Implementing Coarse Priority Schemes

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Tess Brennan

Date

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Abstract

This thesis shows that coarse priority schemes for rationing random supply

can be implemented without prior knowledge of the distribution of valuations

using the VCG mechanism. Coarse priority rationing schemes are those

with fewer priority classes than individuals. Coarse priority schemes are of

interest because full priority schemes may not be feasible in practice and

because coarse priority schemes achieve large parts of the gains from full

priority schemes. Implementation of the coarser priority schemes without

knowledge of the distribution of valuations is useful when offering priority for

the first time, since this information may not be available. I show that with

unit demand, risk neutrality and a known supply distribution, the optimal

priority scheme for a given number of classes can be implemented. I provide

a program to simultaneously solve for the optimal cut-offs and VCG prices

given any discrete valuations and any supply distribution.

Contents

1 Introduction

Priority pricing is often used to ration random supplies of a good that cannot be stored. Priority pricing charges customers a fee for their place in a queue to be served and is used to replicate spot markets or peak load schemes when they are not feasible (Oren, Smith and Wilson, 1985). When an unexpected shortage occurs, the highest ranked customers are served first so that the resource is allocated to the highest valuing users (Wilson, 1989).

Full priority schemes have a unique position for each individual (n or more priority classes, where n is the number of individuals) and serve customers in rank order. If each customer has unit demand and service is in multiples of non-divisible units, each individual is served only if supply is equal to or greater than their position in the queue. Coarse priority schemes have less than n priority classes and serve each class in rank order. Where supply is enough to serve some but not all members of the same priority group, service is randomly allocated within the group. Ultimately, a level of priority represents a customer's probability of service. For full priority schemes this is a unique probability for each customer, for coarse priority schemes it is a probability for each class.

Full priority schemes disconnect customers individually depending on the supply realisation whereas coarse schemes only disconnect groups of customers (Wilson, 1989). Electricity markets, for example, usually have technology that enables individual customers to be selected for immediate disconnection (Chao, Oren, Smith and Wilson, 1986) and full priority schemes are favorable. If it is too costly or not technically feasible to disconnect customers individually a coarse priority scheme might be preferred. Water rationing, for example, is often achieved by public announcements which ban visible outdoor water use by various users (The Productivity Commission, 2008). Road rationing is often achieved by public announcements which ban road use by various vehicle

licence plate numbers (Colombia Reports, 2009). In each example coarse rationing is more feasible because only groups of customers are disconnected and contacting customers individually for each supply realisation is too demanding.

Coarse rationing can in general be introduced provided there is a way to disconnect groups and to allocate randomly or uniformly[1] within groups. In both the road and water applications the public bans can and do uniformly allocate amongst individuals. Sequential choice of licence plates numbers to be banned from the roads (Bogotá D.C, 2009) and alternating house numbers to be banned from outdoor water use (ACTEW Corporation, 2009) allow uniform allocation within the population. All that is required to introduce a coarse priority scheme in these settings is to predetermine various groups of users, for example by providing licences for various groups, and then allow the same public announcements to affect those groups only. For example, cars with blue licence plates can be given priority, but if supply is very low only blue licences ending in 3 may drive.

In 2008 the Productivity Commission suggested in its report *Towards Urban Water reform,* the possibility of introducing a choice of water service probability, where no choice has existed before. Public bans on outdoor water use are enforced by neighborhood observation and so monitoring relies on cheating being visibly detected. With this system in place, a two class scheme could allow for high priority customers to display an exemption on their property so that visible use would be tolerated. It is not easy to imagine how a third class could be introduced and it is possible that a shift from two to three classes would require entirely new monitoring technology. Retaining the current monitoring technology and implementing an extremely coarse priority scheme may therefore be justified in terms of cost.

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[1]With an assumption of risk neutrality introduced later the two will be equivalent.

Further, the introduction of such coarse two tiered priority schemes has been shown in theory to achieve a surprisingly large proportion of the possible gains from full priority schemes.

The thesis extends previous literature on implementing coarse priority schemes to implementation without prior knowledge of the distribution of valuations within the service population. Implementation without knowledge of the distribution will be particularly useful where there are no pre-existing markets for priority and a choice of priority service has never been offered before. In this case it is unlikely that the planner would initially have reliable estimates of the distribution to use.

For example, estimates of willingness to pay for various aspects of water service are numerous and include choice modelling surveys (Hensher, 2006 and Gordon et al., 2001), Marshallian surplus estimation (Quintin and Ward, 2008) and opportunity cost of time estimation (Brennan, Tapsuwan and Ingram, 2007) to name a few. Since a choice of water reliability has never been offered before (Productivity Commission, 2008), it is not surprising that none of these surveys estimate a distribution of preferences; only various measures of average and aggregate social cost.

This thesis shows that the VCG mechanism (Vickrey(1961), Clarke (1971) and Groves (1973) can be used to implement any coarse priority scheme optimally without the planner's prior knowledge of the distribution of valuations. That is, subject to the restriction of a given number of classes, the optimal allocation of individuals to classes can be implemented. The mechanism will ask individuals to report their valuations for service, solve for the socially efficient allocation of priority based on those reports and solve for the associated VCG prices which induce truthful reporting. This process is illustrated using a model for priority service and a program to solve for efficient allocation of priority and to compute VCG prices.

The remainder of the thesis is structured as follows: Section 2 gives an overview of priority schemes literature and VCG literature. Section 3 gives the model for priority service. Section

4 applies the VCG mechanism to the model for priority service. Section 5 gives limitations and suggestions for further research. Section 6 is the conclusion.

2   Literature review

The first part of the literature review demonstrates that the benefit of full priority schemes and coarse priority schemes have been shown provided they can be implemented correctly (efficiently). Implementation has been discussed but knowledge of the distribution of valuations is always required. The second part of the literature shows that general methods for finding the valuations and implementing an efficient social allocation is readily available. This has not yet been applied to the particular problem of priority pricing.

2.1   Priority schemes literature

Wilson (1989) has shown that perfect priority can approximate the first best spot market outcome if each customer is charged for their unique position in a queue. If each customer has unit demand and the order of customers stays the same over time, priority service with n priority classes can exactly replicate the spot market outcome.

The pricing of full priority schemes has also been discussed. Oren, Smith and Wilson (1985) derived the optimal priority pricing for a monopolist. Wilson (1989) derived price schedules to induce socially optimal priority service. Again, customers pay a price for their position in a queue. In both cases the pricing relies on the seller knowing the distribution of valuations in order to offer prices that induce the efficient behaviour.

The benefits of coarse priority schemes have also been discussed. Because coarse priority schemes offers fewer priority classes than there are customers, there is naturally some efficiency

loss as compared with full priority schemes. Wilson (1989) showed that if the optimal pricing can be found, a coarse scheme with m priority classes incurs a loss of order no more than $\frac{1}{m^2}$ compared with perfect priority schemes. This gives some justification for introducing coarse priority schemes, but it relies on being able to know the distribution of valuations because the losses would be larger if the allocation of individuals to classes were not necessarily optimal.

