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Representing Risk Preferences in Expected Utility Based Decision Models

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Abstract: The application and estimation of expected utility based decision models would benefit from having additional simple and flexible functional forms to represent risk preferences. The literature so far has provided these functional forms for the utility function itself. This work shows that functional forms for the marginal utility function are as useful, are easier to provide, and can represent a larger set of risk preferences. Several functional forms for marginal utility are suggested, and how they can be used is discussed. These marginal utility functions represent risk preferences that cannot be represented by any functional form for the utility function.

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1. Introduction

For many years, those working with expected utility (EU) based decision models have recognized that these models would have additional uses and applications if a simple and flexible functional form for the utility function were available. A number of such functional forms have been suggested. Arrow (1965, 1971) and Pratt (1964) identified the constant absolute (CARA) and constant relative (CRRA) averse forms, Merton (1971) the hyperbolic absolute risk averse (HARA) form, Saha (1993) the expo-power (EP) form, Xie (2000) the power risk aversion (PRA) form, and Conniffe (2006) suggests the flexible three parameter (FTP) form. In addition, LiCalzi and Sorato (2006) and Conniffe (2007) have suggested using functional forms originally defined to describe cumulative probability distribution functions as utility functions instead.

As that literature suggests, an ideal functional form for the utility function would be one which is simple, and yet flexible. Both the utility function and the associated risk aversion measures should be simple, and by appropriate choice of parameter values, the functional form should allow a wide variety of risk preferences to be represented. The functional forms for the utility function that have been suggested so far attempt to meet this goal, and each is presented as an improvement upon those that precede it.

The progression of suggested functional forms for the utility function has, in general, reduced the simplicity of the function in favor of increasing the functional form's ability to represent a wider range of risk preferences. The CARA and CRRA forms of Arrow and Pratt are very simple, but only allow the magnitude of risk aversion to vary. The HARA, EP, PRA and FTP forms allow the magnitude of risk aversion to vary, and also allow the slope of the risk aversion measures to be positive or negative. The cost of this increased ability to represent risk preferences, however, is increased complexity. The FTP form is the most general of all of the functional forms, but it is also the most complex.

The goal of this research is also to find functional forms which are simple and flexible. By appropriate choice of parameter values these forms can represent a wide variety of risk preferences. The approach taken here, however, is quite different from that followed in the existing literature. This work does not give another flexible functional form for the utility function. Instead, functional forms for the marginal utility function are presented and discussed. The logical basis for doing this is very simple. It is well known that marginal utility functions can be used, without loss, to represent the risk preferences of an expected utility maximizing decision maker. Therefore, to simply and flexibly represent risk preferences, it is not necessary to specify the utility function, giving the marginal utility function instead will suffice. The basic problem of finding a simple and flexible functional form for representing risk preferences in an EU decision model can be solved by specifying the marginal utility function rather than the utility function itself.

The findings presented here add several things to the existing literature. First, several additional solutions to the basic underlying question are presented. Simple and flexible functional forms for marginal utility are given to represent the risk preferences of EU maximizing decision makers. Second, because the anti-derivatives of these marginal utility functions cannot be represented using elementary functions; that is, there is no closed form for the anti-derivative, the risk preferences these functional forms for marginal utility represent are preferences which cannot be represented by any utility function. This work therefore allows previously excluded risk preferences to be incorporated into an empirical or applied EU decision model. Third, this research explains why finding a simple and flexible functional form for the utility function is a difficult task, while for the marginal utility function, this task is much easier to accomplish. Finally, even though theory indicates that marginal utility is sufficient to represent risk preferences in an EU decision model, this work shows that this is true in practice as well. That is, knowing only the marginal utility function is shown to be sufficient in empirical and applied work. This is accomplished by verifying that marginal utility can be used to address a variety of questions posed in expected utility based decision models

