Measuring Technical Efficiency of Dairy Farms with Imprecise Data: A Fuzzy Data Envelopment Analysis Approach

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This article integrates fuzzy set theory in Data Envelopment Analysis (DEA) framework to compute technical efficiency scores when input and output data are imprecise. The underlying assumption in convectional DEA is that inputs and outputs data are measured with precision. However, production agriculture takes place in an uncertain environment and, in some situations, input and output data may be imprecise. We present an approach of measuring efficiency when data is known to lie within specified intervals and empirically illustrate this approach using a group of 34 dairy producers in Pennsylvania. Compared to the convectional DEA scores that are point estimates, the computed fuzzy efficiency scores allow the decision maker to trace the performance of a decision-making unit at different possibility levels.

Key words: fuzzy set theory, Data Envelopment Analysis, membership function, α-cut level, technical efficiency

JEL Classification: D24, Q12, C02, C44, C61

I. Introduction

The Data Envelopment Analysis (DEA) approach has been extensively applied in agriculture to measure the productive efficiency of production entities. Charnes et al. (1978) developed the DEA methodology for measuring relative efficiencies within a group of decision-making units (DMU's) which utilize several inputs to produce a set of outputs. DEA constructs a nonparametric frontier over data points so that all observations lie on or below the frontier. A competing method for computing technical efficiency scores is the stochastic frontier approach (SFA) developed by Aigner *et al*. (1977) and Meeusen and van den Broeck (1977).

DEA approach has been favored over the SFA for several seasons. First, it requires no assumption about the distribution of the underlying data and deviation from the estimated frontier is interpreted purely as inefficiency. Second, it does not require specification of a functional form for the frontier just as economic theory does not imply a particular functional form. Third, multiple inputs and outputs can be considered simultaneously, and fourth, inputs and outputs can be quantified using different units of measurement.

However, DEA requires detailed data about inputs and outputs. It is based on the assumption that all the input and output data are crisp, i.e., all the observations are considered as feasible with probability one, meaning no noise or measurement error is assumed (Simar 2007, Henderson and Zelenyuk 2007). This assumption may not be realistic in production agriculture where inputs and outputs of a decision making unit (DMU) are ever changing because of weather, seasons, operating states and so on (Guo and Tanaka 2001). The dominance of uncertainty in agricultural production has seen the flourish of studies of production under risk in agricultural economics (Just and Pope 2001). Factors used in production agriculture, such as labor, are sometimes difficult to measure in a precise manner. Input measures are often based on accounting data even though the definition of accounting costs differs from that of economic costs by excluding the opportunity cost (Kuosmanen *et al*., 2007). Producer data may also be available only in linguistic form such as “high yield”, “low yield”, “labor intensive” or “capital intensive.” The convectional DEA[1] approach is very sensitive to data measurement errors

[1] Here we refer to the Charnes, Copper and Rhodes (CCR) model that assume constant return to scale (Charnes *et al*., 1978). The concept presented can equally be extended to

and changes in data, including outliers and missing data, can change the efficient frontier significantly. The DEA model is deterministic in nature, meaning that it does not account for statistical noise.

A number of techniques to account for the deterministic nature have been suggested in the literature, such as the techniques for detecting possible outliers (Cazals *et al.*, 2002) and the stochastic programming approach (Cooper *et al.,* 1998). Notably, Simar and Wilson (1998, 2000a) introduced bootstrapping into the DEA framework to allow for consistent estimation of the production frontier, corresponding efficiency scores, as well as standard errors and confidence intervals. However, as observed by Kousmanen *et al.*, (2007), the statistical properties and hypothesis tests suggested by Simar and Wilson (2000a, 2000b) focus exclusively on the effect of the sampling of firms from the production possibilities set and, hence, the bootstrap approach does not allow for data errors of any kind. Therefore, there is need for a model that can adequately represent the stochastic nature of production data at a micro-level.

This paper introduces fuzzy DEA, an approach advanced in the field of industrial engineering, to measure technical efficiency where data is imprecise. A group of 34 dairy producers in Pennsylvania is used to illustrate how to empirically compute fuzzy technical efficiency scores. The approach incorporates fuzzy set theory and the DEA mathematical programming techniques to compute technical efficiency indices under natural uncertainty inherent in the production processes. Unlike the convectional DEA model, with a fuzzy DEA model the decision maker can consider different degrees of measurement errors (possibilities) when estimating technical efficiency. Expert judgment expressed in

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the Banker, Charnes, and Cooper (BCC) model that assumes variable return to scale (Banker *et al.,* 1984).

linguistic variables can also be incorporated into the fuzzy DEA models (Guo and Tanaka, 2001).

Fuzzy DEA models are rare in the economics or agricultural economics literature. A search for "fuzzy DEA" in the AGRICOLA, AgEcon Search, and EconLit databases returned no items. The only recent application of fuzzy DEA in agriculture is by Hadi-Vencheh and Matin (2011) who compute efficiency scores for wheat provinces in Iran. Other applications of fuzzy set theory in agricultural economics include van Kooten et al (2001) who proposed a fuzzy contingent valuation approach to measure uncertain preferences for non-market goods. Duval and Featherstone (2002) compared compromise programming and fuzzy programming to a traditional mean-variance approach, and Krcmar and Van Kooten (2008) developed a compromise-fuzzy programming framework to analyze trade-offs of economic development prospects of forest dependent aboriginal communities.

Analysis of technical efficiency using fuzzy DEA models is very useful to the decision maker and presents several advantages. First, uncertainty in measurement can be incorporated in DEA model at different degrees. Second, linguistic variables can be incorporated into the DEA model, e.g., expert judgment and environmental variables. Third, fuzzy DEA can be used to deal with missing data, and fourth, the decision maker can trace how the efficiency scores vary at different levels of uncertainty.

In what follows, the convectional DEA model is presented followed by the basic concepts of fuzzy set theory and how those concepts are integrated into the DEA framework. Then, a literature review of numerical and empirical fuzzy DEA models is presented. The data set is discussed next followed by an application of the fuzzy DEA model to that data and discussion of the results. Then, the article concludes.

2. Methodology

Convectional DEA Model

Data Envelopment Analysis (DEA) is a non-parametric methodology for measuring efficiency within a group of decision-making units (DMUs) that utilize several inputs to produce a set of outputs. DEA models provide efficiency scores that assess the performance of different DMUs in terms of either the use of several inputs or the production of certain outputs. The input-oriented DEA scores vary in (0, 1], the unity value indicating the technically efficient units (Leon *et al.,* 2003). The assumption underlying DEA is that all data assume specific numerical values.

