at prices p_1 and p_2. Finally, prices grow at the rate of inflation.

With these assumptions lifetime utility has a unique solution.

$$J(t,W) = e^{-\rho t} A(t)^{1-\alpha} B^{-\alpha} E(t,W)^{\alpha} + V(T,W_T)$$

$$\text{where :}$$

$$A(t) = \tfrac{1}{\delta}\left(1 - e^{-\delta(T-t)}\right); \quad \delta = \tfrac{\rho - \alpha g}{1-\alpha}$$

$$B = \left[\beta_1\left(\frac{p_1}{\beta_1}\right)^{\frac{\nu}{\nu+1}} + \beta_2\left(\frac{p_2}{\beta_2}\right)^{\frac{\nu}{\nu+1}}\right]^{\frac{\nu+1}{\nu}}$$

$$E(t,W) = W - e^{-r(T-t)}W_T + \tfrac{1}{r}\left(1 - e^{-r(T-t)}\right)Y - \tfrac{1}{g}\left(1 - e^{-g(T-t)}\right)(p_1\gamma_1 + p_2\gamma_2)$$

$$\alpha \neq 1; \quad \delta \neq 0; \quad \nu \neq 0; \quad T < \infty$$

Proof is in Appendix 1. A glossary of symbols is in Table 1. As before, our lifetime utility, J, equals a lifetime's utility from consumption plus the utility of bequests, V. Therefore, the first term on the right-hand side is a lifetime's utility of consumption. It depends upon A, B and E. A is an annuity factor which integrates current utility into lifetime utility. Within the annuity factor, δ is the real rate of time preference. The nominal rate of time preference must be positive, but the real rate may be positive or negative. B is the substitution factor. It is dual to the isoquants shown in Figure 2. When the isoquants are linear, B is Leontief and when the isoquants are Leontief, B is linear. E is lifetime expenditures above subsistence.

During our lifetime, demand for commodities is a dynamic version of a generalized constant elasticity of substitution system.

$$Q_1 - \gamma_1 = b_1\frac{E}{A}; \quad b_1 = \left(\frac{p_1}{\beta_1}\right)^{\frac{-1}{\nu+1}} B^{\frac{-\nu}{\nu+1}}$$

$$Q_2 - \gamma_2 = b_2\frac{E}{A}; \quad b_2 = \left(\frac{p_2}{\beta_2}\right)^{\frac{-1}{\nu+1}} B^{\frac{-\nu}{\nu+1}}$$

Table 1: Glossary of Symbols

Description	Symbol
Lifetime utility	J
Current utility	U
Bequests	V
Hamiltonian	H
Annuity factor	A
Substitution factor	B
Lifetime expenditures	E
Wealth	W
Earned income	Y
Consumption of commodity 1	Q_1
Consumption of commodity 2	Q_2
Price of commodity 1	p_1
Price of commodity 2	p_2
Nominal rate of time preference	ρ
Real rate of time preference	δ
Rate of interest	r
Rate of inflation	$r - g$
Elasticity of current utility	α
Elasticity of commodity 1	β_1
Elasticity of commodity 2	β_2
Substitution parameter	ν
Subsistence for commodity 1	γ_1
Subsistence for commodity 2	γ_2
Elasticity of bequests	ω
Slope for utility of bequests	ϕ
Life expectancy	T
Current age	t

On the left hand sides, consumption of commodities is above subsistence. On the right-hand sides, lifetime expenditures above subsistence are divided by the annuity factor and converted into current expenditures. Current expenditures are apportioned between commodities by shares b_1 and b_2. Expenditures grow over time at the nominal rate $r - \delta$. If the interest rate exceeds the real rate of time preference, we save for the future and spend later. The inflation rate is $r - g$ and demand grows at the real rate $g - \delta$. Expenditures and demand both depend upon the real rate of time preference which, in turn, depends upon the parameters which define a monotonic transformation of utility. Unfortunately, a time series of consumption or expenditures can be used to estimate of the real rate of time preference but cannot disentangle the two parameters of the monotonic transformation. Measuring cardinal utility requires more effort.

Measuring Cardinal Utility

Suppose a policy changes the current price of a commodity. Prices are state variables and a change in the current price will set in motion a new evolution of prices and wealth. Welfare analysis must compare lifetime utilities before and after the change.

$$J(t, W - WTP, p) = J\left(t, W, p + \Delta_p\right)$$
$$J(t, W, p) = J\left(t, W + WTA, p + \Delta_p\right)$$

A change in the price of a commodity is Δ_p. Willingness to pay, *WTP*, is an equivalent variation—the amount an individual is willing to pay to avoid the change. Willingness to accept, *WTA*, is a compensating variation—the amount an individual would accept to allow the change (Just *et al.*, 2004). The prices and wealth will be different for lifetime utilities on the left hand and right hand sides of the equations. Before *WTP* and *WTA* can be calculated, therefore, the utility of bequests must be known.

