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A NON-LINEAR MULTI-SECTORAL PLANNING MODEL

by

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A NONLINEAR MULTISECTORAL PLANNING MODEL

This paper presents a nonlinear multisectoral planning model for analysis of development programs of less developed countries.

The model is based on ideas drawn from two recent lines of thought about economy-wide growth models. The first line is that of nonlinear theoretical models designed to analyze the characteristics of an economy in asymptotic optimal growth, viz. Samuelson and Solow [1956], Chakravarty [1965], and Bruno [1966]. The other line is the construction of empirical multisectoral linear programming models to study the development programs of particular countries, viz., the model for Israel by Bruno [1966] and for India by Eckaus and Parikh [1967] and Chakravarty and Lefeber [1966].

With the exception of articles by Chenery and Uzawa [1958], Stoleru [1965], Radner and Friedman [1965], and Kurz [1965], there has as yet been little development of nonlinear growth and planning models for empirical application.

Our purpose is to develop a nonlinear model for empirical implementation and solution with control theory algorithms, and thereby to analyze some of the properties of a growth model with multiple capital good and consumer goods sectors.

The model presented here is ambitious in its scope and serves as a target toward which we will work; however, we expect to begin our computational exercises with a greatly reduced form of this model both in the varieties of constraints and in the number of sectors treated. Also, we hope to make some comparison of the relative usefulness of linear programming and control theory models for economic planning purposes.

The model is specified to maximize a utility functional with consumption levels as the arguments subject to side constraints and boundary conditions. The side constraints represent the production process, availability of produced (capital) and nonproduced (labor) input, and foreign exchange availability. The boundary conditions include specification of (1) initial and terminal capital stocks for each sector or intial capital stocks and post terminal rates of growth and (2) the initial foreign debt of the country.

We proceed with a discussion of (1) the parts of the model, (2) a statement of the first order necessary conditions for a stationary performance index, and (3) some comments on the nature of the solution.

1. Performance Index

The performance index is a utility functional to be maximized over the finite time interval to t_f, i.e.

$$\xi = \int_{t_o}^{t_f} [u(c_1, c_2, \ldots, c_n) - \omega\, 1'h]dt^* \qquad (1.1)$$

where c_i is the consumption of the i^{th} good. Among the functional forms of u with which we expect to experiment are

$$u(c_1, \ldots, c_n) = \sum_i a_i c_i^{b_i} \qquad a_i, c_i, b_i > 0 \qquad (1.2)$$

$$u(c_1, \ldots, c_n) = \sum_i a_i \log c_i \qquad a_i, c_i > 0 \qquad (1.3)$$

The $\omega 1'h$ term in (1.1) is related to the labor force constraints and will be discussed in Section 2.2.

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* 1' is a row vector of ones.

2. The Side Constraints

2.1 Production and Distribution Relations

Assume we know nonlinear production functions*

$$q_i = f_i(s_i, \ell_i, h_i) \qquad i = 1, \ldots, n \tag{2.1}$$

for an n sector economy, where q_i = sector output, s_i = capital stock, ℓ_i = sector skilled labor force, and h_i = sector unskilled labor force.** We assume that new capacity is created from inputs from all other sectors in fixed proportions. Thus, if B is a matrix whose elements b_{ij} are the units of input from the i^{th} sector required per unit addition to capital in the j^{th} sector and g is a vector of new capacities that are to be created, then Bg is the vector of inputs required to build the new capacity.

Furthermore assume that depreciation occurs via a fixed percentage of the capital stock of each sector melting away per unit of time. Then we can write

$$\dot{s} = g - Es \tag{2.2}$$

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* Though we have not been entirely successful in doing so we have endeavored to keep the notation uniform as follows: Matrices are upper case Roman letters, vectors are lower case Roman letters, and scalars are lower case Greek letters. Time subscripts and arguments have been omitted except where essential.

** Two functional forms of f which are frequently used by econometricians are:

$$f(s,\ell,h) = \gamma s^{\alpha}\ell^{\beta}h^{\eta} \qquad \text{(Cobb-Douglas)}$$

$$f(s,\ell,h) = \gamma[\delta_1 s^{-\rho} + \delta_2 \ell^{-\rho} + \delta_3 h^{-\rho}]^{-\frac{\mu}{\rho}}$$

$$\delta_1+\delta_2+\delta_3 = 1 \qquad \text{(Constant elasticity of substitution)}$$

where

E = a diagonal matrix of percentage depreciation rates.

