Weather Forecast Based Conditional Pest Management: A Stochastic Optimal Control Investigation

Liang Lu[1] and Levan Elbakidze[2]

1: Graduate Student, lu5201@vandals.uidaho.edu, Department of Agricultural Economics and Rural Sociology, University of Idaho

2: Assistant Professor, lelbakidze@uidaho.edu, Department of Agricultural Economics and Rural Sociology, University of Idaho

Selected Paper prepared for presentation at the Agricultural & Applied Economics Association's

2011 AAEA & NAREA Joint Annual Meeting, Pittsburgh, Pennsylvania, July 24-26, 2011

Abstract

In this paper, we examine conditional, forecast-based dynamic pest management in agricultural crop production given stochastic pest infestations and stochastic climate dynamics throughout the growing season. Using stochastic optimal control we show that correlation between forecast error for climate prediction and forecast error for pest outbreaks can be used to improve pesticide application efficiency. In the general setting, we apply modified Hamiltonian approach to discuss the steady state equilibrium. Given specific functional forms, a closed form solution can be found for the stochastic optimal control problem. Moreover, we find conditions for model parameters so that the optimal pesticide usage path will be monotonically increasing or decreasing in the correlation coefficient between climate forecast errors and pest growth disturbances.

Introduction

The purpose of this paper is to examine conditional, forecast-based dynamic pest management in agricultural crop production given stochastic pest infestations and stochastic climate dynamics throughout the growing season. Forecasts of pest outbreaks in conjunction with forecasts of climatic conditions can be used to improve effectiveness of pest management decisions. Producers can adjust pesticide application rates depending on the forecasts of pest outbreaks. In this paper we consider that given potential association between pest infestation and climatic conditions and the effects of both of these variables on yields (Elbakidze, Lu and Eigenbrode, 2011; Cobourn et al. 2011), forecasts of both, climatic conditions and pest outbreaks, can be used to optimize applications of pesticides. Furthermore we argue that the prediction errors from climate and pest outbreak forecasts and their correlations can be used to optimize pest management strategies. Using stochastic optimal control we show that correlation between forecast error for climate prediction and forecast error for pest outbreaks can be used to improve pesticide application efficiency.

The literature on pest management typically specifies the pest management problem in terms of damage control inputs within damage (or damage abatement) function in conjunction with the production function (Lichtenberg and Zilberman, 1986; Fox and Weersink, 1995; Saha, Shumway and Havenner, 1997; Carpentier and Weaver, 1997). The advantage of such formulation is that it allows the modeler to separate the effects of direct production inputs from the effectiveness of pest control inputs via damage function specification. While earlier studies focused on static and deterministic specifications, several later studies have extended this approach to dynamic examinations (Zivin, Hueth and Zilberman 2000; Marsh, Huffaker, and long, 2000; Olson and Roy 2002, Zhang and Swinton (2009)). As Olson (2006) pointed out, dynamic models provide more insight than static models in that the value of pesticide application in such models includes not only the benefits of removing the pests in the current period but also the discounted sum of benefits from precluding future pests. Following this logic, we construct a dynamic model corresponding to a planning horizon lasting from planting to harvesting. We assume that the decision on crop acreage has been made, but the decisions about pesticide use remains to be made throughout the growing season.

Another important aspect of pest management problem is uncertainty associated with pest infestation. The dynamics of pest populations can be characterized as a combination of a deterministic population growth pattern and a stochastic fluctuation as a result of unexplained factors that may cause the population of the pest to increase or decrease. In stochastic pest management studies a typical assumption is that the dynamics of pest infestation follows a diffusion process based on Weiner process type of formulation (Saphores 2000; Sunding and Zilvin, 2000; Saphores and Shogren, 2005; Richards et al. 2005). Hertzler (1991) uses stochastic optimal control and Ito stochastic calculus to study dynamic agricultural decisions under risk. He suggests that diffusion process based stochastic dynamic models and Ito calculus can be used for economic studies of pest control in agricultural production. Olson and Roy

(2002) approaches the problem of managing biological invasions in terms of minimizing expected value of discounted sum of costs and damages subject to pest growth dynamics. They solve the minimization problem using stochastic dynamic programming and provide conditions for when it is optimal to eradicate the invasive species. Cobourn (2009) also uses stochastic dynamic programming to study pest management options when activities of heterogeneous producers can influence effectiveness of pesticide use. Kim et al. (2006) study optimal allocation of resources between prevention and control for invasive species management using dynamic formulation of stochastic invasion and subsequent discovery. We extend the previous formulations by incorporating two relevant and related stochastic variables in our optimal control model: climatic conditions and pest invasion. Furthermore, we examine how potential correlation between these stochastic variables may affect optimal pesticide use.

The roles of climate conditions in agriculture (Costello, Adams and Polasky, 1998; Rubas et al 2008; Chen, McCarl and Schimmelpfennig, 2004, Kim and McCarl 2004) as well as the role of climatic condition in pest management (Chen and McCarl 2001, Cobourn et al., 2011) have been addressed by economists. However, the economists have given little attention to optimal dynamic pest management when climatic conditions affect crop growth as well as pest populations simultaneously. Olson and Roy (2002) formulated their model by assuming pest growth is affected by environmental disturbances. Elbakidze, Lu and Eigenbrode (2011) examined the effects of climate and pests on agricultural productivity in a simultaneous fashion taking into account that pest infestations maybe affected by climatic conditions. Their analysis is done in a static context. Cobourn et al. (2011) have also examined how climatic variables may positively affect crop yields which in turn can attract more pests via improved habitat. In this paper we combine the effects of climatic conditions on pest infestations and on crop yields in a stochastic optimal control setting.

The rest of this paper is organized as follows: In the next section, we provide general stochastic optimal control analysis of pest management in the context of stochastic climate and stochastic pest outbreaks. Optimality conditions are discussed and a representative phase diagram is presented. The formulation showing dependence of optimal pest management on correlation between stochastic climate and stochastic pest population is provided. Subsequently, we examine a specific analytical case with specific functional forms. We provide conditions for optimal pesticide use path as a function of the correlation coefficient between pest and climate forecast errors.

The general case

Our framework is based on minimization of total expected costs associated with pests and pest management. Given that crop growth depends on stochastic pest infestation and stochastic climatic conditions, which can be correlated, we formulate dynamic crop growth losses associated pest populations and climate as

$$(1) \quad Y^L = Y^L\left(A, \theta, t\right)$$

where θ denotes a climate index (for example temperature or precipitation), *A* is the pest population, and t denotes time. We assume that crop losses, Y^L, is differentiable with respect to all of its arguments and is increasing in *A*.

Climate index is assumed to be following a diffusion process (Mraoua and Bari, 2005)

$$(2) \quad d\theta = \mu^{\theta}(\theta,t)dt + \sigma^{\theta}(\theta,t)d\tilde{\theta}$$

where μ^{θ} and σ^{θ} are representing expected changes in the climate index over time and standard deviation respectively. $\tilde{\theta}$ is the standard wiener process. μ^{θ} can be interpreted as the predictable change of climate index with standard error σ^{θ}.

Pest population is given by the function:

$$(3) \quad A = A\left(u(t),\theta,t,\tilde{A}\right)$$

where u denotes pesticide use, $\tilde{A}$ represents another Wiener process which can be interpreted as all other uncontrolled factors that affect pest population.

By Ito's Lemma (Hertzler, 1991; Kamien and Schwartz; 2003), we have

$$\begin{aligned}(4) \quad dA &= \left(A_t + A_{\theta}\mu^{\theta} + \frac{1}{2}A_{\theta\theta}\left(\sigma^{\theta}\right)^2 + \frac{1}{2}A_{\tilde{A}\tilde{A}} + A_{\theta\tilde{A}}\rho^{\tilde{\theta}\tilde{A}}\sigma^{\theta}\right)dt + A_{\theta}\sigma^{\theta}(\theta,t)d\tilde{\theta} + A_{\tilde{A}}d\tilde{A} \\ &\triangleq \mu^{A}(u,A,\theta,t)dt + A_{\theta}\sigma^{\theta}(\theta,t)d\tilde{\theta} + A_{\tilde{A}}d\tilde{A}\end{aligned}$$

Then *A* is also following a diffusion process which is similar to Mbah et al. (2010). Here, μ^A denotes the drift for pest growth, or expected change in pest population. μ^A includes the intrinsic deterministic pest growth rate, the deterministic effect of climate on pest growth, and the second order terms by Ito's Lemma. We assume that μ^A is decreasing in the control variable, u. It is clear that the change in pest population will have two sources of uncertainty, one associated with climate uncertainty and the other coming from other unaccounted random environmental and ecological factors. Reflecting a possibility of interaction between climatic index and pest population (Elbakidze, Lu, and Eigenbrode, 2011; Cobourn et al. 2011) beyond deterministic context we assume that $d\tilde{\theta}d\tilde{A} = \rho^{\tilde{\theta}\tilde{A}}dt$, where $\rho^{\tilde{\theta}\tilde{A}}$ denotes the correlation between $\tilde{A}$ and $\tilde{\theta}$. Non zero $\rho^{\tilde{\theta}\tilde{A}}$ implies that deviations from expected (or predicted) levels of pest populations and climate index can be correlated. Again by Ito's lemma, the crop growth dynamics can be expressed as[1]:

$$
\begin{aligned}
(5) \qquad dY^L = & \left(Y_t + Y_\theta \mu^\theta + Y_A \mu^A + \frac{1}{2} Y_{\theta\theta} \left(\sigma^\theta\right)^2 + \frac{1}{2} Y_{AA} \left(\sigma^A\right)^2 + Y_\theta Y_A \rho^{\tilde{\theta}\tilde{A}} \sigma^\theta \sigma^A \right) dt \\
& + \left(Y_\theta \sigma^\theta \left(\theta, t\right) + Y_A A_\theta \sigma^\theta \left(\theta, t\right) \right) d\tilde{\theta} + Y_A A_{\tilde{A}} d\tilde{A} \\
& \triangleq \mu^Y \left(u, A, Y^L, \theta, t\right) dt + \left(Y_\theta \sigma^\theta \left(\theta, t\right) + Y_A A^\theta \sigma^\theta \left(\theta, t\right) \right) d\tilde{\theta} + Y_A A_{\tilde{A}} d\tilde{A}
\end{aligned}
$$