The paper is organized as follows. First, the literature concerning functional forms for the utility function is briefly reviewed. The CARA, CRRA, HARA, EP, PRT, and FTP functional forms are described. Following this, the procedure used to obtain a utility function from a risk aversion measure is discussed in some detail. There are three main steps in this procedure, with the final step being the transformation of the marginal utility function into the utility function. It is observed that this final step is the one that most severely restricts the risk preferences that can be represented by a utility function. Section 4 observes that in theory marginal utility is a complete representation of the risk preferences of an EU decision maker. In addition, however, this section presents a series of EU based decisions, and indicates how marginal utility is sufficient to determine and analyze the particular decision being discussed. Finally, the paper concludes with examples of flexible functional forms for marginal utility functions and the risk preferences that can be represented using them.

2. Functional Forms for the Utility Function

Arrow (1965, 1971) and Pratt (1964) each present the functional form for a utility function which represents risk preferences whose measure of absolute risk aversion is a constant. This constant absolute risk averse (CARA) family of utility functions takes the negative exponential form; that is u(x) = $-e^{-\alpha x}$, where A(x) = α is the level of absolute risk aversion associated with this utility function and these risk preferences. The CARA form allows the user to specify or estimate the level of absolute risk aversion under the assumption that absolute risk aversion is constant.

Arrow and Pratt also present the functional form for utility functions which represent risk preferences whose measure of relative risk aversion is constant. These constant relative risk averse (CRRA) risk preferences are represented by a utility function of the power function form. This utility function is given by $u(x) = \frac{x^{1-\beta}}{1-\beta}$, where β is the level of relative risk aversion, and $A(x) = \frac{\beta}{x}$ is the absolute risk aversion measure. For the case of $\beta = 1$, u(x) = ln x.

These two functional forms for utility presented by Arrow and Pratt allow complete freedom in specifying the magnitude of risk aversion, absolute or relative, but give no flexibility is specifying the slopes of these risk aversion measures. As is well known, the CARA form implies a zero slope for the absolute risk aversion measure, but yields a positive slope for the relative risk aversion measure. Similarly, CRRA implies that the relative risk aversion measure has zero slope, but that the absolute risk aversion measure decreases at rate $\frac{-\beta}{x^2}$.

These two functional forms for utility are each quite simple, involve just one parameter, and make analysis in many economic settings quite tractable. The risk aversion measures associated with these forms are also very simple. As a consequence, these two functional forms for utility and the risk preferences they represent have been frequently used in the literature examining EU based decisions.

Merton (1971) introduces a flexible functional form for utility which allows a wider range of risk preferences to be represented than those available under CARA or CRRA. This functional form involves three parameters rather than one, and is less simple to manipulate and use. The class of risk preferences represented, and the associated set of utility functions, is referred to as the hyperbolic absolute risk averse (HARA) family. These utility functions are given by $u(x) = \frac{(1-\gamma)}{\gamma}(\frac{\beta \cdot x}{(1-\gamma)} + \eta)^{\gamma}$. The parameters are subject to the restrictions that $\gamma \neq 1$, $\beta > 0$, $(\frac{\beta \cdot x}{(1-\gamma)} + \eta) > 0$, and $\eta = 1$ if $\gamma = -\infty$. The measure of absolute risk aversion for this family of utility functions is given by $A(x) = \frac{1}{\frac{x}{1-\gamma} + \frac{\eta}{\beta}}$. The slopes of the absolute and relative risk aversion measures for this functional form can be made positive or negative with appropriately selected values for the three parameters. The CRRA functional form results when the parameter $\eta = 0$, and the CARA form is obtained in the limit as the parameter γ goes to infinity. Merton illustrates the usefulness of this functional form by showing that it is analytically convenient for the particular expected utility based consumption model he is analyzing. Clearly, the HARA functional form can represent a wider range of risk preferences than either the CARA or CRRA forms, but the HARA form is also more complex and less easily manipulated. The absolute risk aversion measure, however, is quite simple, and is a constant divided by a linear function of x.