McAfee (2001) has shown that a two class scheme that uses the mean valuation[2] as a cut-off for high and low priority customers must achieve at least half of the gains from infinitely many classes for any valuation and supply distributions that satisfy common hazard rate assumptions[3]. McAfee suggests that the result will be useful since estimates of the mean might exist. In practice knowledge of both the mean and the location (or percentile) of the mean would be required to implement such a scheme[4] so again some knowledge of the valuation distribution is required. In summary, the result does give a strong justification for using very coarse priority schemes but it does not show how this would be implemented in practice.

The implementation of coarse priority schemes have also been discussed. Wilson (1989) derived price schedules for such coarser schemes as well as the perfect priority schemes discussed

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[2]McAfee's proof also holds when the mean of the function matching service probability to valuations is used as a cut-off although this does not help to sort which customers belong to each group.

[3]The assumptions are that the distribution of valuations $f(v)$ and the function matching service to valuations $g(y)$ satisfy the following hazard rate assumptions:

$$\frac{F(v)}{f(v)} \text{ and } \frac{G(y)}{g(y)} \text{ are increasing and;}$$
$$\frac{1 - F(v)}{f(v)} \text{ and } \frac{1 - G(y)}{g(y)} \text{ are decreasing.}$$

[4]With knowledge of the mean and location of the mean, the number of customers in each service group and the implied probability of service can be found. Without knowledge of the location of the mean it is unknown what level of service is implied by each class.

above. The price menus that achieve an efficient allocation for m priority classes select cut-off valuations so that individuals self-select into classes that maximise the social surplus from the coarse rationing. The pricing again relies on the planner having knowledge of the valuations distribution. The planner must also know the set of customers that are eligible for service for the sake of efficiency, so they must be able to exclude some customers. This set may not be prior knowledge of the planner and it might also not be feasible in terms of monitoring technology. Therefore, the question of how to implement a coarse priority scheme without knowledge of the distribution of valuations remains open and this is the practical problem addressed in this thesis.

2.2 VCG literature

VCG mechanisms are a natural starting point for implementing an optimal social outcome without knowledge of the distribution of valuations because they are dominant strategy implementable and they are efficient. The components of the VCG mechanism and these fundamental properties of the VCG mechanism are summarised here.

Groves (1973) and Clarke (1971) showed that any socially efficient outcome can be implemented provided the scheme does not need to be self-funded. This is because the social planner can simply make transfers to individuals so that maximising total utility is equivalent to maximising individual utility. Specifically, the utility of an individual from the outcome can be reduced to the total utility with a constant term.

Prior to this, Vickrey (1961) established a payment rule that converted these payments into externality payments. The payments were equivalent to charging an individual the total utility of all individuals less the total utility without him/ her, and it was done in the setting of selling goods. Together, the efficient allocation and the combined payments make up the VCG selling mechanism.

Formally, the VCG mechanism is as follows. There is a set $N = \{1, ..., n\}$ of individuals. An outcome $x = (k, t_1, ..., t_n)$ is a choice of an allocation from all feasible allocations $k \in K$ and a set of payments t_i to each individual. The outcome for individual i is $x_i = (k, t_i)$.

Individual types are denoted θ_i, and reported types $\hat{\theta}_i$. Let the vector of true types be $\theta = (\theta_1, ..., \theta_n)$ and the vector if reported types be $\hat{\theta} = (\hat{\theta}_1, ..., \hat{\theta}_n)$. The VCG mechanism is a direct mechanism so individuals will report a type and the mechanism will map reported types to outcomes according to a social choice function $f : \Theta \to X$ where X is the set of all possible outcomes and Θ is the set of all possible types θ.

Individuals are assumed to have utility over outcomes that is quasi-linear (linear in transfers) given by;

$$u_i(x_i, \theta_i) = v_i(k, \theta_i) + t_i.$$

Assuming for now that individuals report truthfully, the VCG mechanism implements the efficient allocation:

$$k_N^*(\theta) = \arg\max_{k \in K} \sum_{i \in N} v_i(k, \theta_i) \tag{1}$$

and makes transfers according to

$$t_i(\theta) = \sum_{j \neq i} [v_j(k_N^*(\theta), \theta_j) - v_j(k_{N \setminus i}^*(\theta_{-i}), \theta_j)] \tag{2}$$

where θ_{-i} is the types of all individuals except i.

The first term on the right represents the welfare of all other individuals at the optimal allocation when i's report is considered. The second term on the right is the welfare of all other individuals at the optimal allocation when i's report is not considered. The payments $t_i(\theta)$ therefore represent the external effect that i's report has on the other individuals.

In showing that the VCG mechanisms are dominant strategy implementable, I follow Milgrom (ch.2, 2004). If $u_i(x, \hat{\theta}_i, \theta_i)$ is the utility for individual i with type θ_i when reporting $\hat{\theta}_i$ and $u_i(x_i, \theta_i)$ is the utility from reporting θ_i truthfully, then the gains to individual i for misreporting $\hat{\theta}_i$ instead of θ_i for any reports of θ_{-i} is

$$
\begin{aligned}
u_i(x, \hat{\theta}_i, \theta_i) - u_i(x_i, \theta_i) &= v_i(k_N^*(\hat{\theta}_i, \theta_{-i}), \theta_i) + t_i(\hat{\theta}) - v_i(k_N^*(\theta), \theta_i) - t_i(\theta) \\
&= v_i(k_N^*(\hat{\theta}_i, \theta_{-i}), \theta_i) + \sum_{j \neq i} [v_j(k_N^*(\hat{\theta}_i, \theta_{-i}), \theta_j) - v_j(k_{N \setminus i}^*(\theta_{-i}), \theta_j)] \\
&\quad - [v_i(k_N^*(\theta), \theta_i) + \sum_{j \neq i} [v_j(k_N^*(\theta), \theta_j) - v_j(k_{N \setminus i}^*(\theta_{-i}), \theta_j)] \\
&= \sum_{j=1} v_j(k_N^*(\hat{\theta}_i, \theta_{-i}), \theta_j) - \sum_{j=1} v_j(k_N^*(\theta), \theta_j) \\
&\leq 0,
\end{aligned}
$$

since

$$
k_N^*(\theta) = \arg \max_{k \in K} \sum_{i \in N} v_i(k, \theta_i).
$$

In other words, the gains from misreporting under the mechanism would only be positive if total utility increased due to the misreport. This is not possible since the mechanism implements the maximum total welfare for truthful reports. As a result reporting $\hat{\theta}_i = \theta_i$ is a dominant strategy for all individuals.

If all individuals play this dominant strategy and report truthfully, the true efficient outcome is implemented by definition in equation (1).