Where $\left(\sigma^A\right)^2 = \left(A_\theta \sigma^\theta \left(\theta, t\right)\right)^2 + \left(A_{\tilde{A}}\right)^2 + A_\theta A_{\tilde{A}} \rho^{\tilde{\theta}\tilde{A}} \sigma^\theta$

μ^Y is assumed to be decreasing in *u;* and variance of dY^L can be expressed as

$$
\begin{aligned}
\sigma^Y &= \left(Y_\theta \sigma^\theta + Y_A A^\theta \sigma^\theta \right)^2 + \left(Y_A A_{\tilde{A}} \right)^2 + \rho \left(Y_\theta \sigma^\theta + Y_A A^\theta \sigma^\theta \right) Y_A A_{\tilde{A}} = \\
&= \left(Y_A \sigma^A \right)^2 + Y_\theta \sigma^\theta \left(Y_\theta \sigma^\theta + 2 Y_A A^\theta \sigma^\theta + Y_A A_{\tilde{A}} \rho^{\tilde{\theta}\tilde{A}} \right)
\end{aligned}
$$

The farmer's problem than is to minimize losses and costs associated with pest infestation and management, which can be expressed as the following stochastic optimal control problem:

$$
\min E \left\{ e^{-rT} p Y^L \left(T\right) + \int_0^T e^{-rt} w u\left(t\right) dt \right\}
$$

subject to (4) and (5)

where T is the terminal crop harvest period, r is discount rate, $u\left(t\right)$ denotes the path of pesticide usage, and *p* and *w* denote the prices of harvested crops and costs of

[1] We use subscripts to denote derivatives throughout the text

pesticide use respectively.

Hamiltionian-Jacobian-Bellman equation is given by:

$$(6) \quad -J_t = \min_u \left\{ wu + J_Y \mu^Y + J_A \mu^A + \frac{1}{2} J_{YY} \left(\sigma^Y\right)^2 + \frac{1}{2} J_{AA} \left(\sigma^A\right)^2 + J_{AY} \sigma^Y \sigma^A \right\}$$

Define the following matrices

$$\mathbf{J}^1 = \left[J_A, J_Y\right]$$

$$\mathbf{J}^2 = \begin{bmatrix} J_{AA} & J_{AY} \\ J_{AY} & J_{YY} \end{bmatrix}$$

$$\boldsymbol{\mu} = \begin{bmatrix} \mu^A\left(u, A, \theta, t\right) \\ \mu^{Y^L}\left(Y^L, A, \theta, t\right) \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \left(\sigma^A\right)^2 & \sigma^A \sigma^Y \rho^{AY} \\ \sigma^A \sigma^Y \rho^{AY} & \left(\sigma^Y\right)^2 \end{bmatrix}$$

Then (6) can be rewritten as:

$$(7) \quad -J_t = \min_u \left\{ wu + \mathbf{J}^1 \boldsymbol{\mu} + \frac{1}{2} trace\left(\mathbf{J}^2 \boldsymbol{\Sigma}\right) \right\}$$

To formulate the Hamiltonian version of the problem, let $\boldsymbol{\lambda}^1 = \left[\lambda^1\left(A, Y^L\right), \lambda^2\left(A, Y^L\right)\right]$ denote the vector of co-state variables for the pest and yield loss state equations respectively, and

$$\boldsymbol{\lambda}^2 = \begin{bmatrix} \lambda_A^1 & \lambda_Y^1 \\ \lambda_A^2 & \lambda_Y^2 \end{bmatrix}$$

The Hamiltonian then can be expressed as

$$(8) \quad H = wu + \boldsymbol{\lambda}^1 \boldsymbol{\mu} + \frac{1}{2} trace\left(\boldsymbol{\lambda}^2 \boldsymbol{\Sigma}\right)$$

By maximum principle,

(9) $$\frac{\partial H}{\partial u} = w + \boldsymbol{\lambda}^1 \frac{\partial \boldsymbol{\mu}}{\partial u} + \frac{1}{2} trace\left(\boldsymbol{\lambda}^2 \frac{\partial \boldsymbol{\Sigma}}{\partial u}\right) = 0$$

Notice that neglect of the stochastic components of the pest and crop growth will remove $\frac{1}{2} trace\left(\boldsymbol{\lambda}^2 \frac{\partial \boldsymbol{\Sigma}}{\partial u}\right)$ from the Hamiltonian. Consequently, the optimal decision rule will be biased.

let $\Lambda = \begin{bmatrix} A_\theta \sigma^\theta(\theta,t) & A_{\tilde{A}} \\ \left(Y_\theta^L \sigma^\theta(\theta,t) + Y_A^L A^\theta \sigma^\theta(\theta,t)\right) & Y_A^L A_{\tilde{A}} \end{bmatrix}$

$d\mathbf{z} = \begin{bmatrix} d\tilde{\theta} \\ d\tilde{A} \end{bmatrix}$

Then the optimaliy of the Hamiltonian implies:

$$\begin{bmatrix} d\lambda_1 \\ d\lambda_2 \end{bmatrix} = \left(r\boldsymbol{\lambda}^{\mathbf{1}^{\mathrm{T}}} - \begin{bmatrix} \frac{\partial H}{\partial A} \\ \frac{\partial H}{\partial Y^L} \end{bmatrix} \right) dt + \boldsymbol{\lambda}^2 \Lambda d\mathbf{z}$$

Suppose $u*$ exists, then the expected steady state can be shown by the following conditions where control variable is a function of co-state variables (Xepapadeas 1997):

(10) $$\frac{dEY^L}{dt} = \mu^{Y^L}\left(u*\left(\lambda^1,\lambda^2\right), A, Y^L, \theta, t\right) = 0$$

(11) $$\frac{dEA}{dt} = \mu^A\left(u*\left(\lambda^1,\lambda^2\right), A, \theta, t\right) = 0$$

(12) $$\frac{dE\lambda^1}{dt} = rE\lambda^1 - E\frac{\partial H\left(u*, A, Y^L, \lambda^1, \lambda^2\right)}{\partial A} = 0$$

(13) $$\frac{dE\lambda^2}{dt} = rE\lambda^2 - E\frac{\partial H\left(u*, A, Y^L, \lambda^1, \lambda^2\right)}{\partial Y^L} = 0$$

Proposition 1：

Suppose the expected steady state equilibrium exists, then it is unique if the following conditions are satisfied:

1. $u*\left(\lambda^1, \lambda^2\right)$ is monotonically increasing in both of its arguments. This condition implies that given an increase in the constraint (either more expected yield loss or pest growth), the optimal decision will always be to use more pesticide.

2. Both μ^{Y^L} and μ^A are monotonic in *A* and *Y*,

3. $\frac{\partial H}{\partial A}$ and $\frac{\partial H}{\partial Y^L}$ are positive implying that the value function is well behaved and increasing in the population of pest and yield loss.

4. The signs of second order partial derivatives of H are fixed.

To prove the above proposition notice that the steady state equilibrium is unique when the intersection of (10) through (13) is unique. Monotonicity of equations (10)-(13) implies that the intersection of (10) through (13) is unique.

Using (10), based on implicit function theorem we have

$$\frac{dY^L}{d\lambda^1} = -\frac{\frac{\partial \mu^Y}{\partial u*}\frac{\partial u*}{\partial \lambda^1}}{\frac{\partial \mu^Y}{\partial Y^L}}$$

The sign of this expression is uniquely identified by the signs of components on the right hand side. Similarly, one can examine the signs of the implicit function in (10) in other subspaces. Each of those relationships will be uniquely signed given fixed signs of factors in on the right hand side. This implies that as long as the signs of components on the RHS don’t change, as assumed in conditions 1 through 4 above, the left hand side signs will not change either. Similar logic can be applied to implicit functions defined by (11), (12), and (13). Therefore equations (10) through (13) are monotonic, implying uniqueness of the solution.

Phase diagram

We can examine the $\left(Y^L,\lambda^1\right)$ subspace of hyper-surface in $\left(A,Y^L,\lambda^1,\lambda^2\right)$ for graphical illustration of the FOC conditions (10-13). This phase (Figure 1) diagram corresponds to given shapes of μ^{Y^L}, μ^A, λ^1, λ^2. Alternative convexity assumptions will change the appearance of the diagram. However, the principle relationships pertinent to this discussion will remain unchanged.