Assume the utility of bequests is a Cobb-Douglas function of terminal wealth at time *T*.

$$V\left(T, W_T\right) = e^{-\rho T}\phi W_T^{\omega}$$

Parameter ω is the elasticity of the utility of bequests and ϕ is the slope. In general, *WTP* and *WTA* are complicated and must be solved numerically. In the simplest case, suppose $\omega = 1$. The utility of bequests will be linear in wealth and *WTP* and *WTA* will have an algebraic solution.

$$WTP = WTA = \tfrac{(1-\alpha)}{\alpha} A\left(\tfrac{1}{\alpha} e^{-(\rho - r)(T-t)}\phi\right)^{\frac{1}{\alpha-1}}\left[B(p)^{\frac{\alpha}{\alpha-1}} - B\left(p + \Delta_p\right)^{\frac{\alpha}{\alpha-1}}\right] + \tfrac{1}{g}\left(1 - e^{-g(T-t)}\right)\Delta_p \gamma_2$$

The derivation is in Appendix 1. The first term on the right hand side is the difference in expenditures and the second term is the difference in wealth. Notice that *WTP* and *WTA* are equal. Also notice that both depend upon the nominal rate of time preference and the elasticity of the utility of consumption. Data on *WTP* and *WTA* are observations of cardinal utility.

Other than the simplest case, *WTA* will diverge from *WTP*. In a static model, the divergence is caused by imperfect substitution among commodities in the utility of consumption. In a dynamic model, this is not true. Regardless of substitution factors $B(p)$ and $B\left(p + \Delta_p\right)$, *WTA* can equal *WTP*. Instead, the divergence is caused by nonlinearity of the utility of bequests. Figure 3 illustrates an increase in the price of commodity 2.

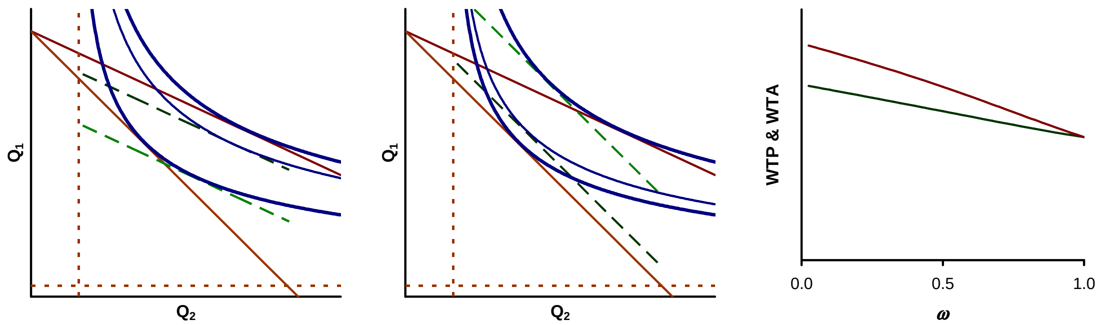

Figure 3. Willingness to Pay and Willingness to Accept

In the first two panels, the curves are isoquants for the utility of consumption. The solid lines are budget constraints that would be binding if the model were static. Before the price increase the budget constraint would be tangent to the highest isoquant. After the price increase the budget constraint would be tangent to the lowest isoquant. The dashed lines compare *WTP* and *WTA* from a static model with *WTP* and *WTA* from a dynamic model. In the first panel, static *WTP* is the parallel shift of the top budget constraint down to the lowest isoquant. In the second panel, static *WTA* is the parallel shift of the lowest budget constraint up to the highest isoquant. Static *WTA* exceeds static *WTP*. For a dynamic model, the budget constraints are not binding and consumers will choose isoquants in between the upper and lower isoquants. The first and second panels illustrate the simplest case in which *WTP* equals *WTA*. Both are much smaller than static *WTP* and *WTA*. The third panel shows *WTP* and *WTA* as the utility of bequests goes from highly nonlinear to linear in terminal wealth. So long as the elasticity is less than one, the utility of bequests is concave and *WTA* exceeds *WTP*.

In a dynamic model, welfare analysis requires utility parameters that can only be estimated from data on *WTP* or *WTA*. We must conduct surveys of consumers. Table 1 shows the data to be collected and the parameters to be estimated. In general, *WTP* and *WTA* do not have algebraic solutions and must be estimated implicitly from lifetime utilities.

Table 1: WTP and WTA Estimation

Data	Parameters
WTP	α
WTA	ρ
p_1	β_1
p_2	β_2
W	γ_1
Y	γ_2
r	ν
g	ω
T	ϕ
t	W_T
Δ_p	

$$0 = J(t, W - WTP, p) - J\left(t, W, p + \Delta_p\right)$$
$$0 = J(t, W, p) - J\left(t, W + WTA, p + \Delta_p\right)$$

Estimating these equations will require nonlinear regression with parameter restrictions across equations.

Static Welfare of Society

To investigate sustainable growth of an economy, social welfare should also be modelled as dynamic. This will be a topic for future research. In this paper, lifetime utilities from the analytical solution in the previous section are incorporated into a static model of social welfare.

$$J(W) = \max_{W_1, W_2} U(W_1, W_2)$$

$$\text{subject to:}$$

$$W = W_1 + W_2$$

In this model, J is social welfare and W_1 and W_2 are society's wealth allocated to categories of individuals 1 and 2. U is the Bergson social welfare function. Many authors have proposed specific functional forms. They all seem to be special cases of a constant elasticity of substitution function, (see Just *et al.*, 2004, section 3.4 for a summary.)

$$U(W_1, W_2) = \left[\beta_1 J_1(W_1)^{-\nu} + \beta_2 J_2(W_2)^{-\nu}\right]^{\frac{-\alpha}{\nu}}$$

Lifetime utilities for categories of individuals are J_1 and J_2. Society's weights for individuals are β_1 and β_2. The elasticity of social welfare is α and the degree of substitution between individuals is ν. If ν = -1, people are biomass and infinite substitutes for each other. This has been called the 'just social welfare function.' If ν goes to positive infinity, people are perfect complements and cannot substitute for each other in any way. This formulation has been used to characterise Rawl's theory of justice in which the least advantaged person in society determines social welfare. In between, people are imperfect substitutes. This has been called 'inequality aversion.'