Saha (1993) introduces the expo-power (EP) functional form for utility. This EP form for utility is given by $u(x) = \theta - \exp(-\beta \cdot x^{\alpha})$, where the three parameters are restricted so that $\theta > 1$, and $\alpha \cdot \beta > 0$. The absolute risk aversion measure is $A(x) = \frac{(1 - \alpha + \alpha \cdot \beta \cdot x^{\alpha})}{x}$. Notice that the parameter θ does not appear in the absolute risk aversion expression. This parameter is simply an additive constant used to alter the magnitude of utility, and is not used to represent a wider variety of risk preferences. Thus, this EP form is actually a two parameter form. As with the HARA form, appropriate choices for the parameters, α and β, allow positive or negative slopes for absolute and relative risk aversion. Saha mentions that this EP functional form reduces to the CARA form with finite parameter values, which has certain advantages over the HARA form of Merton. Saha, Shumway and Talpaz (1994) use this EP functional form when estimating the magnitude and slope of absolute and relative risk aversion for wheat farmers in Kansas.

Xie (2000), without reference to the EP functional form of Saha, defines the power risk aversion (PRA) functional form for utility as an improvement upon the HARA form of Merton. The PRA utility function as given by Xie is $u(x) = \frac{1}{\gamma}\left\{1-\exp\left[-\gamma\left(\frac{x^{1-\sigma}-1}{1-\sigma}\right)\right]\right\}$. There are two parameters in this form and they are restricted so that $\sigma \geq 0$ and $\gamma \geq 0$. This PRT functional form is also sufficiently flexible so that by choosing σ and γ appropriately, the measures of absolute and relative risk aversion can be made to increase or decrease. Xie argues that the PRA form is an improvement over HARA in that it remains well defined for more values for the argument of the utility function. The absolute risk aversion measure for the PRA form is quite simple and is given by $A(x) = \frac{\sigma}{x} + \gamma \cdot x^{-\sigma}$. When one compares the absolute risk aversion measures of the PRA and EP forms, it is clear the PRA form of Xie is the same as the EP form of Saha. To translate parameters, notice that Xie's $\sigma = 1 - \alpha$ of Saha, and Xie's $\gamma = \alpha \cdot \beta$ of Saha.

Conniffe (2006) defines the flexible three parameter (FTP) functional form for utility. This form generalizes the PRA form of Xie. In Conniffe's formulation, utility is given by

$u(x) = \frac{1}{\gamma}\left\{1 - \left[1 - k\gamma\left(\frac{x^{1-\sigma} - 1}{1 - \sigma}\right)\right]^{\frac{1}{k}}\right\}$. When k = 0, this reduces to the PRA form of Xie. The absolute risk aversion measure for the FTP form is given by $A(x) = \frac{\sigma}{x} + \frac{(1 - k)\gamma \cdot x^{-\sigma}}{1 - k\gamma\left(\frac{x^{1-\sigma} - 1}{1 - \sigma}\right)}$, and is quite complex.

Examination of this literature concerning functional forms for utility functions indicates several things. First, the CARA and CRRA forms of Arrow and Pratt are indeed simple and the associated risk aversion measures are also simple, but these functional forms lack the flexibility to represent a sufficiently wide range of risk preferences. Second, the proposed alternatives, HARA, EP/PRT or FTP, are quite complex, but do represent a wider variety of risk preferences. It is also the case that the risk aversion measures associated with the HARA, EP/PRT or FTP functional forms are themselves more complex than for CARA or CRRA. Finally, the literature developing these functions presents very little in the way of discussion concerning how the functional form was determined. Functional forms for utility functions tend to be presented and their properties analyzed, without discussion of their source. How to go about finding utility and marginal utility functions to represent risk preferences is the main topic of the next section.