Using (12), we have

(15) $$E\lambda^1 = \frac{1}{r}\frac{E\partial H\left(u*,A,Y^L,\lambda^1,\lambda^2\right)}{\partial A} > 0$$

Therefore, expected λ^1 is positive

Then the slope of $\frac{dEY^L}{dt}=0$ is determined by the Implicit function defined by (10),

(16) $$\frac{dY^L}{d\lambda^1} = -\frac{\frac{\partial\mu^{Y^L}}{\partial u*}\frac{\partial u*}{\partial\lambda^1}}{\frac{\partial\mu^{Y^L}}{\partial Y^L}} < 0$$

Assuming that $\frac{\partial\mu^{Y^L}}{\partial Y^L}<0$ and given that $\frac{\partial\mu^{Y^L}}{\partial u*}<0$ and $\frac{\partial u*}{\partial\lambda^1}>0$, we get $\frac{dY^L}{d\lambda^1}<0$

The direction of $\frac{dE\lambda^1}{dt}=0$ is determined by the Implicit function defined by (12),

(17) $$\frac{dY^L}{d\lambda^1} = -\frac{r-\frac{\partial^2 H}{\partial A\partial u*}\frac{\partial u*}{\partial\lambda^1}-\frac{\partial^2 H}{\partial A\partial\lambda^1}}{\frac{\partial^2 H}{\partial A\partial Y^L}}$$

Assuming that $\frac{\partial^2 H}{\partial A\partial Y^L}<0$, $\frac{\partial^2 H}{\partial A\partial u*}<0$, $\frac{\partial^2 H}{\partial A\partial\lambda^1}>0$, we get $\frac{dY^L}{d\lambda^1}>0$

From (10) and $\frac{\partial\mu^{Y^L}}{\partial u*}<0$, $\frac{\partial u*}{\partial\lambda^1}>0$, we have that for any $E\lambda^1$ on the right of the

$\frac{dEY^L}{dt} = 0$ curve, it must be the case that $\frac{dEY^L}{dt} < 0$.

Similarly, one can show that for any EY^L below the $\frac{dE\lambda^1}{dt} = 0$ curve,

It must be the case that $\frac{dE\lambda^1}{dt} > 0$

Figure 1 give a summary for the analysis above.

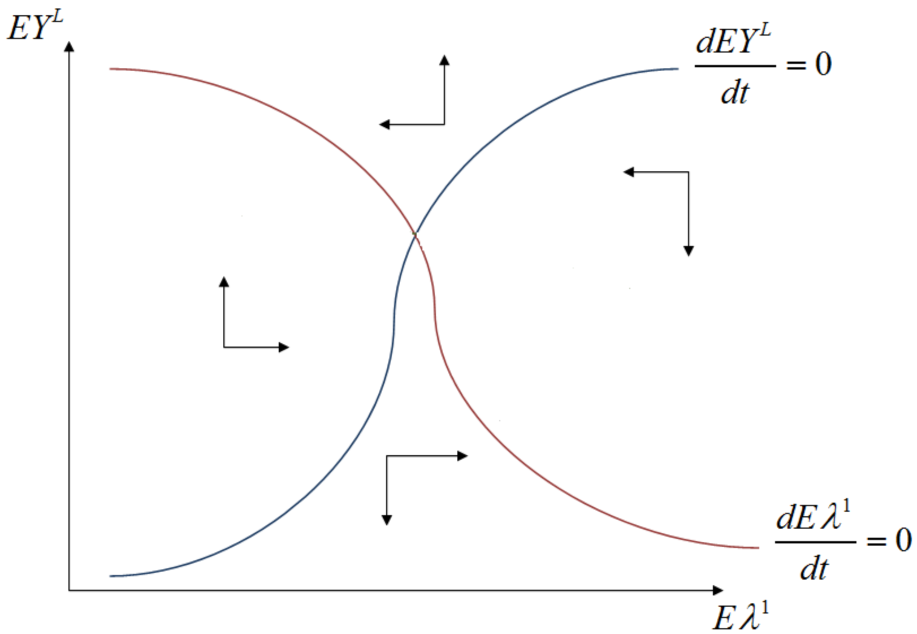

Figure 1 Phase Diagram

The steady state equilibrium is locally stable if and only if, in the neighborhood of the steady state, the Jacobian matrix (M) corresponding to FOCs in (10) to (13) is negative definite (See Lewis and Syrmos, 1995 for reference).

$$
\mathbf{M}\left(\lambda^1,\lambda^2,A,Y^L\right)=\begin{bmatrix}
\frac{\partial\mu^{Y^L}}{\partial u^*}\frac{\partial u^*}{\partial\lambda^1} & \frac{\partial\mu^{Y^L}}{\partial u^*}\frac{\partial u^*}{\partial\lambda^2} & \frac{\partial\mu^{Y^L}}{\partial A} & \frac{\partial\mu^{Y^L}}{\partial Y^L}\\
\frac{\partial\mu^{A}}{\partial u^*}\frac{\partial u^*}{\partial\lambda^1} & \frac{\partial\mu^{A}}{\partial u^*}\frac{\partial u^*}{\partial\lambda^2} & \frac{\partial\mu^{A}}{\partial A} & \frac{\partial\mu^{A}}{\partial Y^L}\\
r-\frac{\partial^2 H}{\partial A\partial\lambda^1}-\frac{\partial^2 H}{\partial A\partial u^*}\frac{\partial u^*}{\partial\lambda^1} & -\frac{\partial^2 H}{\partial A\partial\lambda^2}-\frac{\partial^2 H}{\partial A\partial u^*}\frac{\partial u^*}{\partial\lambda^2} & -\frac{\partial^2 H}{\partial A^2} & -\frac{\partial^2 H}{\partial A\partial Y^L}\\
-\frac{\partial^2 H}{\partial A\partial\lambda^1}-\frac{\partial^2 H}{\partial A\partial u^*}\frac{\partial u^*}{\partial\lambda^1} & r-\frac{\partial^2 H}{\partial A\partial\lambda^2}-\frac{\partial^2 H}{\partial A\partial u^*}\frac{\partial u^*}{\partial\lambda^2} & -\frac{\partial^2 H}{\partial A\partial Y^L} & -\frac{\partial^2 H}{\partial\left(Y^L\right)^2}
\end{bmatrix}
$$

To examine the relationship between optimal pesticide use u* and $\rho^{\tilde{\theta}\tilde{A}}$ – the correlation between climate and pest prediction (expectation) errors – we need to evaluate $\frac{du^*}{d\rho^{\tilde{\theta}\tilde{A}}}$. Since optimal path of the control variable is a function of co state variables (Xepapadeas 1997), $u^*\left(\lambda^1,\lambda^2\right)$, we can use traditional comparative statics approach applied for (10) through (13) to get $\frac{d\lambda^1}{d\rho^{\tilde{\theta}\tilde{A}}}$ and $\frac{d\lambda^2}{d\rho^{\tilde{\theta}\tilde{A}}}$. Then

(18) $$\frac{du^*}{d\rho^{\tilde{\theta}\tilde{A}}}=\frac{\partial u^*}{\partial\lambda^1}\frac{d\lambda^1}{d\rho^{\tilde{\theta}\tilde{A}}}+\frac{\partial u^*}{\partial\lambda^2}\frac{d\lambda^2}{d\rho^{\tilde{\theta}\tilde{A}}}$$

Using traditional comparative statics approach and defining implicit functions (10) to (13) as $F^1\left(\lambda^1,\lambda^2,A,Y^L\right)$ through $F^4\left(\lambda^1,\lambda^2,A,Y^L\right)$ we get:

(19) $$\frac{\partial F^i}{\partial\lambda^1}d\lambda^1+\frac{\partial F^i}{\partial\lambda^2}d\lambda^2+\frac{\partial F^i}{\partial A}dA+\frac{\partial F^i}{\partial Y^L}dY^L=-\frac{\partial F^i}{\partial\rho^{\tilde{\theta}\tilde{A}}}d\rho^{\tilde{\theta}\tilde{A}}$$

for $i=1,2,3,4$.

Using Cramer’s rule and putting the partial derivatives of the implicit functions in matrix form, we have

$$\mathbf{M}^{\rho\lambda^1}=\begin{bmatrix} \frac{\partial\mu^{Y^L}}{\partial\rho^{\tilde{\theta}\tilde{A}}} & \frac{\partial\mu^{Y^L}}{\partial u*}\frac{\partial u*}{\partial\lambda^2} & \frac{\partial\mu^{Y^L}}{\partial A} & \frac{\partial\mu^{Y^L}}{\partial Y^L} \\ \frac{\partial\mu^{A}}{\partial\rho^{\tilde{\theta}\tilde{A}}} & \frac{\partial\mu^{A}}{\partial u*}\frac{\partial u*}{\partial\lambda^2} & \frac{\partial\mu^{A}}{\partial A} & \frac{\partial\mu^{A}}{\partial Y^L} \\ \boldsymbol{\lambda}^1\frac{\partial\boldsymbol{\mu}}{\partial\rho^{\tilde{\theta}\tilde{A}}}+\frac{1}{2}trace\left(\boldsymbol{\lambda}^2\frac{\partial\boldsymbol{\Sigma}}{\partial\rho}\right) & -\frac{\partial^2 H}{\partial A\partial\lambda^2}-\frac{\partial^2 H}{\partial A\partial u*}\frac{\partial u*}{\partial\lambda^2} & -\frac{\partial^2 H}{\partial A^2} & -\frac{\partial^2 H}{\partial A\partial Y^L} \\ \boldsymbol{\lambda}^1\frac{\partial\boldsymbol{\mu}}{\partial\rho^{\tilde{\theta}\tilde{A}}}+\frac{1}{2}trace\left(\boldsymbol{\lambda}^2\frac{\partial\boldsymbol{\Sigma}}{\partial\rho}\right) & r-\frac{\partial^2 H}{\partial A\partial\lambda^2}-\frac{\partial^2 H}{\partial A\partial u*}\frac{\partial u*}{\partial\lambda^2} & -\frac{\partial^2 H}{\partial A\partial Y^L} & -\frac{\partial^2 H}{\partial\left(Y^L\right)^2} \end{bmatrix}$$