Consider n decision-making units, DMU_j, where j =1… n. Each DMU consumes input levels x_{ij}, i = 1… m, to produce outputs levels y_{rj}, r = 1… s. Suppose that $x_{ij} = [x_{ij} ..., x_{mj}]^T$ and $y_{rj} = [y_{rj} ..., y_{sj}]^T$ are the vectors of inputs and outputs values for DMU_j, where $x_j \geq 0$ and $y_j \geq 0$. The relative efficiency score of the $\text{DMU}_o, o \in \{1,...,n\}$, is obtained from the following input-oriented DEA model that aims at reducing the input amounts by as much as possible while keeping at least the present output levels:

$$Min\ Z = \theta \text{ subject to}: \theta x_{io} \geq \sum_{j=1}^{n} \lambda_j x_{ij}, \forall i,; y_{ro} \leq \sum_{j=1}^{n} \lambda_j y_{rj}, \forall r,; \lambda_j \geq 0 \qquad \textbf{(1)}$$

where λ indicates the intensity levels which make the activity of each DMU expand or contract to construct a piecewise linear technology (Färe et al. 1994). The DMU_o is technically efficient if and only if θ =1, otherwise the DMU_o is inefficient. There is an extensive literature on classical DEA models. Cooper *et al.* (2007) provides a comprehensive review of some of the accomplishments and future prospects of DEA. A major drawback of the DEA model is that the computed relative efficiency scores are very sensitive to noise in data. Any outlier or missing value in the data may cause the efficiency measure of most DMUs to change drastically (Kao and Liu, 2000a; Kao and

Liu, 2000b). This makes an approach that is able to deal with inexact numbers, numbers in range or vague measures desirable. Fuzzy set theory can be incorporated in the DEA framework to deal with imprecise data in both the objective function and constraints.

Fuzzy Set Theory

Optimization techniques often used in economics are 'crisp' in that a clear distinction is made in a two-valued way between feasible and infeasible, and between optimal and nonoptimal solutions (Zimmerman, 1994). The techniques do not allow for gradual transition between these categories, a limitation often referred to as the problem of artificial precision in formalized systems (Geyer-Schulz, 1997). Bellman and Zadeh (1970) were the first to suggest modeling goals and/or constraints in optimization problems as fuzzy sets to account for uncertainty and fuzziness of the decision-making environment.

Fuzzy set theory is a generalization of classical set theory in that the domain of the characteristics function is extended from the discrete set {0, 1} to the closed real interval [0, 1]. Zadeh (1965) defined a fuzzy set as a class of objects with continuum grades of membership. Suppose X is a space of objects and x is a generic element of X. A fuzzy set, $\tilde{A}$, in X can be defined as the set of ordered pairs:

$$\tilde{A} = \{(x, u_A(x)) \mid x \in X\}, \qquad \textbf{(2)}$$

where $u_A(x)$: $X \rightarrow M$ is the membership function and M is the membership space that varies in the interval [0, 1]. The closer the value of $u_A(x)$ is to one, the greater the membership degree of X to $\tilde{A}$. However, if $M = \{0, 1\}$, the set A is non-fuzzy[2] (Triantis and Girod, 1998). A fuzzy set $\tilde{A}$ can be defined precisely by associating with each object x a number

[2] This rule outs degree of belongingness. It implies that x belong to the set 100% (1) or is not a member of the set (0).

between 0 and 1, which represents its grade of membership in A. Thus, $u_A(x) = 1$ if x is totally in A, $u_A(x) = 0$ if x is not in A, and $0 < u_A(x) < 1$ if x is partly in A.

Fuzzy set theory[3] is based on several topological concepts that are beyond the scope of this paper. The interested readers are referred to Kaufmann and Gupta (1991) and Zimmerman (1994) for an introduction to fuzzy sets theory and fuzzy mathematical models. However, terms like *fuzzy sets*, *membership functions* and *fuzzy numbers* will be used several times but no real knowledge of the theory of fuzzy sets is required. Basic concepts relevant to understand this paper are defined:

1. A set in convectional set theory, A, such as a set of large dairy farms (x) that produce at least 1000 litres of milk per day is represented as $A = \{x \mid milk(x) \geq 1000\}$. A universal set, U, is the set from which all elements are drawn, for example, all dairy farms. The convectional set is defined such that the elements in a universe are divided into two groups: members (those that do belong to it) and non-members (those that do not belong).
2. A fuzzy set, drawn from U, allows its elements to belong to A at various degrees, with '1' implying a full belongingness and '0' implying no belongingness. For example, from $U = \{x_1 = 500, x_2 = 900, x_3 = 1200\}$, we can have a crisp set $A = \{x_3 = 1200\}$ and fuzzy set $\tilde{A} = \{x_1 = 500/0.5, x_2 = 900/0.9, x_3 = 1200/1\}$. The values 0.5, 0.9 and 1 are membership functions, $u_A(x)$, and represent the grade of membership of x_1, x_2, and x_3 to the set $A = \{x \mid milk(x) \geq 1000\}$. The term "large dairy farms" here is vague and vary depending with the perception of an individual. Therefore, farms x_1 and x_2 can be considered large farms too but with degrees of membership 0.5 and 0.9.

[3] Fuzzy set theory focuses on how to deal with imprecision or inexactness analytically. The imprecision here is non-statistical or non-probabilistic (Levine, 1997).

3. A fuzzy number is a quantity whose value is imprecise, rather than exact as is the case with single-valued numbers. Generally, a fuzzy number is a fuzzy subset of a real number, $\mathbb{R}$, which is both *normal* and *convex* where normal implies that the maximum value of the fuzzy set in $\mathbb{R}$ is 1. It has a peak or plateau with membership grade 1, over which the members of the universe are completely in the set. The membership function is increasing towards the peak and decreasing away from it. Fuzzy numbers can be represented as linear, triangular, trapezoidal, or Gaussian.