Given this functional form, social welfare has a dual solution.

$$J(J_1, J_2) = B^{\alpha} E(J_1, J_2)^{\alpha}$$

$$\text{where:}$$

$$B = \left[\beta_1 (\beta_1 \lambda_1)^{\frac{-\nu}{\nu+1}} + \beta_2 (\beta_2 \lambda_2)^{\frac{-\nu}{\nu+1}}\right]^{-\frac{\nu+1}{\nu}}$$

$$E(J_1, J_2) = \frac{J_1}{\lambda_1} + \frac{J_2}{\lambda_2}$$

$$\alpha \neq 1; \quad \nu \neq 0$$

The derivation is in Appendix 2. B is a substitution factor which is dual to the social welfare function. Within B, λ_1 and λ_2 are the marginal user costs of wealth for categories of individuals 1 and 2. These have the same role as prices in a demand system. Total expenditure for society is E. Even though E doesn't look like an expenditure function, substituting the analytical solution for each category of individuals will show that it is.

Graphing lifetime utilities for all possible distributions of wealth gives a possibility frontier. Any point on the frontier is Pareto efficient. Adding a social welfare function identifies the optimal point. Figure 4 illustrates for homogenous workers, each with a low rate of time preference, nearly linear utility and a fat pay check. Each worker is given equal weight in social welfare.

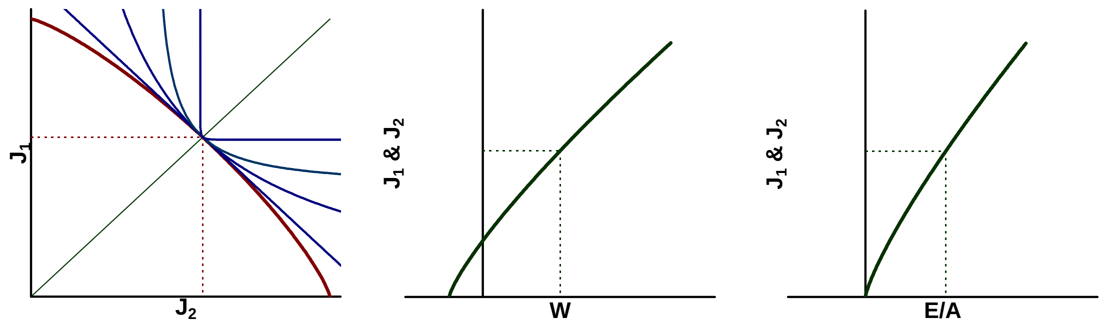

Figure 4. Workers' Paradise

In the first panel, the possibility frontier is concave to the origin. A 45º line emanating from the origin shows equal lifetime utilities. Isoutility curves for the social welfare function are tangent to the frontier at the social optimum. For homogeneous workers, optimal social welfare does not depend upon the degree of substitution. The second and third panels show that workers have the same lifetime utility from the same initial wealth and the same annual expenditure.

Figures 5 and 6 illustrate for workers and retirees. Workers are as before but retirees are older with no pay checks.

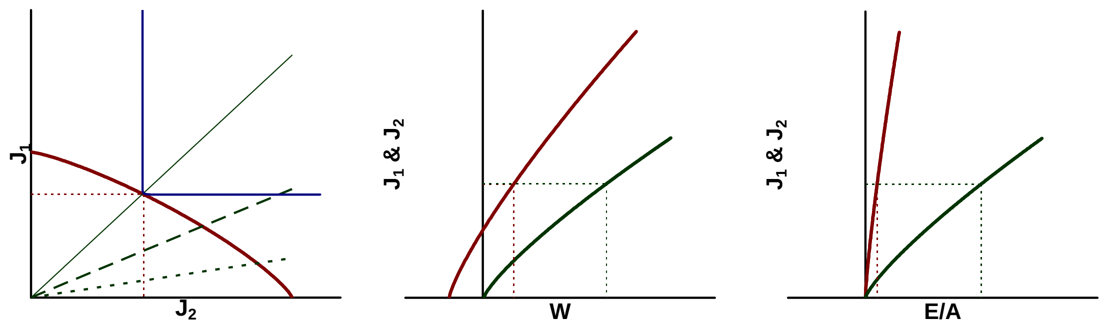

Figure 5. Retiree's Paradise

Retirees are category 1. The possibility frontier shifts down because retirees will fall off the perch soon. The solid line emanating from the origin shows equal lifetime utilities, regardless of age. The dashed line shows utilities if wealth is distributed equally. Above the dashed line, retirees are allotted a greater share. Below the line, workers are allotted a greater share. The dotted line shows utilities if annual expenditures are equal. Above the line, retires spend more. Below the line, workers spend more. The isoutility curve simulates Rawl's theory of justice. Workers and retires are perfect complements with no substitution. The social optimum is a retiree's paradise because they get most of the goodies.

In Figure 6 the isoutility curve simulates the just social welfare function.