3. Finding Functional Forms to Represent Risk Preferences

The risk preferences of an expected utility maximizing decision maker can be represented in a variety of different ways. The usual method of representation is to specify the utility function itself. The utility function completely describes or represents the risk preferences of the decision maker, but as is well known, this representation is not unique. Whenever u(x) represents a set of risk preferences, then so does a + b·u(x), for any value for a, and for b > 0. What is less well known is that there are many sets of risk preferences whose utility function cannot be expressed in terms of elementary functions; that is, there is no closed form for the utility function.

The marginal utility function, u'(x), also completely represents risk preferences. The anti-derivative of marginal utility is the utility function with an arbitrary additive constant. As for the utility function, the marginal utility function is not a unique representation of preferences. Whenever u'(x) represents the risk preferences of a decision maker, then so does b·u'(x) for any b > 0. There are risk preferences for which there is no closed form for the utility function, that can be represented by a marginal utility function of closed form. There are also risk preferences for which there is no marginal utility function of closed form. Higher order derivatives of the utility function do not completely represent risk preferences and therefore are not often used.

In addition to the utility and marginal utility functions, there are several other functions that are also frequently used to represent risk preferences. The two most prominent of these are the absolute risk aversion measure, $A(x) = \frac{-u''(x)}{u'(x)}$, and the relative risk aversion measure, $R(x) = A(x) \cdot x$. These two measures were defined by Pratt and Arrow in part because they are a unique representation of risk preferences, and in part because these representations more clearly are connected to the decision maker's response to risk. There is only one absolute and relative risk aversion measure associated with particular risk preferences. All utility functions or marginal utility functions that represent particular risk preferences have the same measure of absolute and relative risk aversion. Although these two risk aversion measures are a unique representation of risk preferences, they are incomplete. Each represents more than one set of risk preferences. Information in addition to the risk aversion measure is needed to identify which risk preferences are being represented. It is typical in economic analysis to provide that information by assuming that utility is monotonically increasing. This assumption prevents the same absolute risk aversion measure from representing the risk preferences associated with utility functions u(x) and -u(x).

The discussion in this paper and in this section, involves finding simple and flexible functional forms for the representation of risk preferences. The focus in the literature is on finding simple and flexible functional forms for the utility function itself, while the focus here is on finding such functional forms for the marginal utility function instead. For either case, the goal is to have relatively simple absolute and relative risk aversion measures as well. Finding utility functions and marginal utility functions involves the same initial steps. To begin the discussion, a review of how the functional form for the utility function and the absolute risk aversion measure are related is presented.

A functional form for a utility function can be chosen either directly or indirectly. With direct selection, the utility function is specified, and the risk preferences it represents are determined by calculating the absolute risk aversion measure $A(x) = \frac{-u''(x)}{u'(x)}$. As long as the utility function that is selected is differentiable at least twice the functional form for the absolute risk aversion measure can be determined. With experience, and clever choices for u(x), a simple and flexible functional form for utility can be determined in this way using trial and error. Perhaps some of the forms reviewed in section 2 were discovered in this way.

This process can also be reversed. It is possible to begin with an absolute risk aversion measure, and then determine the functional form for the utility function that represents those same risk preferences. Pratt describes a procedure that can be used to determine the utility function associated with any given absolute risk aversion measure. This procedure, however, does not guarantee that the resulting utility or marginal utility function can be expressed in terms of elementary functions; that is, is of closed form. The procedure involves three steps. In step one, the anti-derivative of the absolute risk aversion measure is determined, and its sign is changed. In step two, the result from step one is used as the exponent in an expression whose base is e. Finally, in step three, the anti-derivative of the result from step two is determined. In symbols, this three step process begins with an absolute risk aversion measure $A(x) = \frac{-u''(x)}{u'(x)}$, and ends with u(x), where u(x) is given by $u(x) = \int e^{-\int A(x)} = \int e^{-\int \frac{u''(x)}{u'(x)}}$. The additive and positive multiplicative constants that arise when finding these anti-derivatives are unimportant and are omitted.