4. A triangular fuzzy number, $\tilde{A}$, is a number with piecewise linear membership functions $u_{\tilde{A}}(x)$ defined by:

$$u_{\tilde{A}}(x) = \begin{cases} 0, x < \pi^l \\ \dfrac{x-\pi^l}{\pi^m - \pi^l}, \pi^l \le x \le \pi^m, \\ \dfrac{\pi^m - x}{\pi^u - \pi^m}, \pi^m \le x \le \pi^u, \\ 0, x > \pi^u \end{cases} \qquad \textbf{(3)}$$

This can be denoted as a triplet $\left(\pi^m, \pi^l, \pi^u\right)$ where π^m, π^l, π^u are the centre, left spread, and right spread of the number. Figure 1 illustrates an example of a triangular fuzzy number. Letting $\tilde{A}$ and $\tilde{B}$ to be two triangular fuzzy numbers denoted by $\left(a_l, a_m, a_u\right)$ and $\left(b_l, b_m, b_u\right)$, it follows that $\tilde{A} \le \tilde{B}$ if and only if $a_l \le b_l$, $a_m \le b_m$, and $a_u \le b_u$.

< Insert figure 1>

5. The α-cut level of a fuzzy set is a crisp subset of *X* that contains all the elements of *X* whose membership grades are greater than or equal to the specified value of α. This is denoted by $\tilde{A}_\alpha = \{(x, u_{\tilde{A}}(x)) \ge \alpha \mid x \in X\}$. Each α-cut level of a fuzzy number is a closed interval which can be represented as $\left[L(\alpha), U(\alpha)\right]$, where $L(\alpha)$ is lower bound and $U(\alpha)$ is upper bound at a defined α-cut level, α. A family of α-cut levels determines a fuzzy number.

6. Therefore, the interval of confidence at a given α-cut level, where L is lower bound and U is upper bound, can be characterized as :

   $\forall \alpha \in [0:1], A_\alpha = \left[L = \alpha(\pi^m - \pi^l) + \pi^l, U = \pi^u - \alpha(\pi^u - \pi^m) \right]$.

Fuzzy DEA with Triangular Membership Functions

Consider the convectional DEA model, equation 1, except that the inputs and outputs are fuzzy where, '~', indicates fuzziness. Suppose the input and output are triangular fuzzy numbers represented by $\tilde{x}_{ij} = (x_{ij}^l, x_{ij}^m, x_{ij}^u)$ and $\tilde{y}_{rj} = (y_{rj}^l, y_{rj}^m, y_{rj}^u)$. Kao and Liu (2000a) developed a method to find the membership function of the efficiency scores when the observations are fuzzy numbers based on the idea of the α-cut level and Zadeh's extension principle[4]. The main idea is to transform the levels of inputs and outputs such that the data lie within bounded intervals, i.e. $\tilde{x}_{ij} \in [x_{ij}^L, x_{ij}^U]$ and $\tilde{y}_{rj} \in [y_{rj}^L, y_{rj}^U]$ where L and U represent the lower and upper bounds, respectively. Therefore, equation 1 can be reformulated, taking into consideration the fuzzy data, as:

$$Min\ Z = \tilde{\theta}\ s.t: \tilde{\theta}\tilde{x}_{io} \geq \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij}, \forall i,; \tilde{y}_{ro} \leq \sum_{j=1}^{n} \lambda_j \tilde{y}_{rj}, \forall r,; \lambda_j \geq 0 \qquad \textbf{(4)}$$

The above model can be expanded to indicate the center, lower, and upper bound values as follows:

[4] The extension principle states that the classical results of Boolean logic are recovered from fuzzy logic operations when all fuzzy membership grades are restricted to the classical set {0, 1}.

$$
\begin{aligned}
& Min\ Z = \tilde{\theta}\ s.t: \\
& (\tilde{\theta}x_{io}^{m}, \tilde{\theta}x_{io}^{l}, \tilde{\theta}x_{io}^{u}) \geq \left(\sum_{j=1}^{n} \lambda_j x_{ij}^{m}, \sum_{j=1}^{n} \lambda_j x_{ij}^{l}, \sum_{j=1}^{n} \lambda_j x_{ij}^{u} \right) \forall i, \\
& (y_{ro}^{m}, y_{ro}^{l}, y_{ro}^{u}) \leq \left(\sum_{j=1}^{n} \lambda_j y_{rj}^{m}, \sum_{j=1}^{n} \lambda_j y_{rj}^{l}, \sum_{j=1}^{n} \lambda_j y_{rj}^{u} \right) \forall r, \\
& \lambda_j \geq 0
\end{aligned}
\qquad \textbf{(5)}
$$

This model is fuzzy and the usual linear programming method cannot solve it without being defuzzified. The α-cut level and extension principle is used to defuzzify the model by transforming the fuzzy triangular numbers to 'crisp' intervals that are solvable as a series of conventional DEA models as follows:

$$
\begin{aligned}
& Min\ Z = \theta \text{ subject to:} \\
& [\theta(\alpha x_{io}^{m} + (1-\alpha)x_{io}^{l}), \theta(\alpha x_{io}^{m} + (1-\alpha)x_{io}^{u})] \geq \\
& \left[\sum_{j=1}^{n} \lambda_j (\alpha x_{ij}^{m} + (1-\alpha)x_{ij}^{l}), \sum_{j=1}^{n} \lambda_j (\alpha x_{ij}^{m} + (1-\alpha)x_{ij}^{u}) \right] \forall i, \\
& [\theta(\alpha y_{ro}^{m} + (1-\alpha)y_{ro}^{l}), \theta(\alpha y_{ro}^{m} + (1-\alpha)y_{ro}^{u})] \leq \\
& \left[\sum_{j=1}^{n} \lambda_j (\alpha y_{rj}^{m} + (1-\alpha)y_{rj}^{l}), \sum_{j=1}^{n} \lambda_j (\alpha y_{rj}^{m} + (1-\alpha)y_{rj}^{u}) \right] \forall i, \\
& \lambda_j \geq 0
\end{aligned}
\qquad \textbf{(6)}
$$

The model is solved by means of comparing the left hand side (LHS) and right hand side (RHS) of each equality/inequality constraint. The main advantage of the α-cut level approach used in this paper is that it provides flexibility for the analyst to set their own acceptable possibility levels for decision making in evaluating and comparing DMUs. Zadeh (1978) suggested that fuzzy sets can be used as a basis for the theory of possibility similar to the way that measures theory provides the basis for the theory of probability. The fuzzy variable is associated with a possibility distribution is the same manner that a random variable is associated with a probability distribution. Therefore, the computed fuzzy efficiency scores are viewed as a fuzzy variable in the range [0, 1].