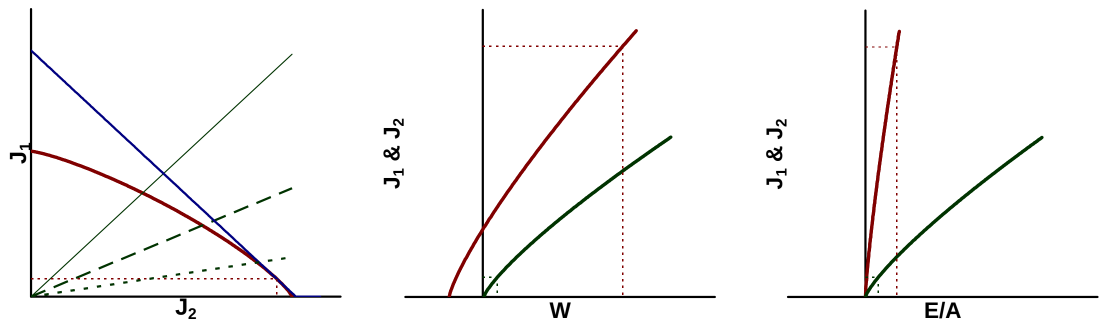

Figure 6. Genteel Poverty

Workers and retirees are biomass with perfect substitution. The social optimum is genteel poverty for retirees and a nice life for workers who hoard the wealth and spend the money.

Figures 7 and 8 illustrate for workers and Uncle Scrooge. Again, workers are as before, but Scrooge is an impatient, grumpy and selfish capitalist. He has a high rate of time preference, a low elasticity of utility from consumption, a very low utility of bequests and earns no income but collects interest on his wealth.

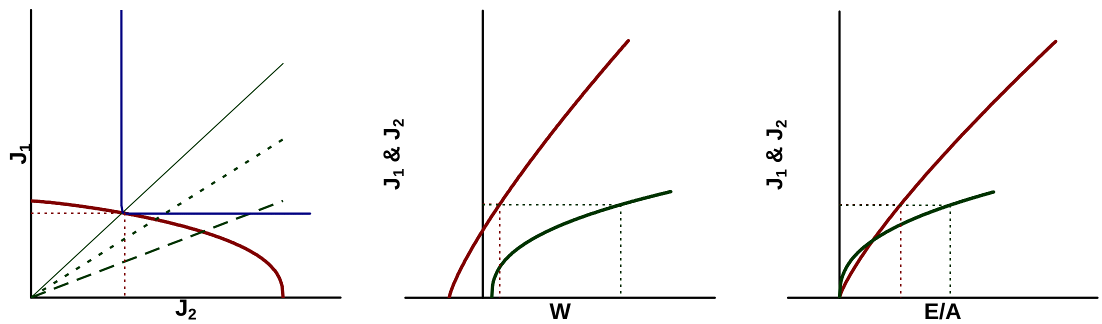

Figure 7. Scrooge's Paradise

Even though Scrooge will have a long and cranky life, he is not a pleasure machine and the frontier shifts down further. In Figure 7, with no substitution, Scrooge is in paradise. The social optimum gives him the gold.

In Figure 6, workers and Scrooge are biomass.

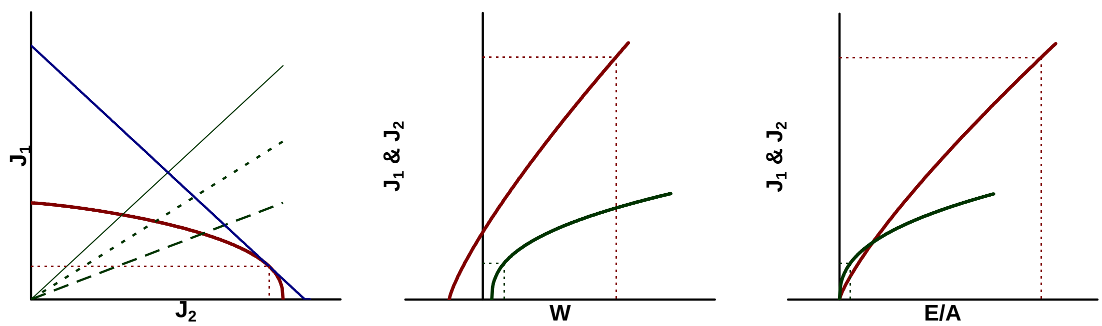

Figure 8. Grinding Poverty

If Scrooge can be easily replaced by a cheerful worker, the social optimum will allocate him a life of grinding poverty.

Finally, suppose Scrooge has a gun and promotes his own theory of justice as in Figure 9.

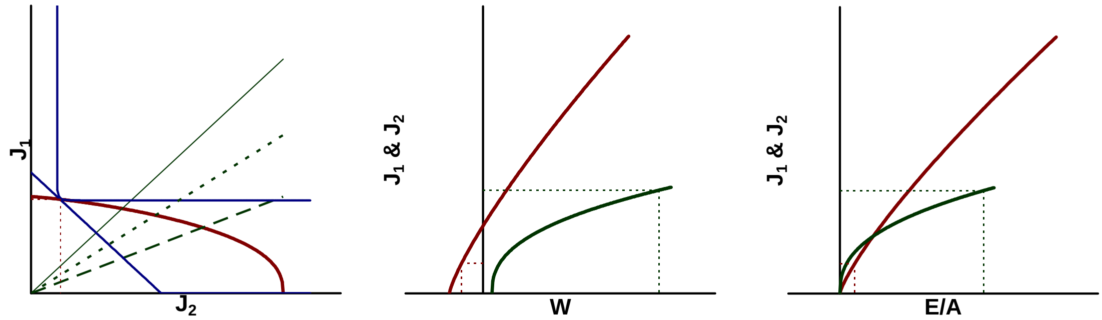

Figure 9. Scrooge's Theory of Justice

Scrooge has no substitutes, but workers are biomass. In addition, Scrooge has a much higher weighting in the social welfare function. At the social optimum, Scrooge has more than all the wealth. Workers owe him their future wages and spend almost nothing.