and

$$\mathbf{M}^{\rho\lambda^2}=\begin{bmatrix} \frac{\partial\mu^{Y^L}}{\partial u*}\frac{\partial u*}{\partial\lambda^1} & \frac{\partial\mu^{Y^L}}{\partial\rho^{\tilde{\theta}\tilde{A}}} & \frac{\partial\mu^{Y^L}}{\partial A} & \frac{\partial\mu^{Y^L}}{\partial Y^L} \\ \frac{\partial\mu^{A}}{\partial u*}\frac{\partial u*}{\partial\lambda^1} & \frac{\partial\mu^{A}}{\partial\rho^{\tilde{\theta}\tilde{A}}} & \frac{\partial\mu^{A}}{\partial A} & \frac{\partial\mu^{A}}{\partial Y^L} \\ r-\frac{\partial^2 H}{\partial A\partial\lambda^1}-\frac{\partial^2 H}{\partial A\partial u*}\frac{\partial u*}{\partial\lambda^1} & \boldsymbol{\lambda}^1\frac{\partial\boldsymbol{\mu}}{\partial\rho}+\frac{1}{2}trace\left(\boldsymbol{\lambda}^2\frac{\partial\boldsymbol{\Sigma}}{\partial\rho^{\tilde{\theta}\tilde{A}}}\right) & -\frac{\partial^2 H}{\partial A^2} & -\frac{\partial^2 H}{\partial A\partial Y^L} \\ -\frac{\partial^2 H}{\partial A\partial\lambda^1}-\frac{\partial^2 H}{\partial A\partial u*}\frac{\partial u*}{\partial\lambda^1} & \boldsymbol{\lambda}^1\frac{\partial\boldsymbol{\mu}}{\partial\rho}+\frac{1}{2}trace\left(\boldsymbol{\lambda}^2\frac{\partial\boldsymbol{\Sigma}}{\partial\rho^{\tilde{\theta}\tilde{A}}}\right) & -\frac{\partial^2 H}{\partial A\partial Y^L} & -\frac{\partial^2 H}{\partial\left(Y^L\right)^2} \end{bmatrix}$$

$\frac{d\lambda_1}{d\rho}=\frac{\left|\mathbf{M}^{\rho\lambda^1}\right|}{|\mathbf{M}|}$ and $\frac{d\lambda_2}{d\rho}=\frac{\left|\mathbf{M}^{\rho\lambda^2}\right|}{|\mathbf{M}|}$

Therefore, we have

$$(20)\qquad \frac{du*}{d\rho}=\frac{\partial u*}{\partial\lambda_1}\frac{\left|\mathbf{M}^{\rho\lambda^1}\right|}{|\mathbf{M}|}+\frac{\partial u*}{\partial\lambda_2}\frac{\left|\mathbf{M}^{\rho\lambda^2}\right|}{|\mathbf{M}|}$$

This identity describes the relationship between optimal control variable level and correlation between climate and pest prediction errors.

The transversality condition for the problem is implied by that $Y^L(T)\le Y^{LT}$ where Y^{LT}

is the maximum attainable yield at the end of the growing season.

$\lambda_2(T) = 0 \quad if \quad Y^{L*}(T) < Y^{LT}$

$\lambda_2(T) > 0 \quad if \quad Y^{L*}(T) = Y^{LT}$

The first of these conditions implies that if, in the terminal period, the yield loss is less than the maximum yield then the shadow price, or value of higher maximum yield, is zero. The second condition says that if the final yield loss equals maximum yield (all yield is lost to pests) then shadow price equals the value of having higher maximum yields and is positive. This can arise when low crop prices and high pesticide prices lead to disuse of pesticides resulting in loss of all yield.

The Specific Case

Following Lichtenberg and Zilberman (1986), crop growth losses due to pests can be expressed as

$$(21) \quad Y^L = f(\theta)D(A)$$

where $f(\theta)$ is maximum yield as a function of climate index, and $D(A)$ denotes the proportional damage function which is assumed to have the following properties:

1. $D(0) = 0$
2. $\lim_{A\to\infty} D(A) = 1$
3. $D(A)$ is non-decreasing in A

Suppose $D(A)$ is linear[2], and there exists a A_{max} such that $D(A_{max}) = 1$.

---

[2] Notice that $D(A)$ could be interpreted as a cumulative distribution function.

Consequently,

$$(22) \qquad D(A) = \frac{A}{A_{\max}}$$

Assuming $f(\theta) = \theta^{\alpha}$ and $0 \le A < A_{\max}$ we can express yield loss as

$$(23) \qquad Y^L = \theta^{\alpha} \frac{A}{A_{\max}}$$

A specific functional form has to be assigned for the climate index dynamics, and we use geometric Brownian motion for the stochastic differential equation:

$$d\theta = \mu^{\theta}\theta dt + \sigma^{\theta}\theta d\tilde{\theta}$$

And similarly, we assume that the pest growth dynamics is following

$$dA = \mu^{A}(A,\theta,u,t)Adt + \sigma^{A}(A,\theta,u,t)Ad\tilde{A}$$

Assuming the standard deviation of pest population growth is constant, $\sigma^{A}(A,\theta,u,t) = \sigma^{A}$, it can be shown (see appendix a1) that the general case becomes:

(24)

$$dY^L = \left[2\left(\alpha\mu^{\theta} + \mu^{A}\right) + \alpha\sigma^{\theta}\sigma^{A}\rho^{\tilde{\theta}\tilde{A}} + \alpha(\alpha-1)\left(\sigma^{\theta}\right)^2\right]Y^L dt + \sigma^{A}Y^L d\tilde{A} + \alpha\sigma^{\theta}Y^L d\tilde{\theta}$$

Solving for *u* requires a functional form for $\mu^{A}(u,\theta,A)$. Assume a simple functional form reflecting decreasing pest population as a function of *u*

$$(25) \qquad \mu^{A}(u,\theta,A) = \mu^{A} - u^{\gamma}A^{\frac{1}{\alpha}}\theta^{\beta}$$

Hamilton-Jacob-Bellman equation for stochastic optimal control problem (Kamien and Schwartz; 2003) can then be given by (See appendix a2)

$$(26) \qquad -J_t = \min_{u}\left\{ e^{-rt}wu + J_Y Y^L \begin{bmatrix} 2\left(\alpha\mu^{\theta} + \mu^{A} - u^{\gamma}A_{\max}^{\frac{\beta}{\alpha}}\left(Y^L\right)^{\frac{\beta}{\alpha}}\right) + \\ \alpha\sigma^{\theta}\sigma^{A}\rho^{\tilde{\theta}\tilde{A}} + \alpha(\alpha-1)\left(\sigma^{\theta}\right)^2 \end{bmatrix} + \frac{J_{Y^L Y^L}}{2}\left(\sigma^{Y^L}Y^L\right)^2 \right\}$$

where

$$\left(\sigma^{Y^L}\right)^2=\left(\sigma^A\right)^2+\left(\alpha\sigma^\theta\right)^2+2\alpha\sigma^\theta\sigma^A\rho^{\tilde{\theta}\tilde{A}}$$

The minimizing *u* is given by (see uppendix):

$$(27)\qquad u=\left(\frac{e^{-rt}wA_{\max}}{2\gamma J_{Y^L}\left(Y^L\right)^2}\right)^{\frac{1}{\gamma-1}}$$

Therefore, the HJB becomes (See appendix):

(28)

$$-J_t=\left(e^{-rt}w\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{1}{J_{Y^L}\left(Y^L\right)^{\frac{\beta}{\alpha}}A_{\max}^{\frac{\beta}{\alpha}}}\right)^{\frac{1}{\gamma-1}}\left[(2\gamma)^{\frac{-1}{\gamma-1}}-(2\gamma)^{\frac{-\gamma}{\gamma-1}}\right]+J_YY^L\left[2\left(\alpha\mu^\theta+\mu^A\right)+\alpha\sigma^\theta\sigma^A\rho^{\tilde{\theta}\tilde{A}}+\alpha(\alpha-1)\left(\sigma^\theta\right)^2\right]+\frac{J_{Y^LY^L}}{2}\left(\sigma^{Y^L}Y^L\right)^2$$

The carefully "guessed" value function that satisfies the above equation is given by

(see appendix):

(29)

$$J\left(Y^L,t\right)=Ce^{-rt}\left(Y^L\right)^{\frac{-\beta}{\alpha\gamma}}+e^{-rT}pY^L(T)-Ce^{-rT}\left(Y^L(T)\right)^{\frac{-\beta}{\alpha\gamma}}$$

Where

$$C=w\left(\frac{\alpha}{\beta}\right)\left(\frac{\left(A_{\max}^{-\frac{\beta}{\alpha}}\right)^{\frac{1}{\gamma-1}}\left[\gamma(-2)^{\frac{-1}{\gamma-1}}-(-2)^{\frac{-\gamma}{\gamma-1}}\right]}{r\gamma+\left(\frac{\beta}{\alpha}\right)\left[2\left(\alpha\mu^\theta+\mu^A\right)+\alpha\sigma^\theta\sigma^A\rho^{\tilde{\theta}\tilde{A}}+\alpha(\alpha-1)\left(\sigma^\theta\right)^2-\left(\frac{\beta-\alpha\gamma}{2\alpha\gamma}\right)\left(\sigma^{Y^L}\right)^2\right]}\right)^{\frac{\gamma-1}{\gamma}}$$