3. Literature Review

Sengupta (1992) was the first to propose a mathematical programming approach where fuzziness was incorporated into DEA by allowing the objective function and the constraints to be fuzzy. The stochastic DEA model was to be solved using chance-constrained programming and required the analyst to supply information on expected values of variables, the variance-covariance matrices of all variables, and the probability levels at which the feasibility constraints are to be satisfied. This method was difficult to implement due to those data requirements.

Triantis and Girod (1998) suggested a mathematical programming approach that transforms fuzzy inputs and outputs into crisp data using membership function values. Efficiency scores would then be computed for different membership functions and averaged. Hougaard (1999) suggested an approach that allows the decision maker to include other sources of information such as expert opinion in technical efficiencies computation. Kao and Liu (2000a) suggested the use of α-cut level sets to transform fuzzy data into interval data so that the fuzzy model becomes a family of convectional crisp DEA models. This approach was much similar to Guo and Tanaka (2001) who proposed a fuzzy CCR model in which fuzzy constraints, including fuzzy equalities and fuzzy inequalities, were all converted to crisp constraints by predefining different possibility levels.

Lertworasirikul et al (2003) proposed a possibility approach in which fuzzy constraints are treated as fuzzy events and fuzzy DEA model is transformed into possibility DEA model by using possibility measures on fuzzy events. Saati (2002) adopted the α-cut level approach, defined the fuzzy CCR model as a possibility-programming problem, and transformed it into an interval programming problem. This model could be solved as a crisp LP model and produce crisp efficiency score for each

DMU and for each given α-cut level. All the above authors used numerical examples to illustrate the application of the proposed fuzzy DEA approach.

Empirical Application of Fuzzy DEA

Empirical application of fuzzy DEA models is still in the infancy stage with only one application in agricultural economics. Hadi-Vencheh and Matin (2011) used an imprecise DEA (IDEA) model to compute the technical efficiency of 15 Iranian wheat producing provinces. Four inputs (acreage, water, wages and number of tractors) and one output (wheat produced) are used. Water and wages are the imprecise variables. The model shows that a DEA model with interval data can be treated as a peculiar DEA model with exact data.

Wu *et al.* (2006) applied a fuzzy DEA model to determine the efficiency of 24 cross-region bank branches in Canada. The authors incorporating fuzzy environmental variables (income level, population density, and the economy) to assess the performance of bank branches from three different regions: Ontario, Quebec, and Alberta. The assumption made was that different regions may face different external environments that exert significant influence to the performance of different branches. The labels of the environmental variables were linguistic, i.e., “high”, “medium”, “very good” and “good.” The possibility approach and α-cut level method as formulated by Lertworasirikul *et al.* (2003) was used with a slight modification where both crisp and fuzzy variables are incorporated into the DEA model. The crisp financial input variables used are personnel, equipment, occupancy and other general expenses. Crisp output variables are term deposits, personal loans, small business loans, non term depots and mortgage. The efficiency scores generated by the classical DEA model are compared to those from the

Fuzzy DEA model. The study finds that the disadvantage posed by the environment contribute to inefficiency besides the inefficiency that is purely operational.

Triantis and Girod (1998) used a three-stage approach to measure the technical efficiency performance of one packaging line that is part of a newspaper preprint insertion process. The model has three fuzzy inputs (direct labor, rework and raw materials) and one fuzzy output (packages). In stage one, the vague input and outputs are expressed in terms of their risk free and impossible bounds[5] and a membership function. In the second stage, the classical DEA models are re-formulated in terms of their risk free and impossible bounds and the membership function for each of the fuzzy input and output variables. The technical efficiency scores are computed in the third stage for different values of the membership function to identify unique sensitive decision making units.

Kao and Liu (2000a) applied the concept of fuzzy set theory for representing three missing values in data when studying the efficiencies of 24 university libraries in Taiwan. A triangular membership function is constructed for the missing values by deriving the smallest possible, most possible, and largest possible values from the observed data. Thus, nine libraries end up having fuzzy efficiency scores. The authors observe that interval estimation is more desirable than point estimation of the efficiency score in the absence of certain data. However, they caution that the number of missing data should be restricted to a level such that the number of DMUs, after taking off DMUs with a lot of missing values, should be at least two to three times of the total number of inputs and outputs specified in the model. This study used the ranking approach to rank fuzzy efficiency scores.

[5] The risk free and impossible bounds represent the production extremes given a fuzzy data. The risk free bound is production scenario that is realistically implementable while the impossible bound is the most realistically non-implementable scenario.

4. Data

Fuzzy DEA is applied to compute the technical efficiency scores of 34 dairy farms in Pennsylvania using the α-cut level approach. The dairy producers use three inputs (land, labor, and cows) to produce two outputs (milk and butterfat). The data is obtained from Stokes *et al*. (2007) who used the convectional DEA to computed technical efficiencies, assuming that either the data is precise or the relationship between inputs and outputs is deterministic. However, the authors hint that the data may not be precise, "*Due to the structure of the data set it was not possible to determine whether all resources such as land or labor were utilized by the dairy operations*." (pp 2558).

To illustrate the application of fuzzy DEA, uncertainty is introduce in the data by representing the inputs and outputs as symmetric triangular fuzzy numbers with a fuzzy interval. The input and output data can be represented as pairs consisting of centers and spreads as $\tilde{x}_{ij} = (x_{ij}^m, \varepsilon_{ij})$ and $\tilde{y}_{rj} = (y_{rj}^m, \beta_{rj})$ respectively[6]. A representation of the input/output relationship is simply:

$$\tilde{Y}(milk, butterfat) = \tilde{X}(land, labor, cows), \qquad \textbf{(7)}$$

where $\tilde{Y}$ and $\tilde{X}$ are matrices of the fuzzy outputs and inputs. The data is listed in Table 1. The spread for each variable is generated as a random number using the random number generator in Microsoft Excel. For the purpose of this study, we assume that the spread for labor is a random number between 0 and 1. The spread for cows is between 1 and 10 and that of land is between 1 and 20. The spread of milk is between 100 and 500 and for butterfat is between 1 and 20[7].

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[6] Symmetric fuzzy numbers means that the upper and lower spreads are equal, i.e., $x_{ij}^l = x_{ij}^u = \varepsilon_{ij}$

[7] Those assumptions are introduced for illustration purposes only. The random numbers around the actual value facilitate the generation of symmetric triangular fuzzy inputs and outputs.