Surely, most democratic societies will have a social optimum which treats people somewhere between perfect complements and perfect substitutes. Each society may have a different culture of welfare and organise itself to suit. As economists we could observe this diversity and apply our empirical skills to estimate social welfare functions.

Conclusions

This paper establishes that preferences must be cardinal over people's lifetimes if they are to choose between saving for the future and spending now. People will choose different preference orderings in response to a policy which changes society's institutions. Dynamic demand and expenditure functions depend upon cardinal preferences which may have scientific merit after all.

Must economists agree on a social welfare function? We don't feel the need to agree on a consumer's utility function and are happy to let consumers determine their own preferences. Why not let societies determine their social welfare functions? Then we can infer social rankings from observable policies. The proposed forms for social welfare functions are special cases of a relatively simple functional form. The two important components are substitutions and weightings among categories of people. Different societies will set these parameters differently and their behaviours can be observed.

The task of ranking all possible outcomes of all possible policies for all possible people may not be as impossible as all that. Witness the large amount of information generated in non market valuation of the environment and the research to transfer benefits from one study area to others.

Should policy-makers also make up the social welfare function? I am a bit worried. In my opinion, the job description of an economist is to create new technologies to increase the efficiency of institutions. At bottom, this means finding ways to replace tort lawyers with contract lawyers and government policy wonks with accountants. Environmental economists have been having some success lately.

Finally, an unintended result of the paper shows that willingness to pay and willingness to accept depend upon cardinal preferences. As a consequence, the main objection to measuring social welfare is also an objection to welfare analysis of any kind. I shudder to think what this might mean for the Slutsky equations.

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Appendix 1: Individual Welfare

Analytical Solution. The dynamic model of demand,

$$J(W_0) = \max_{Q_1,Q_2} \int_0^T U(t,Q_1,Q_2)dt + V(T,W_T)$$

$$\text{subject to:}$$

$$\dot{W} = F(W,Q_1,Q_2)$$

with functional forms,

$$U(t,Q_1,Q_2) = e^{-\rho t}\left[\beta_1(Q_1-\gamma_1)^{-\nu} + \beta_2(Q_2-\gamma_2)^{-\nu}\right]^{\frac{-\alpha}{\nu}}$$

$$F(W,Q_1,Q_2) = rW + Y - p_1Q_1 - p_2Q_2$$

$$p_1 = e^{-(r-g)(T-t)}p_{1T}$$
$$p_2 = e^{-(r-g)(T-t)}p_{2T}$$

has a closed-form solution for lifetime utility:

$$J(t,W) = e^{-\rho t}A(t)^{1-\alpha}B^{-\alpha}E(t,W)^{\alpha} + V(T,W_T)$$

$$\text{where:}$$

$$A(t) = \tfrac{1}{\delta}\left(1-e^{-\delta(T-t)}\right); \quad \delta = \tfrac{\rho-\alpha g}{1-\alpha}$$

$$B = \left[\beta_1\left(\frac{p_1}{\beta_1}\right)^{\frac{\nu}{\nu+1}} + \beta_2\left(\frac{p_2}{\beta_2}\right)^{\frac{\nu}{\nu+1}}\right]^{\frac{\nu+1}{\nu}}$$

$$E(t,W) = W - e^{-r(T-t)}W_T + \tfrac{1}{r}\left(1 - e^{-r(T-t)}\right)Y - \tfrac{1}{g}\left(1 - e^{-g(T-t)}\right)(p_1\gamma_1 + p_2\gamma_2)$$

$$\alpha \neq 1; \quad \delta \neq 0; \quad \nu \neq 0; \quad T < \infty$$

The solution is unique and contains a dynamic demand system,

$$Q_1 - \gamma_1 = b_1 \frac{E}{A}; \quad b_1 = \left(\frac{p_1}{\beta_1}\right)^{\frac{-1}{\nu+1}} B^{\frac{-\nu}{\nu+1}}$$

$$Q_2 - \gamma_2 = b_2 \frac{E}{A}; \quad b_2 = \left(\frac{p_2}{\beta_2}\right)^{\frac{-1}{\nu+1}} B^{\frac{-\nu}{\nu+1}}$$

The change in wealth becomes,

$$\dot{W} = rW + Y - p_1\gamma_1 - p_2\gamma_2 - \frac{E}{A}$$

Proof. Existence and uniqueness are shown by deriving the optimality conditions and then integrating from current time t to final time T. The Hamiltonian at time t is:

$$H = \max_{Q_1,Q_2}\left[e^{-\rho t}\left[\beta_1(Q_1 - \gamma_1)^{-\nu} + \beta_2(Q_2 - \gamma_2)^{-\nu}\right]^{\frac{-\alpha}{\nu}} + \lambda(rW + Y - p_1Q_1 - p_2Q_2)\right]$$

