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A Note on Two-Point Risks, Certainty Equivalents and Quadratic Preference

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Decision makers' preferences can be encoded in utility functions that are unique up to a positive linear transformation. Some applications are aided by algebraic description of such functions. One widely used form is the quadratic function [e.g. 2, pp. 92-6; 5; 7]. In spite of increasing disenchantment with this form on grounds of either its failure to increase monotonically or its implied (irrational) increasing risk aversion [3, 8], practitioners seem destined to sustain interest in the function either because of pragmatic considerations or because of the desire to rationalise mean-variance analysis [1]. The point of this note is to identify a minor technical problem of anomaly that may be encountered in fitting quadratic utilities.[1] This "problem" is resolved in the penultimate paragraph.

A generalized quadratic utility function for an argument X (such as wealth)

(1) $$U' = A + BX + CX^2,$$

can be positive linear transformed [2, p.68] as

(2a) $$U = X + (C/B)X^2 = X + bX^2$$

if $B > 0$ or, if $B < 0$, as

(2b) $$U = -X + (C/(-B))X^2 = -X + bX^2$$

The (2a) form is the one conventionally employed and might be considered the "normal" standard form of the quadratic preference function. Risk aversion is implied if $b < 0$. Failure to recognize when the (2b) form is appropriate can, as is shown directly, lead to an apparent anomaly.

The quadratic might be fitted in several ways: e.g., least squares regression analysis based on "data points" read from (smoothed?) hand-sketched curves [7], "eye-balling" parameter estimation from such data [6] or direct "three-point estimation", on which attention is now focused. Here the case of a two-point risk of x with probability p, and z ($> x$) with probability $q = 1 - p$ is studied. The expected value, E, of the risk is $px + qz$. Let the certainty equivalent (CE) of this risk be y. This is defined as that value which the decision maker is indifferent to confronting with certainty versus the specified risk. Assuming Bernoullian utility maximisation (and the underlying postulates [2, pp. 66-9]) it follows that

(3) $$pU(x) + qU(z) = U(y)$$

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1. This short note grew out of 'detective' work investigating some strange results found in early analysis of data gathered in Brazil [4]. Perhaps through this note I can reduce the probability that other researchers (who are seemingly using variants of the Dillon/Scandizzo [4] methodology) will encounter the perverse case (discussed in the note) of a response of risk preference being measured and categorised as one of risk aversion.

The parameter(s) of the quadratic $U(X)$ that satisfies equation (3) can be found at least three equivalent ways. Most simply, the standard form (2a) can be imposed directly as

(4) $p(x + bx^2) + q(z + bz^2) = y + by^2$,

in which case the standardised quadratic parameter is found as

(5) $b = (y - px - qz)/(px^2 + qz^2 - y^2)$.

The same result comes from the expected utility version [2, p. 93] of (2a) expressed in terms of mean E and variance V, namely $U = E + b(E^2 + V)$, and noting that for the risk, $V = pq(z - x)^2$ and for the certainty, $E = y$ and $V =$ J. Consider, for example, a decision maker [2, p. 113] indifferent between (0.5, 0.5; $6000, $12000) and a certain $8400, in which case the solution of (5) is $b = -(3.086)10^{-5}$, with the negative indicating the anticipated risk aversion (since $E > CE$).

Problems may arise, however, in cases where risk preference is anticipated. Consider, for further example, the case of $p = q = 0.5$, $x = 6$, $z = 12$, and $y = 10$. Here $E = 9 < CE = 10$ so risk preference is anticipated but substitution into (5) yields $b = -0.1$, seemingly and anomalously implying risk *aversion.*

For an explanation, it is convenient to return to the generalised function (1) and thus to the third estimation method. This function can be fitted to the three-point data in a manner analogous to (3) to (5) by imposing an arbitrary scale, e.g. $U(x) = 0$, $U(z) = 1$ so that from (3), $U(y) = q$. Then solution for the parameters A, B and C devolves to the solution of the three linear simultaneous equations.

$$A + Bx + Cx^2 = 0$$
$$A + By + Cy^2 = q$$
$$A + Bz + Cz^2 = 1$$

from which it follows that

$$C = [p(y - x) - q(z - y)]/[(z - y)(y - x)(z - x)],$$

and

$$B = [q(z^2 - y^2) - p(y^2 - x^2)]/[(z - y)(y - x)(z - x)].$$

Conversion to the standard form (2a) is accomplished as $b = C/B$ if $B > 0$ in which case the result is again identical with that of equation (5). However, if $C > 0$ and $B < 0$, the anomalous result of $b < 0$ arises and is seen to be the result of an inadmissible (implicit) negative linear transformation. In such a case, the appropriate "standard" form is (2b) which is found as $b = -C/B$ when $B < 0$. The intermediate "standard" form is where $B = 0$ whence $U = bX^2$.

The simplest check involves inspection of the denominator in (5): as this is $>$, $=$ or < 0, then B is $<$, $=$ or > 0. The standard form (2b) only makes sense as preference a function for a desirable attribute when $E > 1/2b$ (i.e. $\partial U/\partial E > 0$).

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