We follow a three-stage approach to compute the technical efficiency scores. In the first stage, the inputs and outputs are expressed in terms of symmetric triangular fuzzy numbers and membership functions at six different α-cut levels ranging from 0 to 1. Pre-specified intervals of 0.2 are used. In the second stage, the classical DEA model is re-formulated as a series of DEA models in-terms of the membership functions for each of the fuzzy input and output variables following equation (6). The adopted model is presented in the appendix. In the third stage, fuzzy technical efficiency scores are computed for different membership functions to track how the relative efficiency scores of each farm varies at different possibility levels. The FEAR package in R is used to solve the different LP problems.

5. Empirical Results

The lower bound and upper bound input reducing technical efficiency scores $(\theta_{\alpha i})$ are presented in Table 2 and Table 3. The input and output data were assumed to be imprecise and, therefore, the computed efficiency scores are fuzzy too. In general, the lower bound technical efficiency scores $(E_{ji})_{\alpha i}^{L}$ decreases as the membership function shifts the input and output data from the most precise measurement (α = 1) to the most imprecise measurement (α = 0). The upper bound scores $(E_{ji})_{\alpha i}^{U}$ increases as α decrease from 1 to 0. The closer α approaches 1 the greater the level of possibility and the lower the degree of uncertainty is. The fuzzy efficiency score lie in a range and the different α-cut levels indicate those intervals and the uncertainty level associated with precision in data. Specifically, α = 0 has the widest interval. On the other hand, the value of α=1 is the most likely value of efficiency score.

Using the α-cut level approach, the range of a farm's efficiency score at different possibility levels is derived. For example, the efficiency scores for Farm 1 at α-cut level = 1 is 0.740. This deterministic case assumes precision in measurement. At α-cut level =

0.8, the efficiency score range is [0.737, 0.823]. This indicates that it is possible that the efficiency score of Farm 1 will fall between 0.737 and 0.823 at the possibility level 0.8. The range of the efficiency score at the extremes ($\alpha = 0$) is [0.598, 0.829]. This implies that the efficiency score of Farm 1, relative to other farms, will never exceed 0.829 or fall below 0.598. Results of the other farms at different possibility levels can be interpreted in similar manner. Figure 2 illustrates the membership function of the triangular fuzzy efficiency scores for Farm 1. Figure 3 plots the best practice frontiers for the upper bound (dashed lines) and lower bound (dotted lines) membership functions of inputs and outputs at $\alpha = 0$. This represents the extreme range that the frontiers defining the relative technical efficiency scores of each farm are expected to shift due to imprecision in data. The shift of the frontier at $0 < \alpha < 1$ would fall within this range and would keep on narrowing as α approached 1.

The results from the fuzzy DEA model provide more information to the decision maker compared to the point estimates from the convectional DEA model. The analyst can observe the variation of the technical efficiency profile of each farm from the impossible value when α-cut level = 0 to the risk-free value when α-cut level =1. Only four farms, Farm 10, Farm 15, Farm 25 and Farm 30, remain technical efficient at all α-cut levels. Farm 9 becomes technical efficient at the extreme α-cut level = 0.

The computed fuzzy efficiency scores need to be ranked in order to determine how each farm performs relative to the other farms in an uncertain environment. The ranking of the fuzzy efficiency scores can be compared to the ranking of scores of the convectional DEA model in order to discriminate which decision-making units are sensitive to the variation of the inputs/output variable measurement inaccuracy. We use the Chen and Klein (1997) ranking method to compute an index, I, for ranking fuzzy numbers as:

$$I_j = \frac{\sum_{i=0}^{n}((E_j)_{\alpha i}^{U} - c)}{\left[\sum_{i=0}^{n}((E_j)_{\alpha i}^{U} - c) - \sum_{i=0}^{n}((E_j)_{\alpha i}^{L} - d)\right]}, n \to \infty, \quad \textbf{(8)}$$

where $c = \min_{i,j}\left\{(E_{ji})_{\alpha i}^{L}\right\}$ and $d = \max_{i,j}\left\{(E_{ji})_{\alpha i}^{U}\right\}$**.** The lower bound and upper bound efficiency indices are represented by $(E_{ji})_{\alpha i}^{L}$ and $(E_{ji})_{\alpha i}^{U}$. A larger index indicates the fuzzy number is more preferred. The Chen-Klein's method is used to compute the ranking indices for each farm. The ranking is compared to a ranking of the crisp technical efficiency indices from the classical DEA model and the results are presented in Table 4.

The Chen Klein ranking index gives similar results compared to the ranking of crisp technical efficiency scores with one exception. Five farms, Farms 4, 16, 10, 25, and 30, have perfect score of 1, meaning that they are the farms that define the production frontier. The convectional DEA model only identifies Farms 10, 25 and 30 as defining the frontier. The Spearman's rank correlation of the two ranking methods is 0.99 and is significant at less than 1%.

6. Conclusions

The main objective of this paper was to introduce fuzzy DEA models by literature review and application as an alternative for analyzing the productive efficiency of agricultural entities in an uncertain environment. Fuzzy DEA models were found to be applicable when expert judgment or environmental variables (linguistic variables) needs be incorporated into the convectional DEA model, when there are missing data and when the measurement of the data is imprecise.

An empirical example of symmetrical triangular membership functions was used to illustrate the application of fuzzy DEA to a group of 34 dairy farms in Pennsylvania. The α-cut level approach was used to convert the fuzzy DEA scores into crisp scores. The fuzzy DEA model was able to discriminate the farms whose efficiency performance is sensitive to variation in the inputs/outputs. Compared to the classical DEA model, results from the fuzzy DEA model allow for a determination of robustness and lead to recommendations that are more rigorous.

We conclude by arguing here that it will be interesting to apply empirical fuzzy DEA models in the field of agricultural economics using the α-cut level approach. Given the incomplete knowledge of input and output measures often used in DEA models, fuzzy DEA models will provide agricultural economists with an additional tool for efficiency analysis. Uncertainty always exists in human thinking and judgment. Research in efficiency and productivity analysis should apply recent advancements in DEA that address current concerns. Fuzzy DEA can play an important role for evaluation performance of decision-making units when data are imprecise.