This review of the VCG mechanism shows that there is a way to implement an efficient outcome without knowledge of the distribution of valuations. Any optimal project could be implemented provided one can define all feasible projects, allow agents to report their preferences over feasible projects, solve for the optimal project given reports and solve for the externality payments implied by that allocation. As shown, the requirement for efficiency in VCG is a requirement that the

final outcome be the optimal outcome over the feasible outcomes. The fact that a coarse priority scheme is naturally inferior to a full priority schemes merely constrains the feasible allocations.

In summary, the literature shows that (1) pricing schemes for perfect and coarse rationing exist but rely on previous knowledge of the distribution of valuations and (2) there is a general mechanism that allows us to simultaneously reveal preferences and implement a socially optimal allocation without knowledge of the distribution of valuations.

The gap to be filled is to demonstrate how the VCG mechanism can be applied to selling priority so that a coarse priority scheme can be introduced without this prior knowledge of the distribution of valuations.

3   The model of coarse priority service

This section provides a model of selling priority service. The key to applying the VCG mechanism to selling priority service will be to define assumptions on preferences and on service so that we can describe what is meant by a report and by the set of feasible outcomes. Ultimately this will tell us how the social choice function for the VCG mechanism maps reports to allocations in this application. The most important assumptions are transferrable utility, risk neutrality, unit demand and known supply distribution.

The set up used is similar to that given for a two class scheme in McAfee's paper (2001). In this paper the model allows for discrete valuations and for supply to exceed total population demand. The valuation distribution is discrete for the purpose of applying the set up to a finite vector of reports as would be the case in the practical implementation of VCG. If the functional form of a continuous valuation distribution were eventually known it could be incorporated in all of the computations that follow. The supply is allowed to exceed total demand for generality.

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3.1 Supply

There is a good to be rationed by a central planner, whose units y are indivisible and $y \in Y$ gives the set of all possible supply realisations. It's known supply density is g_y which gives the probability of a particular level of supply in any period. That is $g_y = \Pr\left(\tilde{Y} = y\right)$, where supply $\tilde{Y}$ is a random variable. The good cannot be stored and each supply realisation is rationed costlessly by allowing chosen customers to be served.

3.2 Individuals and preferences

There are n individuals. Each customer has unit demand which implies that when served, customers consume one service unit. When customers are not served they consume zero units.

Each individual obtains zero utility from zero service, and has a privately known valuation v_i of service of one unit. In the setting where only one or zero service units can be served, and individuals face a probability of being served one unit, the utility obtained from an outcome, which is a probability and transfer pair (p_i, t_i), is as follows:

$$u_i = u\left(p_i, v_i, t_i\right) = p_i v_i + t_i \tag{3}$$

That is, utility is assumed to be quasilinear as it was for the VCG derivation and agents are assumed to be risk neutral. A reported type $\hat{\theta}_i$ is therefore equivalent to reporting a valuation $\hat{v}_i$ for a service unit in the sense that it conveys all of the necessary information as to the individuals preferences within the scheme to the planner. Once the valuations of individuals are revealed truthfully, individuals are ordered by $v_1 \leq v_2 \leq \ldots \leq v_n$.

3.3 Priority rules

The feasible number of priority classes, m, is given by the environment. Once the size of each priority class is determined, the rules of service are as follows:

1. Each priority class is served in order. A lower priority class is served only if higher priority classes are entirely served.

2. Following the schemes discussed by both Wilson (1989) and McAfee (2001), service is allocated *randomly* (or uniformly) within each class when service is enough to serve some but not all members of a class. If for example there are 20 members in the top class but a supply realisation is 10, no lower class member would be served and the probability of being randomly selected for service as a top priority member, conditional on this realisation, is $\frac{1}{2}$.

3. All individuals in the service population n are assigned to a class. For example, a two class scheme cannot have a high probability, a low probability, and some customers who are not served at all (zero probability). In this setup that is in fact a third class, and serving a class zero probability is certainly feasible, but the existence of a group being served with zero probability is considered a class.

Given this structure of supply and rules of service, the service probabilities for each individual can be calculated. Given the preferences, total utility from these service probabilities can be defined. The case of two priority classes is presented first and examples throughout focus on this two class case because it is the practical focus of the thesis and its applications. For completeness, generalisation to m classes is given in the following section.

3.4 Service probabilities with 2 priority classes

For a two class rationing scheme there will be high and low priority classes of customers. Let c_1 denote the number of customers in the first class, the high priority class. In the case of only two classes this implies $n - c_1$ customers in the low priority class. The probability of service offered to customer i, $p_{i,y}$, can take one of only two values, high probability denoted $p_{1,y}(c_1)$ and low probability denoted $p_{2,y}(c_1)$. This gives an assignment of probabilities to individuals according to:

$$p_{i,y} = \begin{cases} p_{1,y}(c_1), & i = n - c_1 + 1, \ldots, n, \\ \\ p_{2,y}(c_1), & i = 1, \ldots, n - c_1. \end{cases}$$

The priority rules give service probabilities for each group, conditional on supply realisation y, according to:

$$p_{1,y}(c_1) = \begin{cases} y/c_1, & y \leq c_1 \\ \\ 1, & y > c_1 \end{cases}, \quad p_{2,y}(c_1) = \begin{cases} 0, & y \leq c_1 \\ \frac{y-c_1}{n-c_1}, & c_1 < y \leq n \\ 1, & y > n. \end{cases} \tag{4}$$

That is, if supply is less than sufficient for all high priority customers ($y \leq c_1$) then that supply is randomly allocated within the group. Therefore if $y \leq c_1$ the chance of being selected for service for a high priority group member is y/c_1. If $y > c_1$ all high priority customers are served and the probability of being allocated a unit, conditional on this supply realisation is clearly 1.

Likewise for the low priority group, if $y \leq c_1$ there is insufficient supply for the high priority group and so the low group customers are not served at all (the probability of service is zero). For supply $c_1 < y \leq n$, supply is exceeds the needs of the high group and so excess supply, $y - c_1$, is available to be randomly allocated amongst the low group. The probability of service conditional on this supply realisation is the excess supply divided by the number of low customers, $\frac{y-c_1}{n-c_1}$. If supply exceeds the population ($y > n$) then all customers are served with probability 1.

Finally, taking the expectation across all possible realisations of supply we have the following overall probabilities of service in the first (high) and second (low) priority groups:

$$p_1(c_1) = \sum_y g_y p_{1,y}(c_1) = \sum_{y \leq c_1} \frac{y}{c_1} g_y + \sum_{y > c_1} 1 \cdot g_y, \tag{5}$$

$$p_2(c_1) = \sum_y g_y p_{2,y}(c_1) = \sum_{c_1 < y \leq n} \frac{y - c_1}{n - c_1} g_y + \sum_{y \geq n} g_y. \tag{6}$$

If, in the final outcome, an individual were assigned a probability of 0.8, this would imply that the individual had 80% chance of being selected for service of one unit in any given supply period. The conditions given by equations (5) and (6) describe the set of feasible outcomes for a given supply distribution g_y, service population n and for the case of two priority classes.