Skilled labor as it enters the production function can be treated as a factor produced from unskilled labor through a fixed coefficient production function. Let $\sigma(t)$ be the number of unskilled workers to be trained at time t and v be a vector of inputs required per worker trained, so that σv represents the resource utilization for training. The utilization of skilled labor in producing skilled labor is discussed later. It is further assumed that a negligible amount of unskilled labor is used in training skilled labor.

Then letting η represent the total skilled labor force we write

$$\dot{\eta} = \sigma - \zeta\eta \tag{2.3}$$

where ζ is the percentage of the existing skilled workers that leave the labor force during each unit of time.

Given these assumptions we can write distribution equations for the economy as follows:

$$q + m = Aq + Bg + \sigma v + x + c \tag{2.4}$$

where

q = vector of sector outputs

m = vector of "competitive" imports

g = vector of new capacity creation

σ = scalar of skilled workers trained

v = vector of inputs to train skilled workers

x = vector of exports

c = consumption vector

A = flow input/output matrix*

B = matrix of capital coefficients

Solving (2.2) for g and substituting into (2.4) we obtain

$$q + m = Aq + B\dot{s} + BEs + \sigma v + x + c \tag{2.5}$$

which can be solved for $\dot{s}$

$$\dot{s} = B^{-1}(I-A)q + B^{-1}[m-\sigma v-x-c] - Es. \tag{2.6}$$

Letting $R = B^{-1}(I-A)$ and $G = B^{-1}$ (2.6) becomes

$$\dot{s} = Rf(s,\ell,h) + G[m-\sigma v-x-c] - Es. \tag{2.7}$$

Recall

$$\dot{\eta} = \sigma - \zeta\eta \, . \tag{2.8}$$

Then formally we label the capital stock variables s and the skilled labor force variable η as state variables and the variables ℓ, h, m, x, and c, and σ as control variables. Equations (2.7) and (2.8) which are linked by σ are the basic differential equations of the model.

2.2 Labor Force Constraints

The labor force is divided into skilled and unskilled groups. As was discussed above, skilled labor is produced from unskilled labor with other goods and some skilled labor. Thus:

$$1'\ell + \theta\sigma = \eta \tag{2.9}$$

---

*That is, the matrix $[a_{ij}]$ giving the quantity of good i used in production of good j on current account.

where

$$\theta = \text{Number of skilled laborers required per skilled worker trained.}$$

We assume that skilled labor will be fully employed at all times, thus (2.9) is written as an equality relationship.

Unskilled labor includes all of the labor force not in the skilled labor category, i.e.

$$1'h + \eta \leq \overline{P}_t \tag{2.10}$$

where $\overline{P}_t$ is the exogenously given total labor force. Since in many less developed countries there seems to be substantial unemployed or underemployed labor we write (2.10) as an inequality relationship.* However, the production functions f permit unlimited substitution of capital, skilled labor, and unskilled labor for one another, so (2.10) would always be an equality except for the inclusion of the term $(-\omega 1'h)$ in the performance index (1.1). Here ω represents a fixed wage that is established by institutional factors and is at or above the subsistence level. Thus unskilled labor will be used only to a level at which its marginal product equals ω. Of course if ω is sufficiently small, all labor will be employed and (2.10) will hold as an equality.

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* From our definitions, this is an inequality constraint on both state and control variables. Such constraints can be difficult to handle computationally. If this turns out to be the case in our model, we might want to use the simpler constraint

$$1'h \leq \overline{P}_t \tag{2.10a}$$

on the assumption that skilled labor does not comprise a large part of the labor force in a less developed country.

2.3 Balance of Payment Constraint

We write the balance of payment constraint as:

$$d_1'f + d_2'g + d_3'm = 1'w(x) + \emptyset \qquad (2.11)$$

where d_1, d_2, and d_3 are vectors giving the foreign currency costs of production on current account, capital accumulation, and competitive imports, respectively. $w(x)$ is a vector of foreign currency export revenues,* and $\emptyset$ is a scalar representing foreign aid (usually $\emptyset > 0$).