Step one, finding the anti-derivative of $A(x) = \frac{-u''(x)}{u'(x)}$ can be accomplished for all continuous functions A(x), a rather modest requirement. As is well known, this anti-derivative of a function is not unique because of an arbitrary additive constant. Unfortunately, it is possible that the anti-derivative cannot be expressed in terms of elementary functions; that is, it does not have a closed form representation. Since the goal of this research is finding a simple functional form for representing risk preferences, it is necessary that this anti-derivative have a closed form representation. Thus, an A(x) function must be selected so that $-\int A(x) = \int \frac{u''}{u'} =$ ln u'(x), where ln u'(x) is of closed form. Step two can always be carried out, and converts ln u'(x) into u'(x); that is, $e^{\ln u'(x)} = u'(x)$. Thus, completion of steps one and two, determines a closed form solution for the marginal utility function associated with an absolute risk aversion measure. For many absolute risk measures A(x) steps one and two can be carried out and a closed form for u'(x) determined.

Step three requires that the anti-derivative of marginal utility be determined, and to yield a functional form for the utility function, this anti-derivative must be of closed form. For many risk aversion measures, including many very simple ones, no closed form for the utility function results from step three. It is relatively easy to find a function A(x) so that its anti-derivative is of closed form, but it is rather unusual for $e^{-\int A(x)}$ to have an anti-derivative that can be expressed in terms of elementary functions. Theorem 1, given shortly, presents a sufficient condition on A(x) for a closed form for u(x) to exist, but the conditions in Theorem 1 are very strong and severely restrict the risk preferences that can be represented.

In summary, with modest restrictions on the absolute risk aversion measure A(x), a closed form solution for marginal utility can be determined. Only with severe restrictions on A(x), however, can the process determine a closed form for the utility function. Thus, the set of risk preferences that can be represented is expanded if the focus is shifted from utility to marginal utility. Simple and flexible marginal utility functions can represent risk preferences that utility functions can not. Shifting the focus to finding a simple and flexible form for marginal utility makes more options available. This is the main theme of this paper.

As an example of simple risk preferences that can be represented by marginal utility, but cannot be represented by a utility function, consider risk preferences whose absolute risk aversion measures are $A(x) = \frac{\alpha}{x^{\lambda+1}}$ and $R(x) = \frac{\alpha}{x^{\lambda}}$. The α parameter in these expressions can be used to vary the magnitude of risk aversion, and the λ parameter can be used to alter the slopes of the risk aversion measures. For these risk preferences, absolute risk aversion is increasing, constant or decreasing as λ is less than, equal to, or greater than minus one, while relative risk aversion is increasing, constant or decreasing as λ is less than, equal to, or greater than zero.

These are very simple risk preferences with convenient expressions for absolute and relative risk aversion, yet there is no closed form utility function that represents these risk preferences. Marginal utility, on the other hand, is easily obtained in closed form because the anti-derivative of A(x) is just a power function of one higher power. The functional form for marginal utility is u'(x) = $e^{\frac{\alpha}{\lambda x^{\lambda}}}$, quite a simple form. This set of risk preferences and this marginal utility function were used by Meyer and Meyer (2006) in their analysis of the evidence concerning the equity premium puzzle.

By appropriately choosing the absolute risk aversion measure A(x), one can ensure that both the utility function and the marginal utility function are of closed form. Theorem 1, given next, presents a way to identify such A(x) functions. It is the case that the CARA, CRRA, HARA and EP/PRT forms can all be derived using this result.

<u>Theorem 1</u>: If the absolute risk aversion measure A(x) takes the form A(x) = $h(x) - \frac{h'(x)}{h(x)}$ for a function h(x) whose anti-derivative H(x) can be expressed in terms of elementary functions, then u(x) can be expressed in terms of elementary functions and u(x) = $-e^{-H(x)}$.