The implied expected total utility for the two class scheme is:

$$U(c_1) = p_1(c_1) \sum_{i=n-c_1+1}^{n} v_i + p_2(c_1) \sum_{i=1}^{n-c_1} v_i, \tag{7}$$

which is found as the valuation v_i for each individual weighted by their service probability ($p_1(c_1)$ or $p_2(c_1)$ depending on which class they are in) and then aggregated over all n individuals.

3.5   Service probabilities with m priority classes

The model can be extended to the general m class setting for completeness. The allocations of individuals to the m classes can be represented by a vector $c = (c_1, \ldots, c_m)$, where c_j represents the number of individuals allocated to the top j classes for each $j = 1, \ldots, m$. For example c_1 is the number of individuals in the first class and c_2 is the number of individuals in both the first and second classes. All individuals are allocated to a class so $c_m = n$. The number of individuals in the jth class is given by $c_j - c_{j-1}$. The jth top group includes individuals $n - c_j + 1, \ldots, n - c_{j-1}$. This notation allows a neat representation of the service probabilities for m classes.

The priority rules again give the service probability in each group conditional on each supply realisation. The service probability for the jth class with fixed supply y for given c represented by $p_{j,y}(c)$ is summarised in the following table:

Amount of supply (y)	$p_{1,y}(c)$	$p_{2,y}(c)$	$p_{3,y}(c)$		$p_{m,y}(c)$
$y = 0$	0	0	0	$\ldots$	0
$0 < y \leq c_1$	$\frac{y}{c_1}$	0	0	$\ldots$	0
$c_1 < y \leq c_2$	1	$\frac{y-c_1}{c_2-c_1}$	0	$\ldots$	0
$c_2 < y \leq c_3$	1	1	$\frac{y-c_2}{c_3-c_2}$		
$\vdots$	$\vdots$				
$c_{m-1} < y \leq c_m$	1	1	1	$\ldots$	$\frac{y-c_{m-1}}{c_m-c_{m-1}}$
$y > n$	1	1	1	$\ldots$	1

For example, consider the row $c_2 < y \leq c_3$. This means supply is sufficient to completely serve the top two groups (hence $p_{1,y}(c) = p_{2,y}(c) = 1$) and to partially serve the third group. After serving the top two groups (c_2 individuals) there are $y - c_2$ units of supply left to distribute randomly among the $c_3 - c_2$ individuals in the third group, so the third group has service probability $p_{3,y}(c) = \frac{y-c_2}{c_3-c_2}$. Groups $4, \ldots, m$ have zero probability of service in this case.

The general formula for the service probabilities is therefore

$$p_{j,y}(c) = \begin{cases} 0, & y \leq c_{j-1}, \\ \frac{y-c_{j-1}}{c_j-c_{j-1}}, & c_{j-1} < y \leq c_j, \\ 1, & y > c_j, \end{cases} \tag{8}$$

for some fixed supply $y \in Y$. When $j = 1$ we take the value of c_0 to be zero in this formula.

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Finally, taking the expectation across all possible realisations of supply we have the following overall probabilities of service for each class:

$$p_j(c) = \sum_{c_{j-1} < y \leq c_j} \frac{y - c_{j-1}}{c_j - c_{j-1}} g_y + \sum_{y > c_j} g_y. \tag{9}$$

The implied expected total utility for the m class scheme is

$$U(c) = \sum_{j=1}^{m} p_j(c) \sum_{i=n-c_j+1}^{n-c_{j-1}} v_i. \tag{10}$$

The service probabilities (9) generalisation those in equations (5) and (6) to the case of m classes. Similarly equation (10) generalises (7). These equations fully characterise the feasible service probabilities and their implied expected total utility.

4 Applying VCG

With a model for service allocations and preferences the VCG mechanism can now be applied. The practical steps for applying VCG are given first, then the optimal allocation and VCG payments are derived. The mechanism will ask individuals to report their valuations for service, solve for the socially efficient allocation of priority based on those reports and solve for the associated VCG prices to induce this truthful reporting.

To be complete, the practical steps for applying VCG in the sale of priority would be as follows.

1. The planner will announce all of the rules, the known supply distribution, the length of each supply period, the fixed service population size and number of priority classes to participants.

2. Individuals will be asked to report their types, which in this setting are their valuations for service in any supply period. The planner would advise the individuals that it is an optimal strategy to report one's true valuation, independently of the actions of others, and that no individual can be sure that a misrepresentation wouldn't adversely affect their outcome.

3. The reported valuations will be collected and the implied socially optimal assignment of individuals to classes will be solved (see optimal allocation below).

4. Based on the reports and allocation of service probabilities, VCG payments are calculated for each individual (see VCG payments below).

5. The outcome for each individual will be an allocation to a level of priority, and hence a service probability, and a VCG payment for that priority, based on the reports.

The ultimate allocation will indeed be the optimal allocation for the given number of priority classes provided participants' reports are truthful. The process will be assisted by clear understanding of the process and understanding (or belief) that truthful reporting is optimal.

The optimal allocation and VCG payments are now derived for any number of classes. A common hurdle in applying the VCG mechanism is finding a routine to solve for the optimal allocation and to solve a similar optimisation many times for computing the prices (Krishna, 2002). A Gauss program, provided in the Appendix, was written to solve these optimisation problems and to illustrate the VCG mechanism with any number of classes. Specifically, given a supply distribution, individual valuations and number of priority classes, the program computes the optimal allocation, service probabilities and VCG prices. A simple grid search was used to find the optimal cut-offs. The grid search is convenient to program, it works for examples with any numbers of individuals and classes and it can deal with the stepwise nature of the objective

function. Once the optimal cut-offs are found, the service probabilities and VCG payments are computed from the formulae introduced below.

4.1 Optimal allocation

In order to apply the VCG selling mechanism as outlined above, it is necessary to solve for the optimal action $k_N^* (\theta)$ that satisfies (1). That is, it is necessary to solve for the $k_N^* (\theta)$ that maximises total utility of all feasible allocations. In this case, the optimal action consists of the socially optimal allocation of service probabilities. These service probabilities are themselves completely determined once the allocations of individuals to classes is chosen. The resulting service probabilities then constitute the optimal action $k_N^* (\theta)$.

When there are two priority classes, recall from section 3.4 that there are c_1 high priority and $n - c_1$ low priority customers with service probabilities $p_1 (c)$ and $p_2 (c)$ given in (5) and (6) respectively. The optimal allocation problem is formally expressed as

$$c_1^* \in \arg \max_{c_1 \in \{1,\dots,n-1\}} U (c_1), \tag{11}$$

where $U (c_1)$ is given in (7). The resulting service probabilities $p_1 (c_1^*)$ and $p_2 (c_1^*)$ then constitute the optimal action $k_N^* (\theta)$.