This solution holds under the condition that

$$(30)\qquad \frac{w}{p}\leq\frac{Ce^{-rT}Y^L(T)}{e^{-rT}\left(Y^L(T)\right)^{-\frac{1}{\gamma}}-\left(Y^L(0)\right)^{-\frac{1}{\gamma}}}$$

In such case pesticide usage is given by

(31)

$$u=\left(\frac{2\left(A_{\max}^{-\frac{\beta}{\alpha}}\right)(1+2\gamma)}{\frac{\beta}{\alpha}\left(\frac{\beta-\alpha\gamma}{\alpha}\right)\left(\sigma^{Y^L}\right)^2-r\gamma^2-\left(\frac{\beta}{\alpha}\right)\left[2\left(\alpha\mu^{\theta}+\mu^{A}\right)+\alpha\sigma^{\theta}\sigma^{A}\rho^{\tilde{\theta}\tilde{A}}+\alpha\left(\alpha-1\right)\left(\sigma^{\theta}\right)^2\right]}\right)^{-\frac{1}{\gamma}}\left(Y^L\right)^{-\frac{\beta}{\alpha\gamma}}$$

if the condition in equation (30) does not hold, then the pesticide use will be equal to zero.

See appendix a3 and a4.

Proposition 2: Given $(3\alpha\gamma-\beta)\alpha\beta>0$, *the optimal pesticide use is monotonically increasing in the correlation coefficient, i.e.* $\frac{\partial u}{\partial\rho^{\tilde{\theta}\tilde{A}}}>0$

Given $(3\alpha\gamma-\beta)\alpha\beta<0$, *the optimal pesticide usage is monotonically decreasing in the correlation coefficient, i.e.* $\frac{\partial u}{\partial\rho^{\tilde{\theta}\tilde{A}}}<0$

Proof: See appendix a5.

Given the above we can discuss the cases for optimal pesticide use path to be monotonically increasing or decreasing in the correlation coefficient.

Case 1: $\alpha>0$, $\beta>0$, $3\alpha\gamma-\beta>0$, then $\frac{\partial u}{\partial\rho^{\tilde{\theta}\tilde{A}}}>0$

Case 2: $\alpha<0$, $\beta<0$, $3\alpha\gamma-\beta>0$, then $\frac{\partial u}{\partial\rho^{\tilde{\theta}\tilde{A}}}>0$

Case 3: $\alpha<0$, $\beta>0$, then $\frac{\partial u}{\partial\rho^{\tilde{\theta}\tilde{A}}}>0$

Case 4: $\alpha > 0$, $\beta < 0$, then $\dfrac{\partial u}{\partial \rho^{\tilde{\theta}\tilde{A}}} < 0$

Case 5: $\alpha > 0$, $\beta > 0$, $3\alpha\gamma - \beta < 0$, then $\dfrac{\partial u}{\partial \rho^{\tilde{\theta}\tilde{A}}} < 0$

Case 6: $\alpha < 0$, $\beta < 0$, $3\alpha\gamma - \beta < 0$, then $\dfrac{\partial u}{\partial \rho^{\tilde{\theta}\tilde{A}}} < 0$

Case 7: $\alpha < 0$, $\beta > 0$, $3\alpha\gamma - \beta > 0$, which is impossible.

Case 8: $\alpha > 0$, $\beta < 0$, $3\alpha\gamma - \beta < 0$, which is impossible.

Corrolary: If α and β are opposite in sign, the monotonicity of optimal pesticide use path in the correlation coefficient is unambiguous without any further relative magnitude assumptions.

Proof: The statement is trivially true by looking at the sign of the term $\left(3\alpha\gamma - \beta\right)$.

Conclusion/Discussion

In this paper, we examine stochastic dynamic pest management in agricultural crop production under two stochastic factors that influence agricultural productivity: climate and pest populations. Predictions, or expected values, of climatic variables and pest populations can be used to improve pest management practices. We extend this idea by explicitly showing that the pest management practices can be further improved by taking into account potential correlation between prediction errors for climatic variables and pest population.

We first set up a general discounted cost minimization problem with stochastic climate and pest population variables. We provide necessary condition for optimal pesticide use path and discuss properties of the solution. Choosing functional forms that allow for mathematic tractability we find a closed form solution for pesticide use as a function of the correlation coefficient between pest and climate forecast errors. Moreover, we provide conditions for when pesticide use is monotonically increasing, and when it is decreasing in the correlation coefficient.

The model analytically shows the importance of information of stochastic correlation between climate and pest infestation. For instance, if the true correlation coefficient is negative and growers who don't have the information take the correlation coefficient as zero, then over application or inadequate application of pesticide may occur.

For future research we suggest application of these theoretical analyses in an empirical context. For example, one could use data from Elbakidze, Lu and Eigenbrode (2011) and from Clement and Eigenbrode (2007) to simulate crop and pest outbreak predictions with associated prediction errors. These simulations can be combined with pesticide use data to examine how applications of pesticides like dimatoate can be optimized by taking into account correlation between prediction errors for climate and pest population forecasts.

References

Carpentier, A. and R. D. Weaver. “Damage Control Productivity: Why Econometrics Matters.” *Amer. J. Agr. Econ.* 79(Feb., 1997): 47-61.

Chen, C.C., and B.A. McCarl, "Pesticide Usage as Influenced by Climate: A Statistical Investigation," *Climatic Change*, 50, 475-487, 2001.

Chen, C.C., B.A. McCarl, and D.E. Schimmelpfennig, "Yield Variability as Influenced by Climate: A Statistical Investigation," *Climatic Change*, 66(2), 239-261, 2004.

Cobourn, K.M., H.J.Burrack, R.E.Goodhue,J.C.Williams and F.G.Zalom, Implications of simultaneity in a physical damage function, Journal of Environmental Economics and Management (2011), doi:10.1016/j.jeem.2011.02.002

Cobourn. K., M. “Incentives for Individual and Cooperative Management of a Mobile Pest: An Application to the Olive Fruit Fly in California”. PhD. Dissertation. University of California Davis. Department of Agricultural and Resource Economics, 2009

Costello, C. , R. Adams, and S. Polasky. “The value of El Ni ˜no forecasts in the management of salmon: a stochastic dynamic assessment.” *American Journal of Agricultural Economics*. 80 (1998):765–777

Elbakidze, L., L. Lu, S. Eigenbrode, “Evaluating Vector-Virus-Yield Interactions for Peas and Lentils under Climatic Variability: A Limited Dependent Variable Analysis” Working paper, University of Idaho, 2011

Fox, G. and A. Weersink. “Damage Control and Increasing Returns.” *Amer. J. Agr.*

Econ. 77 (Feb., 1995): 33-39.

Hertzler, G. "Dynamic Decisions under Risk: Application of Ito Stochastic Control in Agriculture." *American Journal of Agricultural Economics*. 73 (1991): 1126-1137

Kim, M-K., and B.A. McCarl, "An Investigation of the Yield and Production Effects of the North Atlantic Oscillation," *Climatic Change*, 71, 117-139, 2004.

Kim, C.S., et al. "Prevention or Control: Optimal Government Policies for Invasive Species Management." *Agricultural and Resource Economics Review*. 35 (2006): 29-40

Lewis, F. L. and Syrmos, V. L. "Optimal control"(1995) Publisher: Wiley

Lichtenberg, E. and D. Zilberman. "The Econometrics of Damage Control: Why Specification Matters." *Amer. J. Agr. Econ*. 68(1986): 261-273.

Marsh, T. L., R. G. Huffaker and G. E. Long. "Optimal Control of Vector-Virus-Plant Interactions: The Case of Potato Leafroll Virus Net Necrosis." *Amer. J. Agr. Econ.* 82(2000):556-569.

Mbah, Martial L. Ndeffo, Graeme A. Forster, Justus H. Wesseler and Christopher A. Gilligan. "Economically optimal timing for crop disease control under uncertainty: an options approach." *J. R. Soc. Interface* 7 (2010): 1421-1428

Mraoua, M. and Bari, D. "Temperature stochastic modelling and weather derivatives pricing: empirical study with moroccan data." *Afrika Statistika* (2007) 2(1): 22-43.

Olson, L. “The Economics of Terrestrial Invasive Species: A Review of the Literature.” *Agricultural and Resource Economics Review*. 35 (2006):178-194.

Olson, L. and S. Roy. “The Economics of Controlling A Stochastic Biological Invasion.” *Amer. J. Agr. Econ.* 84(2002):1311-1316.

Richards, T.J., Eaves, J., Manfredo, M., Naranjo, S.E., Chu, C.C. and Henneberry, Thomas, “Spatial-Temporal Model of Insect Growth, Diffusion and Derivative Pricing”. *American Journal of Agricultural Economics*, Vol. 90(2005): 962-978.

Rubas, Debra J., et al. “How adoption rates, timing, and ceilings affect the value of ENSO-based climate forecasts”. *Climatic Change* (2008) 86:235–256

Saha, A., C. R. Shumway, and A. Havenner. “The Economics and Econometrics of Damage Control.” *Amer. J. Agr. Econ.*79(1997): 773-785.