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Table 1. Inputs and Outputs used in the Fuzzy DEA Analysis Models

DMU	Labor (FTE)	Cows	Land (ha)	Milk production (kg/cow)	Butterfat Production (kg/cow)
Farm1	2.66	70	98	10,490	372
Farm2	3.06	67	97	8,736	337
Farm3	3.59	72	38	8,267	319
Farm4	1	60	48	10,010	392
Farm5	2.8	180	166	8,918	330
Farm6	2	112	66	9,953	359
Farm7	1.6	40	109	7,446	302
Farm8	2.28	55	105	9,362	337
Farm9	4.71	118	121	9,016	347
Farm10	1.8	55	19	9,067	317
Farm11	2	58	57	8,605	339
Farm12	2	87	63	9,148	336
Farm13	1.8	40	36	6,802	262
Farm14	2	53	136	8,433	298
Farm15	4.18	249	257	7,339	294
Farm16	1.6	43	40	8,530	303
Farm17	1.38	55	101	6,795	256
Farm18	1.6	36	85	4,870	183
Farm19	1.9	44	60	7,426	297
Farm20	1.51	54	81	8,350	315
Farm21	1	98	121	9,406	365
Farm22	1.65	36	89	7,166	267
Farm23	1.67	54	147	4,391	155
Farm24	3.2	110	127	9,981	349
Farm25	1	64	51	11,438	405
Farm26	3.72	110	42	8,995	352
Farm27	1.93	81	80	11,201	410
Farm28	2.17	56	74	7,015	267
Farm29	2	71	61	6,689	254
Farm30	1	30	45	6,105	245
Farm31	2	82	52	5,379	202
Farm32	2	73	113	7,844	278
Farm33	3	143	126	9,045	353
Farm34	1.15	62	86	8,621	322
Mean	2.15	77.00	88.15	8259.97	309.38
SD	0.92	44.57	46.61	1656.94	59.03
Minimum	1.00	30.00	19.00	4391.00	155.00
Maximum	4.71	249.00	257.00	11438.00	410.00

Table 2. Input Reducing Technical Efficiency Scores at varying α-cut levels

	Lower Bound Membership Function Value $(E_j)^L_{\alpha i}$						
DMU	θ_1	$\theta_{0.8}$	$\theta_{0.6}$	$\theta_{0.4}$	$\theta_{0.2}$	θ_0	**Average**
Farm1	0.740	0.717	0.699	0.673	0.696	0.598	0.687
Farm2	0.642	0.633	0.603	0.571	0.572	0.544	0.594
Farm3	0.719	0.721	0.726	0.667	0.740	0.622	0.699
Farm4	1.000	1.000	0.973	1.000	1.000	1.000	0.996
Farm5	0.291	0.284	0.281	0.275	0.272	0.251	0.276
Farm6	0.591	0.592	0.572	0.522	0.545	0.486	0.551
Farm7	0.924	0.892	0.929	0.798	0.758	0.742	0.840
Farm8	0.836	0.822	0.767	0.710	0.665	0.668	0.745
Farm9	0.407	0.395	0.395	0.378	0.389	0.328	0.382
Farm10	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm11	0.821	0.809	0.821	0.777	0.831	0.690	0.791
Farm12	0.603	0.601	0.608	0.573	0.552	0.546	0.580
Farm13	0.938	0.946	0.929	0.870	0.923	0.976	0.930
Farm14	0.782	0.764	0.723	0.724	0.693	0.598	0.714
Farm15	0.180	0.179	0.176	0.167	0.163	0.154	0.170
Farm16	1.000	1.000	1.000	0.995	1.000	1.000	0.999
Farm17	0.644	0.626	0.628	0.566	0.614	0.595	0.612
Farm18	0.665	0.635	0.658	0.547	0.527	0.505	0.589
Farm19	0.855	0.846	0.838	0.780	0.797	0.736	0.808
Farm20	0.789	0.763	0.749	0.749	0.726	0.634	0.735
Farm21	0.901	0.876	0.902	0.868	0.933	0.762	0.874
Farm22	0.978	0.953	0.960	0.876	0.976	0.901	0.941
Farm23	0.406	0.394	0.376	0.377	0.352	0.325	0.372
Farm24	0.465	0.450	0.436	0.432	0.429	0.408	0.437
Farm25	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm26	0.548	0.552	0.543	0.512	0.524	0.457	0.523
Farm27	0.735	0.720	0.710	0.680	0.703	0.634	0.697
Farm28	0.620	0.600	0.608	0.574	0.593	0.523	0.586
Farm29	0.527	0.516	0.525	0.508	0.497	0.510	0.514
Farm30	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm31	0.393	0.391	0.385	0.362	0.368	0.333	0.372
Farm32	0.550	0.536	0.504	0.487	0.481	0.474	0.505
Farm33	0.366	0.362	0.362	0.347	0.337	0.317	0.348
Farm34	0.779	0.774	0.766	0.803	0.789	0.704	0.769
Average	0.697	0.687	0.681	0.652	0.660	0.618	0.666
Max	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Min	0.180	0.179	0.176	0.167	0.163	0.154	0.170

The table reports the lower bound input reducing technical efficiency scores at various α-levels.

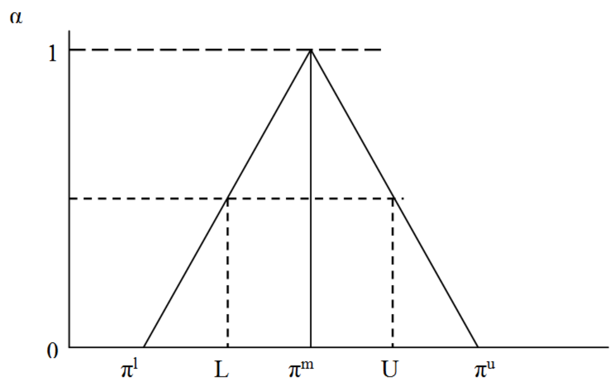
Table 3. Input Reducing Technical Efficiency Scores for Various α-cut levels