The allocation in the case of m classes is determined by $c = (c_0, c_1, \dots, c_m)$. To formally express this optimisation, we define

$$c^* \in \arg \max_{c \in C} U (c), \tag{12}$$

where $C = \{(c_1, \dots, c_m) : c_j > c_{j-1}, c_m = n\}$ represents the set of feasible allocations and $U (c)$ is the total expected valuation defined in (10). The resulting service probabilities $p_j (c^*)$, $j = 1, \dots, m$, then constitute the optimal action $k_N^* (\theta)$. The resulting optimal valuation is $U (c^*)$.

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4.2 Payments

4.2.1 First term

It is clear how to find the first term, $\sum_{j \neq i} v_j(k_N^*(\theta), \theta_j)$, for t_i in equation (2) for the coarse priority schemes. It is the value of the optimal utility for the scheme when everyone is considered less the expected value that i receives from the scheme. The first term of the VCG payment formula (2) is therefore

$$\sum_{j \neq i} v_j(k_N^*(\theta), \theta_j) = U(c^*) - v_i p^{(i)}(c^*),$$

where $p^{(i)}(c^*) = p_j(c^*)$ if individual i is optimally allocated to class j.

4.2.2 Second term

For the second term, $\sum_{j \neq i} v_j(k_{N \setminus i}^*(\theta_{-i}), \theta_j)$, it is not immediately clear how we should treat i to find the optimal allocation without i. Is it found by optimising over the set $N \setminus i$ of individuals or by treating $i's$ valuation as zero and optimising over the full set N? A review of the VCG literature gives mixed definitions for how this term should be solved. Krishna (2002, p.225) notes that the payment is the utility that would result if i reports $v_i = 0$, but that in many settings it is also equivalent to considering utility if i were not present. Milgrom (2004, p.49) also refers to the possibility of excluding individual i when allocating a good or to treat individuals $i's$ valuation as 0 in other circumstances.

In this setting, the two options are not equivalent and the externality payment is correctly found by treating $i's$ valuation as zero. The results from the VCG mechanism rely on making efficient choices from the feasible possibilities. Since all individuals must be assigned to a priority class, excluding i from the service population and optimising over $n - 1$ individuals is not within

the set of feasible outcomes. Therefore the optimal feasible outcome when excluding $i's$ report is found by setting $i's$ report to zero.

Setting i's valuation to zero ensures that individual would be served the lowest offered priority. For the purpose of computing this term of t_i, it is necessary to set $v_i = 0$, reorder the individuals, and recalculate the optimal utility over all n individuals being served.

The second term of the payment formula is therefore

$$\sum_{j \neq i} v_j(k^*_{N \setminus i}(\theta_{-i}), \theta_j) = \sum_{j \neq i} v_j p^{(j)} \left(c^*_{-i} \right),$$

where c^*_{-i} satisfies (12) but with v_i set to zero.

The VCG payments defined in (2) for this application are therefore

$$t_i = \left(U \left(c^*_1 \right) - v_i p^{(i)} \left(c^* \right) \right) - \sum_{j \neq i} v_j p^{(j)} \left(c^*_{-i} \right). \tag{13}$$

4.3 Individual rationality

Since it was assumed that the service population is fixed prior to implementation, it is useful to confirm that the VCG mechanism is individually rational when applied to these coarse priority schemes. To see this note that from equations (10) and (13) that:

$$v_i(p^{(i)} \left(c^* \right) - p^{(j)} \left(c^*_{-i} \right)) + \sum_{j \neq i} v_j(p^{(j)} \left(c^* \right) - p^{(j)} \left(c^*_{-i} \right) \geq 0$$

for any i, so

$$v_i(p^{(i)} \left(c^* \right) + \sum_{j \neq i} v_j[(p^{(j)} \left(c^* \right) - p^{(j)} \left(c^*_{-i} \right)] \geq v_i p^{(j)} \left(c^*_{-i} \right)).$$

This means that individual utility from the scheme is non negative:

$$u_i = v_i p^{(i)} \left(c^* \right) + \sum_{j \neq i} v_j[(p^{(j)} \left(c^* \right) - p^{(j)} \left(c^*_{-i} \right)] \geq 0$$

and the schemes are individually rational.

4.4 Two class example

The procedure is demonstrated using an example for two priority groups, a population $n = 10$, reported valuations

$$v = \{10, 10, 20, 30, 50, 60, 70, 100, 140, 180\},$$

and a truncated Poisson supply distribution with probabilities $g_y = f_y / \sum_{y=0}^{12} f_y$ for $y = 0, 1, \ldots, 12$, where f_y is the probability mass function of the Poisson distribution with parameter $\lambda = 8$. This distribution is shown in Figure 1. These choices of population size, valuations and supply distribution are all arbitrary, the program can accept any other choices.

Here the optimal number of high priority individuals is $c_1^* = 6$. Probability of service in the high and low groups are $p_1(6) = 0.938$ and $p_2(6) = 0.446$. Ultimately this gives overall maximised utility $U(c^*) = 593.78$. By way of comparison, the first best allocation with full priority gives total utility of 614.55 as will be demonstrated later.

To illustrate the calculation of the VCG payments take individual 5, with $v_5 = 50$. This individual is allocated to the high group and so is allocated a probability $p_1(c^*) = 0.938$. The first term in the payment for individual 5 is $U(c^*) - p^{(i)}(c^*) v_i = 593.78 - (0.938).50 = 546.88$. For the calculation of the second term of the payment for individual 5, we find the optimal two tiered scheme where the individual's utility is set to zero. This achieves total utility 567.40. Combined, this gives an overall payment for the 5th individual $t_5 = 546.88 - 567.40 = -20.52$. That is they must pay \$20.52, per supply period, for their final allocation in the optimal two tiered scheme, which was probability of service in each period of $p_1(c^*) = 0.938$.

The full list of payments (in absolute value) are given in Table 1.

Ultimately these prices just reflect each individual's externality which depends on the particular valuations and supply distribution. For example, the fact that the low priority customers

Table 1: Example service probabilities and VCG prices for two classes

Individual	Valuation	Service Probability	VCG Payment
10	180	0.938	16.82
9	140	0.938	17.95
8	100	0.938	19.09
7	70	0.938	19.94
6	60	0.938	20.22
5	50	0.938	20.52
4	30	0.446	0.00
3	20	0.446	0.00
2	10	0.446	0.00
1	10	0.446	0.00

...   First priority class

...   Second priority class

pay nothing here is not always the case; it is just the case for this particular supply and valuation distributions. An individual's externality in a coarse priority scheme includes (1) the effect of occupying a valuable priority position and (2) the impact on the selection of groups. As we will see later, depending on the distributions, all individuals can potentially affect the outcome and therefore be required to make a payment for their externality on others.