<u>Proof</u>:

$$-\int A(x) = -\int h(x) + \int \frac{h'(x)}{h(x)} = -H(x) + \ln h(x)$$

$$e^{-\int A(x)} = e^{-H(x)} \cdot h(x)$$

$$\int e^{-\int A(x)} = \int e^{-H(x)} \cdot h(x) = -e^{-H(x)}$$

QED

The CARA functional form for the utility function is derived using Theorem 1 and choosing h(x) = α. Choosing $h(x) = \frac{\beta}{x}$ and applying Theorem 1 yields the CRRA form for utility. The HARA utility function results from choosing $h(x) = \frac{\frac{-\gamma}{1-\gamma}}{\frac{\eta}{\beta} + \frac{x}{1-\gamma}}$. Finally, the PRT form results from choosing $h(x) = \gamma \cdot x^{-\sigma}$.

This section has shown that there exist closed form marginal utility functions that represent risk preferences in a simple and flexible way, and that those same risk preferences cannot be represented by a closed form utility function, much less one of simple form. In addition, an explanation of why this might be the case is given. It is apparent that there may be substantial gains from representing risk preferences using marginal utility rather than utility. The next section, verifies that these gains can be acquired in practice as well as in theory.

4. Using Marginal Utility in Expected Utility Decision Models.

All twice differentiable utility functions u(x) represent risk preferences whose absolute risk aversion measure is $A(x) = \frac{-u''(x)}{u'(x)}$. These same risk preferences can instead be represented by marginal utility u'(x). That is, u(x) and u'(x) are each sufficient to determine the absolute risk aversion measure A(x). The opposite is not true. All differentiable marginal utility functions represent risk preferences given by $A(x) = \frac{-u''(x)}{u'(x)}$, but u'(x) may or may not have a closed form anti-derivative; that is, the associated utility function u(x) may or may not be of closed form. The topic discussed in this section is how to use u'(x), whether u(x) is of closed form or not, to represent the risk preferences of the decision maker in various EU decision models. The conclusion drawn from the examples presented is that in general u'(x) is sufficient, although in some instances some effort is needed when reformulating the decision model so that this is so.

Example 1: One of the basic uses of expected utility is to choose from, or to rank a finite set of alternatives. A simple integration by parts argument indicates that this application can be accomplished just as easily with marginal utility as with the utility function itself. Assume F(x) and G(x) are two cumulative distribution functions with support in some interval [a, b]. Then $EU_F \geq EU_G$ if and only if $\int_a^b u(x)dF(x) \geq \int_a^b u(x)dG(x)$. This expresses the ranking of F and G using the utility function. Integrating by parts and recognizing that F(a) = G(a) = 0 and F(b) = G(b) = 1 changes the inequality to $\int_a^b u'(x)F(x)dx \leq \int_a^b u'(x)G(x)dx$. Thus, to compare or rank a pair of alternatives using expected utility, only the marginal utility function u'(x) is required if the alternatives are represented using their cumulative distribution functions. Any finite set of alternatives can be ordered using the marginal utility function and this procedure.

Example 2: Another important category of expected utility decision models are those where the decision maker chooses the value of a parameter that can be continuously varied. In general notation, the decision maker chooses a value for α to maximize Eu(z) where $z = z(\alpha, x)$, and x is a random variable. The first order condition for this optimization is $Eu'(z) \cdot z_\alpha(\alpha, x) = 0$, which can obviously be evaluated and analyzed using only the marginal utility function. Comparative static analysis within such a decision model always begins with this first order condition expression. Thus, in this application of expected utility, it has been the case all along that the marginal utility function rather than the utility function has been used. The standard portfolio model and the model of the competitive firm facing output price uncertainty are examples of decision models of this type.