	Upper Bound Membership Function Value $(E_j)^U_{\alpha i}$						
DMU	θ_1	$\theta_{0.8}$	$\theta_{0.6}$	$\theta_{0.4}$	$\theta_{0.2}$	θ_0	**Average**
Farm1	0.740	0.761	0.778	0.755	0.701	0.829	0.760
Farm2	0.642	0.650	0.683	0.712	0.688	0.747	0.687
Farm3	0.719	0.718	0.710	0.753	0.693	0.795	0.731
Farm4	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm5	0.291	0.298	0.302	0.304	0.308	0.355	0.310
Farm6	0.591	0.589	0.600	0.648	0.628	0.636	0.615
Farm7	0.924	0.956	0.930	1.000	1.000	1.000	0.968
Farm8	0.836	0.849	0.904	0.864	0.867	0.918	0.873
Farm9	0.407	0.417	0.417	0.404	0.397	0.470	0.419
Farm10	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm11	0.821	0.831	0.812	0.819	0.780	0.903	0.828
Farm12	0.603	0.605	0.595	0.618	0.649	0.647	0.619
Farm13	0.938	0.934	0.951	0.949	0.909	0.913	0.932
Farm14	0.782	0.798	0.839	0.758	0.733	0.887	0.799
Farm15	0.180	0.181	0.181	0.196	0.191	0.206	0.189
Farm16	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm17	0.644	0.661	0.655	0.716	0.663	0.672	0.668
Farm18	0.665	0.693	0.671	0.714	0.707	0.766	0.702
Farm19	0.855	0.863	0.873	0.958	0.942	0.976	0.911
Farm20	0.789	0.814	0.822	0.806	0.817	0.908	0.826
Farm21	0.901	0.927	0.900	0.918	0.858	0.906	0.902
Farm22	0.978	1.000	0.994	0.967	0.859	0.959	0.960
Farm23	0.406	0.418	0.431	0.417	0.419	0.464	0.426
Farm24	0.465	0.481	0.494	0.488	0.482	0.501	0.485
Farm25	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm26	0.548	0.544	0.551	0.581	0.571	0.769	0.594
Farm27	0.735	0.751	0.758	0.769	0.777	0.836	0.771
Farm28	0.620	0.640	0.639	0.663	0.633	0.731	0.655
Farm29	0.527	0.536	0.522	0.515	0.538	0.536	0.529
Farm30	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm31	0.393	0.394	0.393	0.416	0.417	0.454	0.411
Farm32	0.550	0.565	0.597	0.587	0.564	0.588	0.575
Farm33	0.366	0.369	0.365	0.367	0.390	0.406	0.377
Farm34	0.779	0.785	0.801	0.774	0.795	0.859	0.799
Average	0.697	0.707	0.711	0.719	0.705	0.754	0.715
Max	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Min	0.180	0.181	0.181	0.196	0.191	0.206	0.189

The table reports the upper bound input reducing technical efficiency scores at various α-levels.

Table 4. Ranking of the Crisp and Fuzzy Efficiency Scores

Rank	DMU	Chen-Klien Index	CCR Technical Efficiency
1	Farm15	0.023	0.180
2	Farm5	0.162	0.291
3	Farm33	0.241	0.366
4	Farm31	0.277	0.393
5	Farm9	0.287	0.407
6	Farm23	0.290	0.406
7	Farm24	0.359	0.465
8	Farm29	0.425	0.527
9	Farm32	0.450	0.550
10	Farm26	0.470	0.548
11	Farm6	0.498	0.591
12	Farm12	0.517	0.603
13	Farm28	0.540	0.620
14	Farm2	0.560	0.642
15	Farm17	0.562	0.644
16	Farm18	0.565	0.665
17	Farm3	0.651	0.719
18	Farm1	0.654	0.740
19	Farm27	0.665	0.735
20	Farm14	0.688	0.782
21	Farm20	0.712	0.789
22	Farm34	0.731	0.779
23	Farm8	0.734	0.836
24	Farm11	0.759	0.821
25	Farm19	0.795	0.855
26	Farm7	0.833	0.924
27	Farm21	0.853	0.901
28	Farm13	0.916	0.938
29	Farm22	0.930	0.978
30	Farm4	0.995	1.000
31	Farm16	0.999	1.000
32	Farm10	1.000	1.000
33	Farm25	1.000	1.000
34	Farm30	1.000	1.000



Note: $L = \pi^l + \alpha(\pi^m - \pi^l)$ and $U = \pi^u + \alpha(\pi^u - \pi^m)$