Several properties of the prices are always seen. Firstly, notice that high priority customers are indeed the customers paying significantly more than the low priority customers and this is always the case. When we move to more than two classes we will similarly see that higher groups always pay more than lower groups.

Secondly, notice that each individual does indeed obtain positive utility from the scheme above. Take individual 5, with the highest payment, valuation 50, probability of service 0.938 and payment $20.52. Expected utility from the scheme is therefore $u_5 = 50(0.938) - 20.52 > 0$. As seen by looking at individual 10, the expected utility can indeed be substantial.

Often the prices actually increase from the top within a group, as happens in this example for the top priority customers. Notice above that the 6th ranked member of the high group pays a higher price than the 1st ranked individual although the 6th ranked individual has a lower valuation. This seems unusual but it reflects the fact that including a marginal member to the high group can impose a relatively large and negative external effect on existing high group members by lowering the overall service in the group.

Finally, these results confirm what has been said previously about the relative efficiency of these coarser schemes. McAfee proved that a simple two class scheme that used the mean as a cut-off would achieve at least 50% of the possible gains from random allocation to full priority. In this

particular example, a mean cut-off scheme would actually achieve 76.2% of the improvements[5] if it could be implemented[6]. The optimal two class scheme in this example achieves 82.4% of the gains from full priority. These coarse two class schemes do achieve surprisingly large gains. The diminishing returns to increasing the number of classes suggested by Wilson (1989) will be illustrated as we look at the extension of the example to m priority classes.

4.4.1 m class example

Continuing with the example given for two classes, Table 2 gives utility and VCG prices for $m = 2$, $m = 3$, $m = 4$ classes and for perfect rationing, $m = 10$ classes (with $n = 10$).

For $m = 3$ the optimal allocation in this case is specified by cut-offs at $c^* = (3, 6, 10)$ implying a top priority class with 3 individuals, a middle class also with 3 individuals and a low priority class with 4 individuals. The expected service probabilities are respectively 0.994, 0.881 and 0.446 for these classes. The maximised expected utility with 3 classes is $U(c^*) = 607.29$. For $m = 4$ classes the optimal allocation in this case is specified by $c^* = (3, 6, 8, 10)$. The maximised expected utility with 4 classes is $U(c^*) = 611.64$.

---

[5] In this setting where supply can exceed total demand, random rationing implies that for given supply y, each individual is served probability (y/n) if $y \leq n$, and 1 if $y > n$. The expected probability of service is

$$p = \frac{1}{n} \sum_{0 \leq y \leq n} y g_y + \sum_{y > n} g_y.$$

The expected utility obtained from random rationing is simply $U = p \sum_{i=1}^{n} v_i$.

[6] Notice that it would be difficult to implement McAfee's mean cut-off scheme using the VCG mechanism because it is hard to find constraints on the feasible projects K that would implement the mean as the socially optimal outcome. It is straight forward to implement a median cutoff using the VCG mechanism; simply by using the constraint on feasible projects that there is an equal number in each group. Having said that, since we can achieve the "optimal" two class scheme, why wouldn't you?

Table 2: Example service probabilities and VCG prices

		2 classes		3 classes		4 classes		10 classes	
$U(c^*)$		593.78		607.29		611.64		614.55	
% of full priority		82.4%		93.9%		97.5%			

i	v_i	$p_i(c^*)$	t_i	$p_i(c^*)$	t_i	$p_i(c^*)$	t_i	$p_i(c^*)$	t_i
1	180	0.938	16.82	0.994	21.68	0.994	21.01	1.000	21.96
2	140	0.938	17.95	0.994	21.28	0.994	21.01	0.997	21.56
3	100	0.938	19.09	0.994	20.95	0.994	21.18	0.985	20.41
4	70	0.938	19.94	0.881	13.07	0.881	13.13	0.955	18.27
5	60	0.938	20.22	0.881	13.75	0.881	13.56	0.894	14.60
6	50	0.938	20.52	0.881	13.07	0.881	13.99	0.796	9.71
7	30	0.446	0.00	0.446	0.00	0.591	2.93	0.665	5.80
8	20	0.446	0.00	0.446	1.18	0.591	3.67	0.516	2.82
9	10	0.446	0.00	0.446	0.44	0.301	0.00	0.367	1.33
10	10	0.446	0.00	0.446	0.44	0.301	0.00	0.235	0.00

... First priority class

... Second priority class

... Third priority class

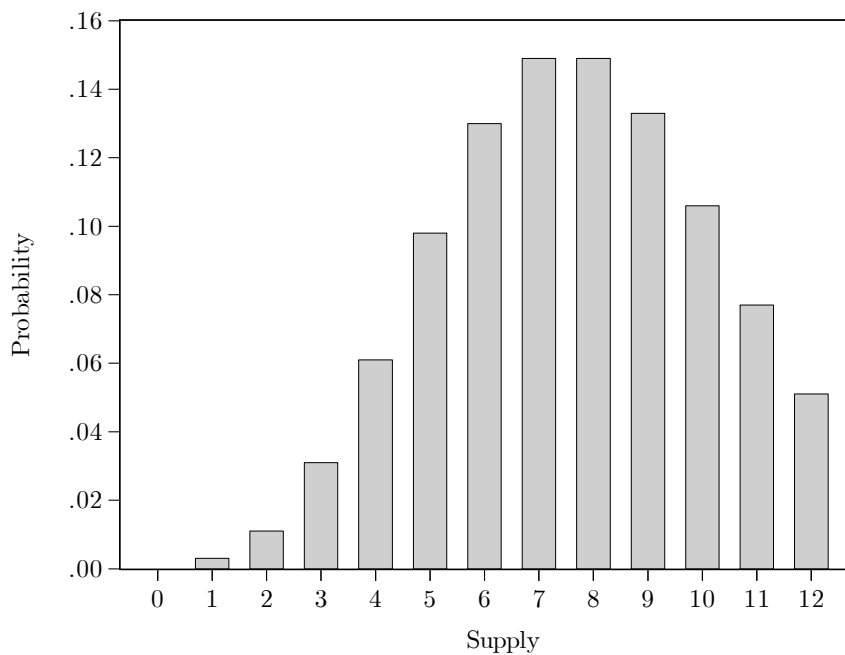... Fourth priority class

Figure 1: Truncated Poisson supply distribution

For $m = 10$, perfect rationing, each individual has a unique service probability and a unique VCG payment. Individuals no longer exert pooling externalities so that the VCG payments only reflect the fact that each individual occupies a valuable position in a queue. This represents the maximum obtainable utility and provides a useful point of comparison for the results based on coarse rationing.