Example 3: A simple but often performed task that uses expected utility is to determine the certainty equivalent (CE) of a particular lottery or gamble. Suppose x represents the payoff from a gamble or lottery, and Eu(x) is the expected utility from this gamble. Then the usual definition of the CE is given by CE = $u^{-1}(Eu(x))$. Finding the CE this way requires knowledge of the utility function and its inverse. The CE can also be defined as the amount for certain which gives the same expected utility as does the gamble. Using this definition eliminates the need for the utility function or its inverse. To see this, let F(x) denote the cumulative distribution function for the outcome of the gamble, and $F_{CE}(x)$ denote the cumulative distribution function whose value is 0 for $x < CE$ and is 1 for $x \geq CE$. The certainty equivalent for the gamble is the value for CE solving the equation $\int_a^b u'(x)F(x)dx = \int_a^b u'(x)F_{CE}(x)dx = \int_{CE}^b u'(x)dx$. Thus, knowing only u'(x), the certainty equivalent of a gamble can be determined.

Example 4: Quadratic approximations to the utility function were used extensively by Pratt and by Arrow to give meaning to the absolute and relative risk aversion measures they defined. In this example, the probability premium and its relationship to the absolute risk aversion measure and the size of the gamble is discussed. The example shows that the probability premium and its relationship to the absolute risk aversion measure can be obtained using only the marginal utility function. Let w be nonrandom initial wealth, and consider a gamble which either adds h or subtracts h from this initial wealth. The probability premium is defined to be the difference between the probability of winning and losing that makes the decision maker indifferent between accepting the gamble or not. The cumulative probability distribution for not accepting the gamble is denoted $F_w(x)$ and is given by 0 for $x < w$ and 1 for $x \geq w$. This is being compared with the cumulative distribution F(x) from accepting the gamble. F(x) is equal to 0 for $x < w - h$, $\frac{1 - p}{2}$ for $w - h \leq x < w + h$, and 1 for $x \geq w + h$. Requiring these two alternatives to give the same expected utility requires that $\int_a^b u'(x)[F_w(x) - F(x)]dx = 0$. Using the definitions of F(x) and $F_w(x)$ this implies that $\int_{w-h}^{w} u'(x)\frac{(p - 1)}{2}dx + \int_{w}^{w+h} u'(x)\frac{(p + 1)}{2}dx = 0$. Rewriting, this becomes $\frac{p}{2}\int_{w-h}^{w+h} u'(x)dx - \frac{1}{2}\int_{w-h}^{w} u'(x)dx + \frac{1}{2}\int_{w}^{w+h} u'(x)dx = 0$. Now if one linearly approximates u'(x) using $u'(x) = u'(w) + u''(w)(x - w)$, this equation reduces to $p \cdot u'(w) \cdot h + u''(w)\frac{h^2}{2} = 0$. Solving for p gives $p = \frac{-u''(w)}{u'(w)} \cdot \frac{h}{2}$, the relationship between p, h and absolute risk aversion as Pratt determined in his original formulation of the probability premium.

These examples indicate that in a variety of applications of expected utility, knowing the utility function is unimportant, and that the marginal utility function can be used in its place if the decision model is appropriately reformulated. The examples also show, that to do this, it is often the case that the random alternative must be described using the cumulative distribution function rather than the density or probability function. For empirical applications, this is not likely to be an issue since often the empirical distribution function is employed.

5. Functional Forms for Marginal Utility

This section describes several simple and flexible functional forms for marginal utility. In each case, the anti-derivative of the marginal utility function is not of closed form, and hence the risk preferences being represented cannot be represented by a functional form for utility.

A functional form for the marginal utility function was presented as an example in section 3. For that example, the absolute risk aversion measure is A(x) = $\frac{\alpha}{x^{\lambda+1}}$, and the relative risk aversion measure is R(x) = $\frac{\alpha}{x^{\lambda}}$. This absolute risk aversion measure has a closed form anti-derivative, and implies that the functional form for marginal utility is u'(x) = $e^{\frac{\alpha}{\lambda x^{\lambda}}}$.