The first-order conditions for the controls, state and costate are:

$$\frac{\partial H}{\partial Q_1} = 0 = \alpha e^{-\rho t}\left[\beta_1(Q_1 - \gamma_1)^{-\nu} + \beta_2(Q_2 - \gamma_2)^{-\nu}\right]^{\frac{-\alpha}{\nu}-1} \beta_1(Q_1 - \gamma_1)^{-\nu-1} - \lambda p_1$$

$$\frac{\partial H}{\partial Q_2} = 0 = \alpha e^{-\rho t}\left[\beta_1(Q_1 - \gamma_1)^{-\nu} + \beta_2(Q_2 - \gamma_2)^{-\nu}\right]^{\frac{-\alpha}{\nu}-1} \beta_2(Q_2 - \gamma_2)^{-\nu-1} - \lambda p_2$$

$$-\frac{\partial H}{\partial W} = \dot{\lambda} = -\lambda r$$

$$\frac{\partial H}{\partial \lambda} = \dot{W} = rW + Y - p_1Q_1 - p_2Q_2$$

The Hamiltonian is concave and the second-order conditions are satisfied. In addition, the costate satisfies the transversality condition,

$$\lambda_T = \frac{\partial V}{\partial M_T}$$

and the solution must satisfy the terminal condition,

$$J(T,W_T) = V(T,W_T)$$

To integrate the first-order conditions, first integrate the costate and obtain a particular solution using the transversality condition.

$$\lambda = e^{r(T-t)}\lambda_T = e^{r(T-t)}\frac{\partial V}{\partial M_T}$$

Solve the controls as a function of the costate.

$$Q_1 - \gamma_1 = \left(\tfrac{1}{\alpha} e^{\rho t} \lambda\right)^{\frac{1}{\alpha-1}} \left(\frac{p_1}{\beta_1}\right)^{\frac{-1}{\nu+1}} B^{\frac{\alpha+\nu}{(\alpha-1)(\nu+1)}}$$

$$Q_2 - \gamma_2 = \left(\tfrac{1}{\alpha} e^{\rho t} \lambda\right)^{\frac{1}{\alpha-1}} \left(\frac{p_2}{\beta_2}\right)^{\frac{-1}{\nu+1}} B^{\frac{\alpha+\nu}{(\alpha-1)(\nu+1)}}$$

Substitute these controls into current utility.

$$U(t,W_t) = e^{-\rho t}\left(\tfrac{1}{\alpha} e^{\rho t} \lambda\right)^{\frac{\alpha}{\alpha-1}} B^{\frac{\alpha}{\alpha-1}}$$

Substitute current utility into lifetime utility, beginning at time t.

$$J(t,W_t) = \int_t^T e^{-\rho s}\left(\tfrac{1}{\alpha} e^{\rho s} \lambda\right)^{\frac{\alpha}{\alpha-1}} B^{\frac{\alpha}{\alpha-1}} ds + V(T,W_T)$$

Substitute the costate and prices for all future times as s goes from t to T. Integrate.

$$\int_t^T e^{\frac{\rho}{\alpha-1}s}\left(\tfrac{1}{\alpha} e^{r(T-\tau)} \lambda_T\right)^{\frac{\alpha}{\alpha-1}} \left[\beta_1\left(\frac{e^{-(r-g)(T-\tau)} p_{1T}}{\beta_1}\right)^{\frac{\nu}{\nu+1}} + \beta_2\left(\frac{e^{-(r-g)(T-\tau)} p_{2T}}{\beta_2}\right)^{\frac{\nu}{\nu+1}}\right]^{\frac{\alpha(\nu+1)}{(\alpha-1)\nu}} ds$$

$$= \left(\tfrac{1}{\alpha}\lambda_T\right)^{\frac{\alpha}{\alpha-1}} \left[\beta_1\left(\frac{p_{1T}}{\beta_1}\right)^{\frac{\nu}{\nu+1}} + \beta_2\left(\frac{p_{2T}}{\beta_2}\right)^{\frac{\nu}{\nu+1}}\right]^{\frac{\alpha(\nu+1)}{(\alpha-1)\nu}} e^{\frac{\rho}{\alpha-1}(T-t)}\left(\tfrac{\alpha-1}{\rho-\alpha g}\right)\left(e^{\frac{\rho}{1-\alpha}(T-t)} - e^{\frac{\alpha g}{1-\alpha}(T-t)}\right)$$

Then replace the costate and prices at terminal time T with the costate and prices at current time t and simplify.

$$J(t,W_t) = e^{-\rho t}\left(\tfrac{1}{\alpha} e^{\rho t} \lambda\right)^{\frac{\alpha}{\alpha-1}} A B^{\frac{\alpha}{\alpha-1}} + V(T,W_T)$$

Next, integrate the state variable. First, split total expenditures into expenditures on subsistence and expenditures above subsistence.

$$\dot{W} = rW + Y - p_1\gamma_1 - p_2\gamma_2 - p_1(Q_1 - \gamma_1) - p_2(Q_2 - \gamma_2)$$

Substitute the first-order conditions for the controls.

$$\dot{W} = rW + Y - p_1\gamma_1 - p_2\gamma_2 - \left(\tfrac{1}{\alpha} e^{\rho t} \lambda\right)^{\frac{1}{\alpha-1}} B^{\frac{\alpha}{\alpha-1}}$$

Solve the differential equation to find terminal wealth.

$$W_T = e^{r(T-t)}\left[W + \int_t^T e^{-r(s-t)}\left[Y - p_1\gamma_1 - p_2\gamma_2\right]ds - \int_t^T e^{-r(s-t)}\left(\tfrac{1}{\alpha} e^{\rho s} \lambda\right)^{\frac{1}{\alpha-1}} B^{\frac{\alpha}{\alpha-1}} ds\right]$$

The second integral is lifetime expenditures above subsistence. Identify it as the function E.