Saphores, Jean-Daniel M. “The Economic Threshold with a Stochastic Pest Population: A Real Options Approach.” *American Journal of Agricultural Economics*, Vol. 82, No. 3 (Aug., 2000), pp. 541-555

Saphores, Jean-Daniel M. and J. F. Shogren. “Managing exotic pests under uncertainty: optimal control actions and bioeconomic investigations.” *Ecological Economics.* 52 (2005): 327-339

Sunding, D., and J. Zivin. “Insect Population Dynamics, Pesticide Use, and Farmworker Health.” *American Journal of Agricultural Economics* 82(2000): 527–40.

Xepapadeas. Advanced principles in environmental policy.(1997) Publisher: E.Elgar

Zhang, W. and S. M. Swinton. “Incorporating natural enemies in an economic threshold for dynamically optimal pest management.” *Ecological Modelling*. 220 (2009): 1315–1324

Zivin, J., B. M. Hueth, D. Zilberman. “Managing a Multiple-Use Resource: The Case of Feral Pig Management in California Rangeland.” *Journal of Environmental Economics and Management*. 39 (2000): 189-204

Appendix A1

Derivation for (24), Y^L dynamics

the components of 3.1.4 can be derived as follows

$$\frac{\partial Y^L}{\partial t} = \alpha \frac{\partial \theta}{\partial t} \frac{\theta^{\alpha-1} A}{A_{\max}} + \frac{\theta^{\alpha}}{A_{\max}} \frac{\partial A}{\partial t}$$

$$= \alpha \mu^{\theta} \frac{\theta^{\alpha} A}{A_{\max}} + \mu^{A} \frac{\theta^{\alpha} A}{A_{\max}} = \left(\alpha \mu^{\theta} + \mu^{A}\right) Y^L \qquad (A.1.1)$$

$$Y_A^L dA = \theta^{\alpha} \frac{1}{A_{\max}} \left(\mu^A A dt + \sigma^A A d\tilde{A}\right) = \mu^A Y^L dt + \sigma^A Y^L d\tilde{A} \qquad (A.1.2)$$

$$Y_{\theta}^L d\theta = \alpha \frac{\theta^{\alpha-1} A}{A_{\max}} \left(\mu^{\theta} \theta dt + \sigma^{\theta} \theta d\tilde{\theta}\right) = \alpha \mu^{\theta} Y^L dt + \alpha \sigma^{\theta} Y^L d\tilde{\theta} \qquad (A.1.3)$$

$$Y_{A\theta}^L \sigma^{\theta} \sigma^{A} \rho^{\tilde{\theta}\tilde{A}} A \theta dt = \alpha \frac{\sigma^{\theta} \sigma^{A} \rho^{\tilde{\theta}\tilde{A}} A \theta^{\alpha}}{A_{\max}} dt = \alpha \sigma^{\theta} \sigma^{A} \rho^{\tilde{\theta}\tilde{A}} Y^L dt \qquad (A.1.4)$$

$Y_{AA}^L = 0$ and $Y_{\theta\theta}^L \left(\sigma^{\theta} \theta\right)^2 dt = \alpha \left(\alpha - 1\right) A \theta^{\alpha} \frac{1}{A_{\max}} = \alpha \left(\alpha - 1\right) Y^L dt$

Therefore, we get

$$dY^L = \left[2\left(\alpha \mu^{\theta} + \mu^{A}\right) + \alpha \sigma^{\theta} \sigma^{A} \rho^{\tilde{\theta}\tilde{A}} + \alpha \left(\alpha - 1\right)\left(\sigma^{\theta}\right)^2\right] Y^L dt + \sigma^A Y^L d\tilde{A} + \alpha \sigma^{\theta} Y^L d\tilde{\theta}$$

A2 the minimizing u formula (28) and (29)

Given (26) and (27) first order condition for minimizing u is:

$$e^{-rt} w - 2\gamma J_{Y^L} \left(Y^L\right)^{\frac{\alpha+\beta}{\alpha}} u^{\gamma-1} A_{\max}^{\frac{\beta}{\alpha}} = 0 \qquad (A.2.1)$$

Solving A2.1 gives

$$u^{\gamma-1} = \frac{e^{-rt} w}{2\gamma J_{Y^L} \left(Y^L\right)^{\frac{\alpha+\beta}{\alpha}} A_{\max}^{\frac{\beta}{\alpha}}} \qquad (A.2.2)$$

which is equivalent to (28).

Putting (28) into(26), we have:

$$-J_t = e^{-rt}w\left(\frac{e^{-rt}w}{2\gamma J_{Y^L}\left(Y^L\right)^{\frac{\alpha+\beta}{\alpha}}A_{\max}^{\frac{\beta}{\alpha}}}\right)^{\frac{1}{\gamma-1}} + J_Y Y^L\left[\begin{array}{l}2\left(\alpha\mu^{\theta}+\mu^{A}-\left(\frac{e^{-rt}w}{2\gamma J_{Y^L}\left(Y^L\right)^{\frac{\alpha+\beta}{\alpha}}A_{\max}^{\frac{\beta}{\alpha}}}\right)^{\frac{\gamma}{\gamma-1}}A_{\max}^{\frac{\beta}{\alpha}}\left(Y^L\right)^{\frac{\beta}{\alpha}}\right)\\ +\alpha\sigma^{\theta}\sigma^{A}\rho^{\tilde{\theta}\tilde{A}}+\alpha\left(\alpha-1\right)\left(\sigma^{\theta}\right)^2\end{array}\right]+\frac{J_{Y^LY^L}}{2}\left(\sigma^Y Y^L\right)^2 \qquad (A.2.3)$$

Then, we simplify the right hand side expression of (A.2.3)

$$=\left(e^{-rt}w\right)^{\frac{1}{\gamma-1}+\frac{\gamma-1}{\gamma-1}}\left(\frac{1}{2\gamma J_{Y^L}\left(Y^L\right)^{\frac{\alpha+\beta}{\alpha}}A_{\max}^{\frac{\beta}{\alpha}}}\right)^{\frac{1}{\gamma-1}}-\left(\frac{1}{J_{Y^L}\left(Y^L\right)^{\frac{\alpha+\beta}{\alpha}}A_{\max}^{\frac{\beta}{\alpha}}}\right)^{-\frac{\gamma-1}{\gamma-1}}\left(\frac{e^{-rt}w}{2\gamma J_{Y^L}\left(Y^L\right)^{\frac{\alpha+\beta}{\alpha}}A_{\max}^{\frac{\beta}{\alpha}}}\right)^{\frac{\gamma}{\gamma-1}}$$

$$+J_Y Y^L\left[2\left(\alpha\mu^{\theta}+\mu^{A}\right)+\alpha\sigma^{\theta}\sigma^{A}\rho^{\tilde{\theta}\tilde{A}}+\alpha\left(\alpha-1\right)\left(\sigma^{\theta}\right)^2\right]+\frac{J_{YY}}{2}\left(\sigma^Y Y^L\right)^2$$

$$=\left(e^{-rt}w\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{1}{2\gamma J_{Y^L}\left(Y^L\right)^{\frac{\alpha+\beta}{\alpha}}A_{\max}^{\frac{\beta}{\alpha}}}\right)^{\frac{1}{\gamma-1}}-\left(\frac{1}{J_{Y^L}\left(Y^L\right)^{\frac{\alpha+\beta}{\alpha}}A_{\max}^{\frac{\beta}{\alpha}}}\right)^{\frac{1}{\gamma-1}}\left(e^{-rt}w\right)^{\frac{\gamma}{\gamma-1}}\left(2\gamma\right)^{\frac{-\gamma}{\gamma-1}}$$

$$+J_Y Y^L\left[2\left(\alpha\mu^{\theta}+\mu^{A}\right)+\alpha\sigma^{\theta}\sigma^{A}\rho^{\tilde{\theta}\tilde{A}}+\alpha\left(\alpha-1\right)\left(\sigma^{\theta}\right)^2\right]+\frac{J_{Y^LY^L}}{2}\left(\sigma^Y Y^L\right)^2$$

$$=\left(e^{-rt}w\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{1}{J_{Y^L}\left(Y^L\right)^{\frac{\alpha+\beta}{\alpha}}A_{\max}^{\frac{\beta}{\alpha}}}\right)^{\frac{1}{\gamma-1}}\left[\left(2\gamma\right)^{\frac{-1}{\gamma-1}}-\left(2\gamma\right)^{\frac{-\gamma}{\gamma-1}}\right]+J_{Y^L}Y^L\left[2\left(\alpha\mu^{\theta}+\mu^{A}\right)+\alpha\sigma^{\theta}\sigma^{A}\rho^{\tilde{\theta}\tilde{A}}+\alpha\left(\alpha-1\right)\left(\sigma^{\theta}\right)^2\right]+\frac{J_{Y^LY^L}}{2}\left(\sigma^Y Y^L\right)^2$$

which is the same as in (29)

A3 the solution formula (30) and final solution for u

We first conjecture that the value function should have the form:

$$J\left(Y^L,t\right)=Ce^{-rt}\left(Y^L\right)^{\frac{-\beta}{\alpha\gamma}}+C_0 \qquad (A.3.1)$$