As suggested above, the efficiency improvements as reported in the top section of Table 2 for increasing the number of classes is diminishing rapidly. For example, in this particular case, rationing with 3 classes provides 93.9% of the gains from random to full priority rationing. Since fewer classes will make administration and implementation simpler, and since in many circum-stances only a limited number of classes is feasible, it is extremely useful to know that these coarse schemes are so effective.

5 Limitations and further research

VCG prices are commonly volatile and seemingly unfair (Milgrom, 2004). This is evidently an issue when applying VCG to priority service since we saw prices that could move in any way depending on the supply and valuation distribution and often decrease for higher valuing customers. Further, the mechanism might seem complex to participants. In this particular VCG mechanism, reporting types, a single valuation, is relatively simple. The allocation decision on the other hand is not simple.

Whilst fewer classes make the mechanism less complex, finding a more familiar institution than the VCG mechanism shown here is an important direction for future research. A uniform or ascending price format that is outcome equivalent would be highly desirable. An extension of the Ausubel auction, an ascending price auction that is outcome equivalent to the Vickrey auction (Krishna, 2002, p173), might be feasible in this setting. In any event, a practical format that is equivalent but more familiar to participants would be a useful extension of this research.

Trials of the mechanism in small service districts would help to test the functioning of the mechanism in each particular case. This would be informative for both participants and the planner, since the process will be new to both. The trials may be all that is necessary since once the valuation distribution is revealed there is no need to continue the process. Previously derived pricing schedules could be used for selling priority or even simpler schemes may become apparent once the distribution is known, particularly if there is some obvious pooling of individuals.

Although this paper actually makes less assumptions than previous literature, in that it does not require knowledge of the valuation distribution, the assumptions including known supply distribution and unit demand will need to be reasonable for the mechanism to be appropriate in practice. The suitability of these assumptions will depend on each case.

Completing the application of water rationing, water authorities closely monitor the probability of various supply shortfalls and an example of a supply distribution for the ACT region is given by ABARE (2008). Unit demand, or that households consume approximately equivalent volumes of outdoor water when on the same level of restrictions, is actually assumed often in current planning (see ACTEW Corporation (2005), and DSE (2007)). This would clearly be an approximation. However households with extremely large outdoor areas could be asked to pay a premium for their larger use. For the purpose of planning and managing water supply, these assumptions of known supply and unit demand appear to be plausible.

Further assumptions were risk neutrality and quasilinear preferences. The assumption of quasilinear preferences is standard in the literature and is reasonable if individuals do not face significant budget constraints (Milgrom, 2004, p.46). It is useful to note that in the case of water rationing, the average household water bill is less than 1% of overall household consumption (ABS, 2004). Especially with the payments for priority ultimately being very small relative to reports, it is unlikely that budget constraints would be of concern when selling priority for water customers.

Addressing the assumption of risk neutrality could be an avenue for further research, although it is difficult to imagine definitions for optimal allocations of service probability and a practical way to report types where individuals are not assumed to be risk neutral. A consideration of bidding behaviour and optimal allocations within these coarse priority schemes when agents are not risk neutral remains open for future research.

6   Conclusion

The results have shown a theoretical way to implement coarse priority schemes without prior knowledge of the valuation distribution. Implementation without prior knowledge of the valuation distribution had not been discussed previously in the literature and the VCG mechanism was

a useful starting point for these problems. The approach demonstrated in this thesis will be particularly useful where a priority scheme is being introduced for the first time and this information is unknown.

Coarse priority schemes were chosen because in practice only few classes may be feasible. Previous literature suggests that coarse schemes achieve large parts of the gains from full priority schemes and my results reflect this fact.

I have set up a model of coarse rationing schemes for any number of priority classes and for a finite population. I have applied the VCG mechanism to the selling of priority for a fixed number of priority classes and a given supply distribution. I have provided numerical illustration of the feasibility of the implementation including examples of optimal allocation and VCG prices for 2, 3, 4 and 10 class priority schemes. I have provided a program that can solve the optimal allocation of priority and the appropriate VCG prices to implement that optimal allocation for any number of priority classes.

Using this VCG application to selling priority will not only ensure such schemes are implemented optimally, it will reveal information that is extremely useful for future planning decisions. The benefits of increasing the number of priority classes are precisely shown using this machinery. The willingness to pay for overall supply upgrades can easily be seen and can also be illustrated using this machinery. Whether such a scheme is used in practice or in trials, implementation without prior knowledge of the valuation distribution could be extremely useful when considering introducing a coarse priority scheme.

In 2008 the Productivity Commission suggested introducing a choice of water service probability, where no choice has existed before. Due to the current monitoring technology, only a two tiered scheme was likely to succeed at first. Not surprisingly previous surveys did not measure

the distribution of valuations because a choice of service had not been an important issue. The procedure for implementing this optimal two class scheme that was emphasised throughout the paper could readily be used to finally offer urban water customers this choice in their probability of water service.

References

ABS (2004) Water Account Australia 2000-01Cat. no. 4610.0,2005, 2003-2004 Household Expenditure Survey, Canberra. Canberra

ACTEW CORPORATION, A. (2005) Future water options for the ACT region – implementation plan: a summary of the recommended strategy to increase the ACT's water supply.

AFACCHE, P. & MENDELSON, H. (2004) Pricing and Priority Auctions in Queueing Systems with a Generalized Delay Cost Structure. Management Science, 50, 869-882.

ALLEN CONSULTING GROUP, (2007) Saying goodbye to permanent water restrictions in Australia's cities: key priorities for achieving water security.

BOGOTA D.C. (2009) Pico y placa 2009 - 2010. Retrieved October 19, 2009 from http://www.bogota-dc.com/trans/bog-tra.htm

BLAMEY, R., GORDON, J. & CHAPMAN, R. (1999) Choice modelling: assessing the environmental values of water supply options. Australian Journal of Agricultural & Resource Economics, 43, 337.

BRENNAN, D., TAPSUWAN, S. & INGRAM, G. (2007) The welfare costs of outdoor water restrictions. Australian Journal of Agricultural and Resource Economics, 51, 243-61.

CLARKE, E. H. (1971) Multipart pricing of public goods. Public Choice, 11, 17-33.

COLOMBIA REPORTS (2009) Unrest in Bogotá over "pico y placa". Retrieved October 19, 3009 from http://www.colombiareports.com/colombia-news/news/2958-unrest-in-bogota-over-pico-y-placa.html

GRAFTON, R., QUENTIN & WARD, M. B. (2008) Prices versus rationing: Marshallian surplus and mandatory water restrictions. Economic Record, 84, 57-65.

GRIFFIN, R. C. & MJELDE, J. W. (2000) VALUING WATER SUPPLY RELIABILITY. American Journal of Agricultural Economics, 82, 414.

GROVES, T. (1973) Incentives in Teams. Econometrica, 41, 617-631.