$$E(t,W_t) = \int_t^T e^{-r(s-t)}\left(\tfrac{1}{\alpha}e^{\rho s}\lambda\right)^{\frac{1}{\alpha-1}} B^{\frac{\alpha}{\alpha-1}} ds$$

As with lifetime utility, substitute the costate and prices for all future times and integrate. Then replace the costate and prices at the terminal time with the costate and prices at the current time and simplify.

$$E(t,W_t) = \left(\tfrac{1}{\alpha}e^{\rho t}\lambda\right)^{\frac{1}{\alpha-1}} A B^{\frac{\alpha}{\alpha-1}}$$

Rearrange the formula for terminal wealth to obtain a second expression for lifetime expenditures above subsistence.

$$E(t,W_t) = W - e^{-r(T-t)}W_T + \int_t^T e^{-r(s-t)}\left[Y - p_1\gamma_1 - p_2\gamma_2\right]ds$$

Evaluate the integral for lifetime income above subsistence.

$$E(t,W_t) = W - e^{-r(T-t)}W_T + \tfrac{1}{r}\left(1 - e^{-r(T-t)}\right)Y - \tfrac{1}{g}\left(1 - e^{-g(T-t)}\right)\left(p_1\gamma_1 + p_2\gamma_2\right)$$

With two expressions for lifetime expenditures, the first expression can be solved for the costate.

$$\lambda = \alpha e^{-\rho t} A^{1-\alpha} B^{-\alpha} E^{\alpha-1}$$

Use the costate to obtain the final solution. Substitute into lifetime utility, current utility, the demand system and the change in wealth,

$$J(t,W) = e^{-\rho t}\left(\tfrac{1}{\alpha}e^{\rho t}\lambda\right)^{\frac{\alpha}{\alpha-1}} A B^{\frac{\alpha}{\alpha-1}} + V(T,W_T) = e^{-\rho t}A^{1-\alpha}B^{-\alpha}E^{\alpha} + V(T,W_T)$$

$$U(t,W) = e^{-\rho t}\left(\tfrac{1}{\alpha}e^{\rho t}\lambda\right)^{\frac{\alpha}{\alpha-1}} B^{\frac{\alpha}{\alpha-1}} = e^{-\rho t}A^{-\alpha}\, B^{-\alpha}E^{\alpha}$$

$$Q_1 - \gamma_1 = \left(\tfrac{1}{\alpha}e^{\rho t}\lambda\right)^{\frac{1}{\alpha-1}} \left(\frac{p_1}{\beta_1}\right)^{\frac{-1}{\nu+1}} B^{\frac{\alpha+\nu}{(\alpha-1)(\nu+1)}} = \left(\frac{p_1}{\beta_1}\right)^{\frac{-1}{\nu+1}} B^{\frac{-\nu}{\nu+1}} \frac{E}{A} = b_1\frac{E}{A}$$

$$Q_2 - \gamma_2 = \left(\tfrac{1}{\alpha}e^{\rho t}\lambda\right)^{\frac{1}{\alpha-1}} \left(\frac{p_2}{\beta_2}\right)^{\frac{-1}{\nu+1}} B^{\frac{\alpha+\nu}{(\alpha-1)(\nu+1)}} = \left(\frac{p_2}{\beta_2}\right)^{\frac{-1}{\nu+1}} B^{\frac{-\nu}{\nu+1}} \frac{E}{A} = b_2\frac{E}{A}$$

$$\dot{W} = rW + Y - p_1\gamma_1 - p_2\gamma_2 - \left(\tfrac{1}{\alpha}e^{\rho t}\lambda\right)^{\frac{1}{\alpha-1}} B^{\frac{\alpha}{\alpha-1}} = rW + Y - p_1\gamma_1 - p_2\gamma_2 - \frac{E}{A}$$

The solution has singularities at $\alpha = 1$, $\delta = 0$, $\nu = 0$ and $T = \infty$. The first three singularities can be avoided by taking limits or by setting the parameters to be $\pm\varepsilon$ away from the singularity.

Welfare Analysis. Define willingness to pay and willingness to accept as equivalent and compensating variations for a discrete change in prices, Δ_p.

$$J(t,W - WTP, p) = J\left(t,W, p + \Delta_p\right)$$
$$J(t,W, p) = J\left(t,W + WTA, p + \Delta_p\right)$$

In general, these equations are highly nonlinear and must be solved numerically, which requires a functional form for the terminal value. Assume a Cobb-Douglas function.

$$V(T,W_T) = e^{-\rho T}\phi W_T^{\omega}$$

One special case has an algebraic solution. Differentiate to find the terminal costate.

$$\lambda_T = \omega e^{-\rho T}\phi W_T^{\omega-1}$$

Multiply by both sides of this equation by wealth and solve for the terminal value.

$$V(T,W_T) = \frac{\lambda}{\omega} e^{-r(T-t)} W_T$$

From the derivative, costate λ will be independent of stocks if $\omega = 1$. In addition, expenditures will be independent.

$$E(t,W) = \left(\tfrac{1}{\alpha} e^{\rho t}\lambda\right)^{\frac{1}{\alpha-1}} AB^{\frac{\alpha}{\alpha-1}}$$

Substituting expenditures and terminal wealth into lifetime utilities and simplifying gives an algebraic solution.

$$WTP = WTA = \tfrac{(1-\alpha)}{\alpha} A\left(\tfrac{1}{\alpha} e^{-(\rho-r)(T-t)}\phi\right)^{\frac{1}{\alpha-1}}\left[B(p)^{\frac{\alpha}{\alpha-1}} - B(p+\Delta_p)^{\frac{\alpha}{\alpha-1}}\right] + \tfrac{1}{g}\left(1-e^{-g(T-t)}\right)\Delta_p\gamma_2$$