Then it follows that

$$J_{Y^L}=-\frac{C\beta}{\alpha\gamma}e^{-rt}\left(Y^L\right)^{\frac{-\beta}{\alpha\gamma}-1} \qquad (A.3.2)$$

$$J_{Y^LY^L}=\frac{C\beta}{\alpha\gamma}\left(\frac{\beta-\alpha\gamma}{\alpha\gamma}\right)e^{-rt}\left(Y^L\right)^{\frac{-\beta}{\alpha\gamma}-2} \qquad (A.3.3)$$

$$J_t = -rCe^{-rt}\left(Y^L\right)^{\frac{-\beta}{\alpha\gamma}} \qquad (A.3.4)$$

Putting equations (A.3.2) to (A.3.4) into (29), we have:

$$-J_t = rCe^{-rt}\left(Y^L\right)^{\frac{-\beta}{\alpha\gamma}}$$

Putting into the first term on the right hand side of 29 gives:

$$\left(e^{-rt}w\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{1}{-\frac{C\beta}{\alpha\gamma}e^{-rt}\left(Y^L\right)^{\frac{-\beta}{\alpha\gamma}-1}\left(Y^L\right)^{\frac{\alpha+\beta}{\alpha}}A_{\max}^{\frac{\beta}{\alpha}}}\right)^{\frac{1}{\gamma-1}}\left[(2\gamma)^{\frac{-1}{\gamma-1}}-(2\gamma)^{\frac{-\gamma}{\gamma-1}}\right]$$

$$=\left(e^{-rt}w\right)^{\frac{\gamma}{\gamma-1}}\left(e^{-rt}\right)^{\frac{-1}{\gamma-1}}\left(\frac{1}{A_{\max}^{\frac{\beta}{\alpha}}\frac{C\beta}{\alpha}}\right)^{\frac{1}{\gamma-1}}\left[\left(Y^L\right)^{-\frac{\beta(\gamma-1)}{\alpha\gamma}}\right]^{\frac{1}{\gamma-1}}(-\gamma)^{\frac{1}{\gamma-1}}\left[(2\gamma)^{\frac{-1}{\gamma-1}}-(2\gamma)^{\frac{-\gamma}{\gamma-1}}\right]$$

$$=e^{-rt}\left(Y^L\right)^{\frac{-\beta}{\alpha\gamma}}(w)^{\frac{\gamma}{\gamma-1}}\left(\frac{1}{A_{\max}^{\frac{\beta}{\alpha}}\frac{C\beta}{\alpha}}\right)^{\frac{1}{\gamma-1}}\left[(-2)^{\frac{-1}{\gamma-1}}-\frac{1}{\gamma}(-2)^{\frac{-\gamma}{\gamma-1}}\right]$$

The second term on the right hand side becomes:

$$J_{Y^L}Y^L\left[2\left(\alpha\mu^\theta+\mu^A\right)+\alpha\sigma^\theta\sigma^A\rho^{\tilde{\theta}\tilde{A}}+\alpha(\alpha-1)\left(\sigma^\theta\right)^2\right]$$

$$=-\frac{C\beta}{\alpha\gamma}e^{-rt}\left(Y^L\right)^{\frac{-\beta}{\alpha\gamma}-1}Y^L\left[2\left(\alpha\mu^\theta+\mu^A\right)+\alpha\sigma^\theta\sigma^A\rho^{\tilde{\theta}\tilde{A}}+\alpha(\alpha-1)\left(\sigma^\theta\right)^2\right]$$

$$=e^{-rt}\left(Y^L\right)^{\frac{-\beta}{\alpha\gamma}}\left(-\frac{C\beta}{\alpha\gamma}\right)\left[2\left(\alpha\mu^\theta+\mu^A\right)+\alpha\sigma^\theta\sigma^A\rho^{\tilde{\theta}\tilde{A}}+\alpha(\alpha-1)\left(\sigma^\theta\right)^2\right]$$

The third term on the right hand side becomes:

$$\frac{J_{Y^LY^L}}{2}\left(\sigma^{Y^L}Y^L\right)^2$$

$$=\frac{C\beta}{2\alpha\gamma}\left(\frac{\beta-\alpha\gamma}{\alpha\gamma}\right)e^{-rt}\left(Y^L\right)^{\frac{-\beta}{\alpha\gamma}}\left(\sigma^Y\right)^2$$

Then, the equation (29) becomes:

$$rCe^{-rt}\left(Y^{L}\right)^{\frac{-\beta}{\alpha\gamma}}=e^{-rt}\left(Y^{L}\right)^{\frac{-\beta}{\alpha\gamma}}\left(w\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{1}{A_{\max}^{\frac{\beta}{\alpha}}\frac{C\beta}{\alpha}}\right)^{\frac{1}{\gamma-1}}\left[\left(-2\right)^{\frac{-1}{\gamma-1}}-\frac{1}{\gamma}\left(-2\right)^{\frac{-\gamma}{\gamma-1}}\right]+$$
$$e^{-rt}\left(Y^{L}\right)^{\frac{-\beta}{\alpha\gamma}}\left(-\frac{C\beta}{\alpha\gamma}\right)\left[2\left(\alpha\mu^{\theta}+\mu^{A}\right)+\alpha\sigma^{\theta}\sigma^{A}\rho^{\tilde{\theta}\tilde{A}}+\alpha\left(\alpha-1\right)\left(\sigma^{\theta}\right)^{2}\right]+$$
$$\frac{C\beta}{2\alpha\gamma}\left(\frac{\beta-\alpha\gamma}{\alpha\gamma}\right)e^{-rt}\left(Y^{L}\right)^{\frac{-\beta}{\alpha\gamma}}\left(\sigma^{Y^{L}}\right)^{2} \qquad (A.3.5)$$

Dividing $Ce^{-rt}\left(Y^{L}\right)^{\frac{-\beta}{\alpha\gamma}}$ on both sides, (A.3.5) becomes:

$$r=\left(w\right)^{\frac{\gamma}{\gamma-1}}\left(A_{\max}^{-\frac{\beta}{\alpha}}\right)^{\frac{1}{\gamma-1}}\left(\frac{C\beta}{\alpha}\right)^{-\frac{\gamma}{\gamma-1}}\left[\left(-2\right)^{\frac{-1}{\gamma-1}}-\frac{1}{\gamma}\left(-2\right)^{\frac{-\gamma}{\gamma-1}}\right]+$$
$$\left(-\frac{\beta}{\alpha\gamma}\right)\left[2\left(\alpha\mu^{\theta}+\mu^{A}\right)+\alpha\sigma^{\theta}\sigma^{A}\rho^{\tilde{\theta}\tilde{A}}+\alpha\left(\alpha-1\right)\left(\sigma^{\theta}\right)^{2}\right]+\frac{\beta}{2\alpha\gamma}\left(\frac{\beta-\alpha\gamma}{\alpha\gamma}\right)\left(\sigma^{Y}\right)^{2} \quad (A.3.6)$$

Now, in order to solve for C, we rewrite (A.3.6) as:

$$\left(w\right)^{\frac{\gamma}{\gamma-1}}\left(A_{\max}^{-\frac{\beta}{\alpha}}\right)^{\frac{1}{\gamma-1}}\left(\frac{C\beta}{\alpha}\right)^{-\frac{\gamma}{\gamma-1}}\left[\left(-2\right)^{\frac{-1}{\gamma-1}}-\frac{1}{\gamma}\left(-2\right)^{\frac{-\gamma}{\gamma-1}}\right]=r+\left(\frac{\beta}{\alpha\gamma}\right)\left[2\left(\alpha\mu^{\theta}+\mu^{A}\right)+\alpha\sigma^{\theta}\sigma^{A}\rho^{\tilde{\theta}\tilde{A}}+\alpha\left(\alpha-1\right)\left(\sigma^{\theta}\right)^{2}\right]$$
$$-\frac{\beta}{2\alpha\gamma}\left(\frac{\beta-\alpha\gamma}{\alpha\gamma}\right)\left(\sigma^{Y}\right)^{2}$$

$$C^{\frac{\gamma}{\gamma-1}}=\frac{\left(w\right)^{\frac{\gamma}{\gamma-1}}\left(A_{\max}^{-\frac{\beta}{\alpha}}\right)^{\frac{1}{\gamma-1}}\left[\gamma\left(-2\right)^{\frac{-1}{\gamma-1}}-\left(-2\right)^{\frac{-\gamma}{\gamma-1}}\right]}{r\gamma\left(\frac{\beta}{\alpha}\right)^{\frac{\gamma}{\gamma-1}}+\left(\frac{\beta}{\alpha}\right)^{\frac{\gamma}{\gamma-1}+1}\left[2\left(\alpha\mu^{\theta}+\mu^{A}\right)+\alpha\sigma^{\theta}\sigma^{A}\rho^{\tilde{\theta}\tilde{A}}+\alpha\left(\alpha-1\right)\left(\sigma^{\theta}\right)^{2}-\left(\frac{\beta-\alpha\gamma}{2\alpha\gamma}\right)\left(\sigma^{Y}\right)^{2}\right]}$$

Which is the desired constant.