Substituting for g from (2.2) into (2.11) we obtain

$$d_1'f + d_2'\dot{s} + d_2'Es + d_3'm = 1'w(x) + \emptyset. \qquad (2.12)$$

The foreign aid constraint is specified as an integral over time, viz. a country might desire to pay off all or a part of its foreign debt by the end of the plan period. Thus we write

$$\int_{t_o}^{t_f} \emptyset(t)dt = \Psi(t_f) - \Psi(t_o) \qquad (2.13)$$

where

$\Psi(t_o)$ = foreign debt at time t_o

$\Psi(t_f)$ = total foreign debt at the end of the planning period**

The integral constraint (2.13) can be transformed to a differential equation with $\emptyset$ as an additional state variable as follows:***

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* One can write revenue = quantity of export x times price of x, and approximate the price function by $x^{-\gamma}$ $(\gamma<1)$. Thus, $w_i(x_i)=x_i^{1-\gamma}$.

** Alternatively $\Psi(t_f)$ could be assigned a cost and added to the performance index.

*** This method of treating an integral constraint is discussed in Bryson and Ho [1966], Section 3.1.

$$\dot{\emptyset} = d_1'f + d_2'\dot{s} + d_2'Es + d_3'm - 1'w(x) \tag{2.14}$$

or substituting for $\dot{s}$, as

$$\dot{\emptyset} = d_1'f + d_2'[Rf(s,\ell,h) + G(m-\sigma v-x-c) - Es] + d_2'Es$$

$$+ d_3'm - 1'w(x) \tag{2.15}$$

or

$$\dot{\emptyset} = (d_1'+d_2'R)f + (d_2'G+d_3')m - d_2'G(\sigma v+x+x) - 1'w(x) \tag{2.16}$$

3. Boundary Conditions

We specify initial and terminal stocks of capital $s(t_o)$ and $s(t_f)$, terminal net foreign debt, $\emptyset(t_f) = \Psi(t_t) - \Psi(t_o)$,** and number of skilled laborers $\eta(t_o)$ and $\eta(t_f)$. Alternatively we might set the initial conditions as above but specify cost functions on the terminal conditions instead of requiring that they be met exactly.

A third method might be used for the terminal conditions on the capital stocks. This is to specify post terminal rates of growth for the control variables and thereby prevent the terminal capital stocks from being run down.*

4. Solution of the Model

The variational Hamiltonian for the problem as stated is:

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* See S. Chakravarty and R. S. Eckaus, "An Approach to a Multi-sectoral Intertemporal Planning Model," in Capital Formation and Economic Development, P. N. Rosenstein-Rodan (ed.), M.I.T. Press, Cambridge, Mass., 1964.

** By definition, $\emptyset(t_o = 0$.

$$H = u-\omega 1'h + p'[Rf(s,\ell,h) + G(m-\sigma v-x-c)-Es] + \lambda[\sigma-\zeta\eta] + \pi[1'\ell + \theta\sigma - \eta] + \mu[1'h+\eta-\overline{P}] + \nu[(d_1'+d_2'R)f(s,\ell,h) + (d_2'G+d_3')m - d_2'G(\sigma v+x+c) -1'w(x)] \quad (4.1)$$

Where the p's can be interpreted as the factor prices for capital stocks, λ as the value of investment in skilled labor, π as the wage rate for skilled labor, μ as the wage rate for unskilled labor (when $\mu \neq 0$), and ν is the shadow price on foreign aid.

The first order conditions for a stationary performance index may be written:

$$\dot{s} = Rf(s,\ell,h) + G(m-\sigma v-x-c) - Es \quad (2.7)$$

$$\dot{\eta} = \sigma - \zeta\eta \quad (2.8)$$

$$\dot{p}' = -\frac{\partial H}{\partial s} = -\left[p'(R\frac{\partial f}{\partial s} - E) + \nu\left(d_1'+d_2'R\right)\frac{\partial f}{\partial s}\right]$$

$$\dot{p}' = p'E - [\nu d_1' + (p'+\nu d_2')R]\begin{bmatrix} \frac{\partial f_1}{\partial s_1} & & 0 \\ & \ddots & \\ 0 & & \frac{\partial f_n}{\partial s_n} \end{bmatrix} \quad (4.2)$$

$$\dot{\lambda} = -\frac{\partial H}{\partial \eta} = -[-\lambda\zeta-\pi+\mu] \quad (4.3)$$

$$\frac{\partial H}{\partial \ell} = 0 = \pi 1' + [\nu d_1' + (p'+\nu d_2')R]\begin{bmatrix} \frac{\partial f_1}{\partial \ell_1} & & 0 \\ & \ddots & \\ 0 & & \frac{\partial f_n}{\partial \ell_n} \end{bmatrix} \quad (4.4)$$

$$\frac{\partial H}{\partial h} = 0 = -\omega 1' + \mu 1' + [\nu d_1' + (p' + \nu d_2')R] \begin{bmatrix} \frac{\partial f_1}{\partial h_1} & & 0 \\ & \ddots & \\ 0 & & \frac{\partial f_n}{\partial h_n} \end{bmatrix} \tag{4.5}$$

when $1'h + \eta - \overline{P} < 0$ (2.10)