HENSHER, D., SHORE, N. & TRAIN, K. (2005) Households' Willingness to Pay for Water Service Attributes. Environmental and Resource Economics, 32, 509-531.

HENSHER, D., SHORE, N. & TRAIN, K. (2006) Water Supply Security and Willingness to Pay to Avoid Drought Restrictions. Economic Record, 82, 56-66.

HUNG-PO CHAO, S. O., STEPHEN SMITH AND ROBERT WILSON ((1986)) Multilevel demand subscription Pricing for Electric Power. Energy Economics, 4, 199-217.

KRISHNA, V. (2002) Auction Theory, California, Academic Press.

MCAFEE, R. P. (2002) Coarse Matching. Econometrica, 70, 2025-2034.

MILGROM, P. (2004) Putting Auction Theory to Work, New York, Cambridge University Press.

OREN, S., SMITH, S. & WILSON, R. (1985) Econometrica. Econometrica, 53.

THE PRODUCTIVITY COMMISSION, (2008) Towards Urban Water Reform: A Discussion Paper. IN COMMISSION, T. P. (Ed.) Melbourne, The Productivity Commission.

SHIMER, R. A. S., LONES (2001) Matching, Search, and Heterogeneity. Advances in Macroeconomics, 1, 5.

VICKREY, W. (1961) Counterspeculation, Auctions and Competitive Sealed Tenders. Journal of Finance, 16, 8-37.

VICTORIAN GOVERNMENT, (2003) Securing Our Water Future. IN ENVIRONMENT, D. O. S. A. (Ed.) Melbourne.

VICTORIAN GOVERNMENT, (2007) Our water, our future: the next stage of the government's water plan. IN ENVIRONMENT, D. O. S. A.

VICTORIAN WATER INDUSTRY ASSOCIATION INC (2005) Victorian Uniform Drought Water Restriction Guidelines. IN ENVIRONMENT, D. O. S. A. (Ed.).

WILSON, R. (1989) Efficient and Competitive Rationing. Econometrica, 57, 1-40.

Appendix

Gauss program for the optimal allocation and VCG prices

```
new; library pgraph; graphset; rndseed 42; format /rd 10,3; et=hsec;


// Number of individuals and their vector of valuations

N = 10;

vv = {1,1,2,3,5,6,7,10,14,18}; vv = vv*10; vv=rev(vv);


// Poisson supply distribution

U = 12; x = seqa(0,1,U+1); theta = 8;

g = exp(-theta + x.*ln(theta) - lnfact(x));

g = g/sumc(g);


// 1 class

hmax = n;

{EUkmax,pmax} = EUk(hmax,vv);

print; "*********************************"; rows(hmax) "classes";

"Umax" EUkmax;

" Class No.indiv.  Prob.  service";

seqa(1,1,rows(hmax))~hmax~pmax;

// Total maximised utility minus vector of individual utilities

t1 = zeros(n,1); i=1;

for j (1,rows(hmax),1);

 for k (1,hmax[j],1);

 t1[i] = EUkmax - vv[i]*pmax[j];

 i = i + 1;
```

```
 endfor;

endfor;

// Re-maximise excluding each individual one at a time

EUiv = zeros(n,1);

for i (1,n,1);

 vvi = vv; vvi[i] = 0; vvi = trimr(rev(sortc(vvi,1)),0,0);

 {EUiv[i],pmax} = EUk(n,vvi);

endfor;

// Prices, rounded to 10d.p.

price = round((EUiv-t1)*1e10)*1e-10;

print;" Individual Valuation Price";;

seqa(1,1,N)~rev(vv~price);


// 3 classes

EUkmax = 0;

for h1 (1,n-2,1);

 for h2 (1,n-h1-1,1);

 hv = h1|h2|(n-h1-h2);

 {EUh,ph} = EUk(hv,vv);

 if EUh > EUkmax; EUkmax = EUh; hmax = hv; pmax=ph; endif;

 endfor;

endfor;

print;"*********************************"; rows(hmax) "classes";

"Umax" EUkmax;

" Class No.indiv.  Prob.   service";;
```

```
seqa(1,1,rows(hmax))~hmax~pmax;

// Total maximised utility minus vector of individual utilities

t1 = zeros(n,1); i=1;

for j (1,rows(hmax),1);

 for k (1,hmax[j],1);

 t1[i] = EUkmax - vv[i]*pmax[j];

 i = i + 1;

 endfor;

endfor;

// Re-maximise excluding each individual one at a time

EUiv = zeros(n,1);

for i (1,n,1);

 vvi = vv; vvi[i] = 0; vvi = rev(sortc(vvi,1));

 for h1 (1,n-2,1);

 for h2 (1,n-h1-1,1);

 hv = h1|h2|(n-h1-h2);

 {EUh,ph} = EUk(hv,vvi);

 if EUh > EUiv[i]; EUiv[i] = EUh; endif;

 endfor;

 endfor;

endfor;

// Prices, rounded to 10d.p.

price = round((EUiv-t1)*1e10)*1e-10;

print;" Individual Valuation Price";;

seqa(1,1,N)~rev(vv~price);
```

```
print; "*********************************"; "n groups (i.e.  perfect)";;

hmax = ones(n,1);

{EUkmax,pmax} = EUk(hmax,vv);

print; rows(hmax) "classes"; "Umax" EUkmax; " Class No.indiv.  Prob.  service";;

seqa(1,1,rows(hmax))~hmax~pmax;

// Total maximised utility minus vector of individual utilities

t1 = zeros(n,1); i=1;

for j (1,rows(hmax),1);

 for k (1,hmax[j],1);

 t1[i] = EUkmax - vv[i]*pmax[j];

 i = i + 1;

 endfor;

endfor;

// Re-maximise excluding each individual one at a time

EUiv = zeros(n,1);

for i (1,n,1);

 vvi = vv; vvi[i] = 0; vvi = trimr(rev(sortc(vvi,1)),0,0);

 {EUiv[i],pmax} = EUk(ones(n,1),vvi);

endfor;

// Prices, rounded to 10d.p.

price = round((EUiv-t1)*1e10)*1e-10;

print;" Individual Valuation Price";; seqa(1,1,N)~rev(vv~price);

end;


proc 2=EUk(h,vv);
```

```
local p,hh,x,j,EU;

p = zeros(U+1,rows(h)); hh = 0|cumsumc(h);

for x (1,U,1);

 for j (1,rows(h),1);

 if (x>hh[j]) and (x le hh[j+1]);

 p[x+1,j] = (x-hh[j])/h[j];

 elseif (x>hh[j]);

 p[x+1,j] = 1;

 endif;

 endfor;

endfor;

p=p'g; EU = 0;

for j (1,rows(h),1);

 EU = EU + p[j]*sumc(vv[hh[j]+1:hh[j+1]]);

endfor;

retp(EU,p);

endp;
```