For comparison, *WTP* and *WTA* from a static model have algebraic solutions which depend upon the ratio of substitution factors.

$$WTP = \left(1 - \frac{B(p)}{B(p+\Delta_p)}\right)\left[M + \tfrac{1}{r}\left[Y - p_1\gamma_1 + p_2\gamma_2\right]\right] + \frac{B(p)}{B(p+\Delta_p)}\tfrac{1}{r}\Delta_p\gamma_2$$

$$WTA = \frac{B(p+\Delta_p)}{B(p)} WTP$$

The static model is a special case if the terminal value can be chosen freely, a steady state prevails before the change and a new steady state follows immediately after the change.

Appendix 2: Social Welfare

Dual Solution. The static model of social welfare,

$$J(W) = \max_{W_1,W_2} U(W_1,W_2)$$

$$\text{subject to:}$$

$$W = W_1 + W_2$$

with social welfare functional

$$U(W_1,W_2) = \left[\beta_1 J_1(W_1)^{-\nu} + \beta_2 J_2(W_2)^{-\nu}\right]^{\frac{-\alpha}{\nu}}$$

and J_1 and J_2 as lifetime utilities for categories of individuals 1 and 2, has a dual solution:

$$J(J_1,J_2) = B^{\alpha}E(J_1,J_2)^{\alpha}$$

where :

$$B = \left[\beta_1(\beta_1\lambda_1)^{\frac{-\nu}{\nu+1}} + \beta_2(\beta_2\lambda_2)^{\frac{-\nu}{\nu+1}}\right]^{-\frac{\nu+1}{\nu}}$$

$$E(J_1,J_2) = \frac{J_1}{\lambda_1} + \frac{J_2}{\lambda_2}$$

$$\alpha \neq 1; \quad \nu \neq 0$$

λ_1 and λ_2 are costates for categories of individuals.

Proof. First derive the optimality conditions. The Lagrangian is:

$$L = \max_{W_1,W_2}\left[\left[\beta_1 J_1(W_1)^{-\nu} + \beta_2 J_2(W_2)^{-\nu}\right]^{\frac{-\alpha}{\nu}} + \theta(W - W_1 - W_2)\right]$$

The first-order conditions are:

$$\frac{\partial L}{\partial W_1} = 0 = \alpha\left[\beta_1 J_1(W_1)^{-\nu} + \beta_2 J_2(W_2)^{-\nu}\right]^{\frac{-\alpha}{\nu}-1}\beta_1 J_1{}^{-\nu-1}\lambda_1 - \theta$$
$$\frac{\partial L}{\partial W_2} = 0 = \alpha\left[\beta_1 J_1(W_1)^{-\nu} + \beta_2 J_2(W_2)^{-\nu}\right]^{\frac{-\alpha}{\nu}-1}\beta_2 J_2{}^{-\nu-1}\lambda_2 - \theta$$

$$\frac{\partial L}{\partial \theta} = 0 = W - W_1 - W_2$$

The Lagrangian is concave and the second-order conditions are satisfied. Solve for the lifetime utilities for categories of individuals as functions of the Lagrange multiplier.

$$J_1 = \left(\tfrac{1}{\alpha}\theta\right)^{\frac{1}{\alpha-1}}(\beta_1\lambda_1)^{\frac{1}{\nu+1}} B^{\frac{\alpha+\nu}{(\alpha-1)\nu}}$$
$$J_2 = \left(\tfrac{1}{\alpha}\theta\right)^{\frac{1}{\alpha-1}}(\beta_2\lambda_2)^{\frac{1}{\nu+1}} B^{\frac{\alpha+\nu}{(\alpha-1)\nu}}$$

Substitute these into the welfare functional and simplify.

$$U(J_1,J_2) = \left(\tfrac{1}{\alpha}\theta\right)^{\frac{\alpha}{\alpha-1}} B^{\frac{\alpha}{\alpha-1}}$$

Normalize by defining the social welfare equivalent of an expenditure function.

$$E(J_1,J_2) = \frac{J_1}{\lambda_1} + \frac{J_2}{\lambda_2}$$

Substitute in the lifetime utilities for categories of individuals and simplify.

$$E(J_1, J_2) = \left(\tfrac{1}{\alpha}\theta\right)^{\frac{1}{\alpha-1}} B^{\frac{-\alpha}{\alpha-1}}$$

Alternatively, substitute lifetime utilities and costates from solutions for categories of individuals in Appendix 1. Simplify.

$$E(J_1, J_2) = \tfrac{1}{\alpha_1} E_1 + \tfrac{1}{\omega_1} e^{-r(T-t)} W_{1T} + \tfrac{1}{\alpha_2} E_2 + \tfrac{1}{\omega_2} e^{-r(T-t)} W_{2T}$$

With two expressions for expenditures, solve the first expression for the Lagrange multiplier.

$$\theta = \alpha B^{\alpha} E^{\alpha-1}$$

Use the Lagrange multiplier to obtain the dual solution for static social welfare.

$$J(J_1, J_2) = \left(\tfrac{1}{\alpha}\theta\right)^{\frac{\alpha}{\alpha-1}} B^{\frac{\alpha}{\alpha-1}} = B^{\alpha} E^{\alpha}$$

The dual solution has singularities at $\alpha = 1$ and $\nu = 0$.