$\mu = 0$

and (4.5) determines h

when $1'h + \eta - \overline{P} = 0$ (2.10)

$\mu > 0$

and (2.10) and (4.5) together determine h.*

$$\frac{\partial H}{\partial \sigma} = 0 = -p'Gv + \lambda + \pi\theta - \nu d_2'Gv \tag{4.6}$$

$$\frac{\partial H}{\partial m} = 0 = p'G + \nu(d_2'G + d_3') \tag{4.7}$$

$$\frac{\partial H}{\partial x} = 0 = -p'G - \nu d_2'G - \nu 1' \begin{bmatrix} \frac{\partial w}{\partial x_1} & & 0 \\ & \ddots & \\ 0 & & \frac{\partial w}{\omega x_n} \end{bmatrix} \tag{4.8}$$

$$\frac{\partial H}{\partial c} = 0 = -p'G - \nu d_2'G + \begin{bmatrix} \frac{\partial u}{\partial c_1} \\ \vdots \\ \frac{\partial u}{\partial c_n} \end{bmatrix} \tag{4.9}$$

$$\dot{\nu} = -\frac{\partial H}{\partial \emptyset} = 0 \tag{4.10}$$

$\therefore$ ν is a constant

π determined by (2.9 and (4.3), (4.4), and (4.6).

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* See Bryson and Ho [1966], Section 3.10

$$\mu \begin{cases} > 0 \quad , & 1'h + \eta - \overline{P} = 0 \\ = 0 \quad , & 1'h + \eta - \overline{P} < 0 \,. \end{cases} \tag{4.11}$$

For $\mu > 0$ equations (2.10) and (4.5) together determine μ along with h. (Note λ is partially determined by $\mu > 0$ from (4.3).)

There are 2n+2 differential equations (2.7), (2.8), (4.2), and (4.3) for an n-sector model, along with 5n+1 optimality conditions (4.4) through (4.9), two labor constraints (2.9) and (2.10), and an integral constraint (2.13) to determine 5n+1 control variables (ℓ, h, m, x, c, σ), three influence numbers (π, μ, ν), n+1 state variables (s, η), and n+1 price variables (p, λ).

5. Comments

5.1 The Size of the Model and the Number of Controls

As far as detail in planning goes, the more sectors we use in the model, the better. However, for illustrative purposes (and all we can hope to do at this stage is illustrate) three or four sectors would be thoroughly adequate. Ten to twenty-five "years" would make a reasonable planning period.

The model as presented here involves 5n+1 controls for n sectors, but in a sense this is an overstatement. For example, some sectors may not enter at all into international trade, which means that the corresponding m_i and x_i need not be considered at all. Similarly, other sectors need not produce consumable goods directly, or may not require skilled labor in any large amounts.

5.2 Avoiding Corners

Since we are presumably modeling a "real" economy, with a great deal of built-in friction, we would like to avoid large discontinuities in the control variables, especially the consumptions (c_i) and sectoral labor forces (l_i). It is not immediately clear whether or not the model is stable enough to avoid such corners, and probably only experimentation will tell. One way to avoid large discontinuities in c and l would be to define control variables c* and l* so that

$$\dot{c} = c^*$$

$$\dot{l} = l^*$$

and then put inequality constraints on c* and l* (or else put them in the Lagrangian with penalties on the squared magnitudes.)

5.3 Possible Reformulations

The model as given here is structurally rather similar to linear programming planning models. This is an advantage, particularly in the early stages of analysis, since the structural similarity allows us to use accumulated knowledge about what patterns of sectoral growth to expect in interpreting our results. Also, we can evaluate the relative computational efficiency of the two approaches.

On the other hand, the formulation given here does not take full advantage of the possibilities for realistic modeling which come from being able to use nonlinear functions at will. For example, one could specify explicit functions for governmental revenue using variable multiplicative tax rates as controls. Or if multiple exchange rates were being considered as a governmental policy, one could reformulate the balance-of-payments constraint to try to work out the optimal exchange rate policy. Along with these two examples, other extensions of the model to incorporate actual policy instruments along with our rather formal controls can in principle be made, and should be made as economists' experience with control theory models increases.

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