Notice that the value function has to satisfy the end point condition:

$J\left(Y^L\left(T\right),T\right)=e^{-rT}pY^L\left(T\right)$ and

$$J\left(Y^L\left(T\right),T\right)=Ce^{-rT}\left(Y^L\left(T\right)\right)^{\frac{-\beta}{\alpha\gamma}}+C_0$$

Then $C_0=e^{-rT}pY^L\left(T\right)-Ce^{-rT}\left(Y^L\left(T\right)\right)^{\frac{-\beta}{\alpha\gamma}}$

In sum, $J\left(Y^L,t\right)=Ce^{-rt}\left(Y^L\right)^{\frac{-\beta}{\alpha\gamma}}+e^{-rT}pY^L\left(T\right)-Ce^{-rT}\left(Y^L\left(T\right)\right)^{\frac{-\beta}{\alpha\gamma}}$

Moreover, J has to be nonnegative:

$$J\left(Y^L\left(0\right),0\right)=C\left(Y^L\left(0\right)\right)^{\frac{-\beta}{\alpha\gamma}}+e^{-rT}pY^L\left(T\right)-Ce^{-rT}\left(Y^L\left(T\right)\right)^{\frac{-\beta}{\alpha\gamma}}\geq 0$$

Then, the parameter C has to satisfy:

$$Ce^{-rT}\left(Y^L\left(T\right)\right)^{\frac{-\beta}{\alpha\gamma}}-C\left(Y^L\left(0\right)\right)^{\frac{-\beta}{\alpha\gamma}}\leq e^{-rT}pY^L\left(T\right)$$

$$C\left[e^{-rT}\left(Y^L\left(T\right)\right)^{\frac{-\beta}{\alpha\gamma}}-\left(Y^L\left(0\right)\right)^{\frac{-\beta}{\alpha\gamma}}\right]\leq e^{-rT}pY^L\left(T\right)$$

Then $C\leq\dfrac{e^{-rT}pY^L\left(T\right)}{e^{-rT}\left(Y^L\left(T\right)\right)^{\frac{-\beta}{\alpha\gamma}}-\left(Y^L\left(0\right)\right)^{\frac{-\beta}{\alpha\gamma}}}$

Since

$$C=w\left(\frac{\alpha}{\beta}\right)\left(\frac{\left(A_{\max}^{-\frac{\beta}{\alpha}}\right)^{\frac{1}{\gamma-1}}\left[\gamma\left(-2\right)^{\frac{-1}{\gamma-1}}-\left(-2\right)^{\frac{-\gamma}{\gamma-1}}\right]}{r\gamma+\left(\frac{\beta}{\alpha}\right)\left[2\left(\alpha\mu^\theta+\mu^A\right)+\alpha\sigma^\theta\sigma^A\rho^{\tilde{\theta}\tilde{A}}+\alpha\left(\alpha-1\right)\left(\sigma^\theta\right)^2-\left(\frac{\beta-\alpha\gamma}{2\alpha\gamma}\right)\left(\sigma^{Y^L}\right)^2\right]}\right)^{\frac{\gamma-1}{\gamma}}$$

The condition can be stated as

$$\frac{w}{p} \le \frac{e^{-rT} Y^L(T)}{e^{-rT}\left(Y^L(T)\right)^{-\frac{1}{\gamma}} - \left(Y^L(0)\right)^{-\frac{1}{\gamma}}}$$

$$\times\left(\frac{\alpha}{\beta}\right)\left(\frac{\left(A_{\max}^{-\frac{\beta}{\alpha}}\right)^{\frac{1}{\gamma-1}}\left[\gamma(-2)^{\frac{-1}{\gamma-1}} - (-2)^{\frac{-\gamma}{\gamma-1}}\right]}{r\gamma + \left(\frac{\beta}{\alpha}\right)\left[2\left(\alpha\mu^\theta + \mu^A\right) + \alpha\sigma^\theta\sigma^A\rho^{\tilde{\theta}\tilde{A}} + \alpha(\alpha-1)\left(\sigma^\theta\right)^2 - \left(\frac{\beta-\alpha\gamma}{2\alpha\gamma}\right)\left(\sigma^{Y^L}\right)^2\right]}\right)^{\frac{\gamma-1}{\gamma}}$$

Which can be interpreted as the factor cost cannot be too much or the output price cannot be too low.

When the condition does not hold,

$$J\left(Y^L, t\right) = C_0$$

will be the solution to the PDE.

By (A.2.2)

$$u^{\gamma-1} = \frac{e^{-rt} w}{2\gamma J_{Y^L}\left(Y^L\right)^{\frac{\alpha+\beta}{\alpha}} A_{\max}^{\frac{\beta}{\alpha}}}$$

Then

$$u^{\gamma-1} = \frac{e^{-rt} w}{-2\gamma A_{\max}^{\frac{\beta}{\alpha}} \frac{C\beta}{\alpha\gamma} e^{-rt}\left(Y^L\right)^{\frac{\beta(\gamma-1)}{\alpha\gamma}}}$$

$$= \left(\frac{2\left(A_{\max}^{-\frac{\beta}{\alpha}}\right)(1+2\gamma)}{r\gamma + \left(\frac{\beta}{\alpha}\right)\left[2\left(\alpha\mu^\theta + \mu^A\right) + \alpha\sigma^\theta\sigma^A\rho^{\tilde{\theta}\tilde{A}} + \alpha(\alpha-1)\left(\sigma^\theta\right)^2 - \left(\frac{\beta-\alpha\gamma}{2\alpha\gamma}\right)\left(\sigma^{Y^L}\right)^2\right]}\right)^{-\frac{1}{\gamma}} \left(Y^L\right)^{-\frac{\beta}{\alpha\gamma}} \quad (A.3.7)$$

A5 Proof for proposition 2

$$u=\left(\frac{2\left(A_{\max}^{-\frac{\beta}{\alpha}}\right)(1+2\gamma)}{r\gamma+\left(\frac{\beta}{\alpha}\right)\left[2\left(\alpha\mu^{\theta}+\mu^{A}\right)+\alpha\sigma^{\theta}\sigma^{A}\rho^{\tilde{\theta}\tilde{A}}+\alpha(\alpha-1)\left(\sigma^{\theta}\right)^{2}-\left(\frac{\beta-\alpha\gamma}{2\alpha\gamma}\right)\left(\sigma^{Y^{L}}\right)^{2}\right]}\right)^{-\frac{1}{\gamma}}\left(Y^{L}\right)^{-\frac{\beta}{\alpha\gamma}}$$

$$\frac{\partial u}{\partial\rho^{\tilde{\theta}\tilde{A}}}=-\frac{1}{\gamma}\left(\frac{2\left(A_{\max}^{-\frac{\beta}{\alpha}}\right)(1+2\gamma)}{r\gamma+\left(\frac{\beta}{\alpha}\right)\left[2\left(\alpha\mu^{\theta}+\mu^{A}\right)+\alpha\sigma^{\theta}\sigma^{A}\rho^{\tilde{\theta}\tilde{A}}+\alpha(\alpha-1)\left(\sigma^{\theta}\right)^{2}-\left(\frac{\beta-\alpha\gamma}{2\alpha\gamma}\right)\left(\sigma^{Y}\right)^{2}\right]}\right)^{-\frac{1}{\gamma}-1}\left(Y^{L}\right)^{-\frac{\beta}{\alpha\gamma}}\times$$

$$\frac{\frac{\beta}{\alpha}\left(2\alpha\sigma^{\theta}\sigma^{A}-\left(\frac{\beta-\alpha\gamma}{2\alpha\gamma}\right)2\alpha\sigma^{\theta}\sigma^{A}\right)}{\left\{\frac{\beta}{\alpha}\left(\frac{\beta-\alpha\gamma}{\alpha}\right)\left(\sigma^{Y^{L}}\right)^{2}-r\gamma^{2}-\left(\frac{\beta\gamma}{\alpha}\right)\left[2\left(\alpha\mu^{\theta}+\mu^{A}\right)+\alpha\sigma^{\theta}\sigma^{A}\rho^{\tilde{\theta}\tilde{A}}+\alpha(\alpha-1)\left(\sigma^{\theta}\right)^{2}\right]\right\}^{2}}$$

$$=-\frac{1}{\gamma}\left(\frac{\beta}{\alpha}\frac{(3\alpha\gamma-\beta)}{\gamma}\sigma^{\theta}\sigma^{A}\right)\times$$

$$\frac{\left[2(1+2\gamma)\right]^{-\frac{1}{\gamma}-1}}{\left\{\frac{\beta}{\alpha}\left(\frac{\beta-\alpha\gamma}{\alpha}\right)\left(\sigma^{Y^{L}}\right)^{2}-r\gamma^{2}-\left(\frac{\beta\gamma}{\alpha}\right)\left[2\left(\alpha\mu^{\theta}+\mu^{A}\right)+\alpha\sigma^{\theta}\sigma^{A}\rho^{\tilde{\theta}\tilde{A}}+\alpha(\alpha-1)\left(\sigma^{\theta}\right)^{2}\right]\right\}^{-\frac{1}{\gamma}+1}}\left(Y^{L}\right)^{-\frac{\beta}{\alpha\gamma}}$$

$$=-\frac{\beta(3\alpha\gamma-\beta)\sigma^{\theta}\sigma^{A}}{\alpha\gamma^{2}}\times$$

$$\frac{\left[2(1+2\gamma)\right]^{-\frac{1}{\gamma}-1}}{\left\{\frac{\beta}{\alpha}\left(\frac{\beta-\alpha\gamma}{\alpha}\right)\left(\sigma^{Y^{L}}\right)^{2}-r\gamma^{2}-\left(\frac{\beta\gamma}{\alpha}\right)\left[2\left(\alpha\mu^{\theta}+\mu^{A}\right)+\alpha\sigma^{\theta}\sigma^{A}\rho^{\tilde{\theta}\tilde{A}}+\alpha(\alpha-1)\left(\sigma^{\theta}\right)^{2}\right]\right\}^{-\frac{1}{\gamma}+1}}\left(Y^{L}\right)^{-\frac{\beta}{\alpha\gamma}}$$