Figure 1. A triangular fuzzy number

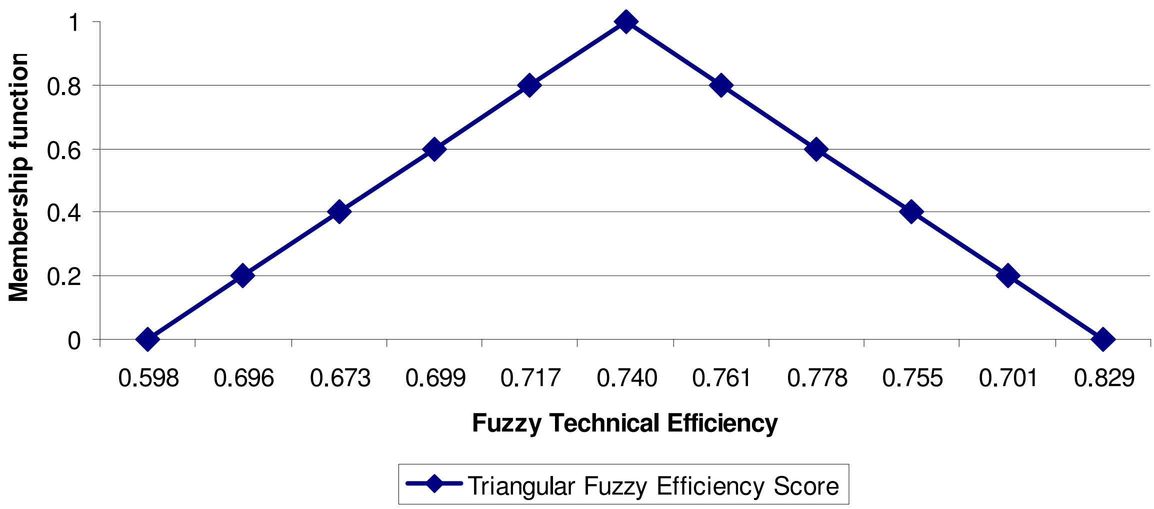
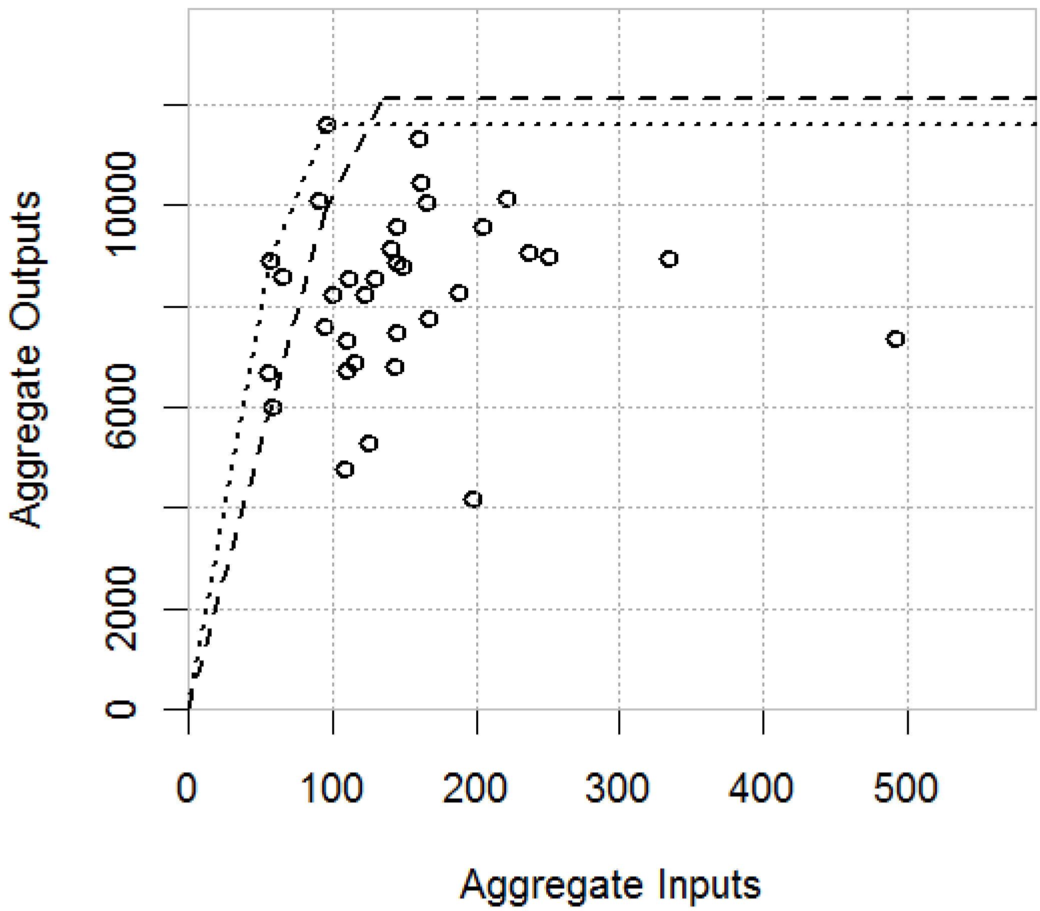


Figure 2. Triangular fuzzy efficiency scores for Dairy Farm 1



Note

1. The dotted line represent the lower non-increasing returns to scale frontier at α-level=0
2. The dashed line represent upper non-increasing returns to scale frontier at α-level=0

Figure 3. Best-practice frontiers at α-level=0

Appendix

$$TE(LN, LB, CW, MK, BT) = Min\ \theta$$

subject to:

$$\text{Constriants}\begin{cases} \theta(\alpha LN_{io}^{m} + (1-\alpha)LN_{io}^{l}) \geq \sum_{j=1}^{34} \lambda_j (\alpha LN_{ij}^{m} + (1-\alpha)LN_{ij}^{l}), \\ \theta(\alpha LB_{io}^{m} + (1-\alpha)LB_{io}^{l}) \geq \sum_{j=1}^{34} \lambda_j (\alpha LB_{ij}^{m} + (1-\alpha)LB_{ij}^{l}), \\ \theta(\alpha CW_{io}^{m} + (1-\alpha)CW_{io}^{l}) \geq \sum_{j=1}^{34} \lambda_j (\alpha CW_{ij}^{m} + (1-\alpha)CW_{ij}^{l}), \\ \theta(\alpha MK_{i0}^{m} + (1-\alpha)MK_{io}^{l}) \leq \sum_{j=1}^{34} \lambda_j (\alpha MK_{ij}^{m} + (1-\alpha)MK_{ij}^{l}), \\ \theta(\alpha BF_{io}^{m} + (1-\alpha)BF_{io}^{l}) \leq \sum_{j=1}^{34} \lambda_j (\alpha BF_{ij}^{m} + (1-\alpha)BF_{ij}^{l}), \\ \lambda_j \geq 0, \theta_j \geq 0 \end{cases}$$

$$TE(LN, LB, CW, MK, BT) = Min\ \theta$$

subject to:

$$\text{Constriants}\begin{cases} \theta(\alpha LN_{io}^{m} + (1-\alpha)LN_{io}^{u}) \geq \sum_{j=1}^{34} \lambda_j (\alpha LN_{ij}^{m} + (1-\alpha)LN_{ij}^{u}), \\ \theta(\alpha LB_{io}^{m} + (1-\alpha)LB_{io}^{u}) \geq \sum_{j=1}^{34} \lambda_j (\alpha LB_{ij}^{m} + (1-\alpha)LB_{ij}^{u}), \\ \theta(\alpha CW_{io}^{m} + (1-\alpha)CW_{io}^{u}) \geq \sum_{j=1}^{34} \lambda_j (\alpha CW_{ij}^{m} + (1-\alpha)CW_{ij}^{u}), \\ \theta(\alpha MK_{io}^{m} + (1-\alpha)MK_{io}^{u}) \leq \sum_{j=1}^{34} \lambda_j (\alpha MK_{ij}^{m} + (1-\alpha)MK_{ij}^{u}), \\ \theta(\alpha BF_{io}^{m} + (1-\alpha)BF_{io}^{u}) \leq \sum_{j=1}^{34} \lambda_j (\alpha BF_{ij}^{m} + (1-\alpha)BF_{ij}^{u}), \\ \lambda_j \geq 0, \theta_j \geq 0 \end{cases}$$

where LN = Land, LB = Labor, CW = Cows, MK = Milk and BF = Butterfat, $0 \leq \alpha \leq 1$ is the α-cut level, $0 < \theta \leq 1$ is the efficiency index, subscripts *l*, *m*, and *u* indicate the lower, center, and upper bounds of the fuzzy number.