For these risk preferences, the risk aversion measures and the marginal utility function are each quite simple. These functional forms are also flexible in that the magnitude and slope of risk aversion are easily adjusted by altering the values for α and λ. The elasticity of the absolute risk aversion measure is constant and equal to (-λ - 1). The elasticity of the relative risk aversion measure is also constant and is equal to -λ. The CARA for where λ = -1 and the CRRA form where λ = 0 appear to be the only two forms where the utility function can be expressed in terms of elementary functions. These risk preferences represented by this functional form for marginal utility are referred to as risk preferences with isoelastic risk aversion measures.

Kocherlakota (1996) used the CRRA version of this form (λ = 0) when presenting an empirical verification of the equity premium puzzle described by Mehra and Prescott (1985).

Kocherlakota showed that under CRRA, no level of relative risk aversion simultaneously explains the risk premiums and risk free rates of interest observed in the data. Meyer and Meyer (2006) use this marginal utility function with various values for $\lambda > 0$, which represent decreasing relative risk aversion on the part of the decision maker. Meyer and Meyer show that under DRRA, there are magnitudes for risk aversion such that the equity premium puzzle is no longer verified. Specifically, if λ is increased from the CRRA value of 0 to DRRA values such as 1 or 2, then there are levels of risk aversion such that the risk premiums and risk free interest rates that have been observed historically are consistent with optimal portfolio composition and consumption over time.

The marginal utility function just described was obtained by choosing a simple and flexible form for absolute risk aversion and then determining the anti-derivative of that function. The two examples presented next are formed in a slightly different manner. They are derived by building upon a well known result, and using information concerning other absolute risk aversion and marginal utility function pairs. Many marginal utility functions can be formed in this way.

It is well known that whenever $A(x) = A_1(x) + A_2(x)$, where the $A_i(x)$ are absolute risk aversion measures, then $u'(x) = [u_1'(x)][u_2'(x)]$ for the corresponding marginal utility functions. That is, adding absolute risk aversion measures is equivalent to multiplying their marginal utility functions together. This observation allows the marginal utility function associated with an absolute risk aversion measure which is the sum of other simple absolute risk aversion measures, to be determined quite easily.

For example, when $A_1(x) = \alpha$ so that $u_1'(x) = \alpha e^{-\alpha x}$, and $A_2(x) = \frac{\beta}{x}$ so that $u_2'(x) = x^{-\beta}$, then for $A(x) = \alpha + \frac{\beta}{x}$, marginal utility $u'(x) = \alpha e^{-\alpha x} \cdot x^{-\beta}$. The marginal utilities $u_1'(x)$ and

$u_2'(x)$, are those associated with CARA and CRRA respectively. For these two marginal utility functions, the functional form for utility can also be given. The marginal utility function for the sum of the two risk aversion measures, however, does not, in general, have a closed form anti-derivative.

A three parameter marginal utility function that can display CARA, CRRA, or increasing or decreasing relative or absolute risk aversion can be formed using this additive procedure. This function can serve as an alternative to the HARA, EP and PRA functional forms. The absolute risk aversion measure is quite simple in form and given by $A(x) = \alpha + \frac{\beta}{x} + \frac{\gamma}{x^2}$. It is composed of a CARA component, a CRRA component and finally a DRRA component. Using the known marginal utility functions for each of the three additive terms in A(x) it is easy to verify that the associated marginal utility function takes the form $u'(x) = \alpha e^{-\alpha x} \cdot x^{-\beta} \cdot e^{\frac{\gamma}{x}} = \alpha e^{-\alpha x + \frac{\gamma}{x}} \cdot x^{-\beta}$. This three parameter functional form reduces to CARA when β and λ are zero, and to CRRA when α and λ are zero.

Several marginal utility functions that represent simple and flexible forms for the absolute and relative risk aversion measures were presented in this section. More importantly, the process for finding such functional forms was described and verified as quite easy to carry out. Many more simple and flexible functional forms for marginal utility can be determined using the procedures just described. For the marginal utility function representation, more simple and flexible functional forms are available than for utility function representation. Those applying expected utility decision models to specific settings should be able to take advantage of that.

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