IDENTIFICATION OF ECONOMIES OF SCOPE IN STOCHASTIC PRODUCTION TECHNOLOGIES

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ABSTRACT

Three restricted specifications of a stochastic technology that enable economies of scope to be identified are defined in the study. For each specification, the results of this study exactly parallel those obtained by Panzar and Willig (1981) for defining economies of scope using a non-stochastic technology. Sub-additivity of the individual ex ante cost function of each output is a sufficient condition for multi-output firms operating in a stochastic production environment.

Ⅰ. Introduction

Under certainty, the basic and intuitively appealing property of multi-product firms is "economies of scope" (rather than of scale) (see Panzar and Willig 1981).[1] Whenever the cost of providing

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[1] Many economists use the terminology of economies of scale" and economies of scope" interchangeably. However, they are not same thing even though the most convenient measures of these phenomena coincide at cost minimizing points. In general, the elasticity of scale($=\varepsilon$) measures how output responds as one moves out along a scale line from the origine in input space and if $\varepsilon > 1$, the production function exhibits increasing returns

the services of the sharable input to two or more product lines are sub-additive, the multi-product cost function exhibits economies of scope. An input joint technology underpins economies of scope, which exists if the sum of the costs from producing multi-outputs individually exceeds the cost of producing the same output jointly. The existence of shared inputs across outputs or fixed costs are common sources of economies of scope (Panzar 1988; Gorman 1985), while the absence of economies of scope in production is associated with a cost function that is strongly separable with respect to output.

To date, economies of scope has not been precisely defined under production uncertainty. Seminal study on stochastic technologies instead initially focused on separating preferences from the technology for the single output case using an ex ante cost function. For example, Pope and Chavas (1994) showed that modeling all moments of the distribution of output would separate preferences from technology while Chambers and Quiggin (1998) showed that modeling all states of nature achieve the same result.

In order to define economies of scope for stochastic technologies, it is necessary to first extend the definition of the single-output stochastic technology to joint and non-joint stochastic technologies. Given properties of a multi-output stochastic technology, this study then demonstrates that the single output approach for separating preferences from the technology is not sufficient for the identification of properties of a multi-output technology using an ex ante cost function. In particular, if the effect of the stochastic factor in the production environment is

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to scale. In this case, there is an incentive to centralize production operations and one might say, so to speak, that the entrepreneur faces economies of scale. The elasticity of scope (or size)(= η) measures the cost response associated with movements along the locus of cost-minimizing points in input space, that is, the expansion path. If $\eta < 1$, there are apparently cost advantages to be reaped from centralizing production, and thus the firm exhibit economies of scope. By necessity, therefore, the two measures are generally based on different input combinations.

correlated across outputs (e.g., the effect of weather in multiple crops), because of stochastic dependence results, it may not be possible to identify input non-jointness in an ex ante cost function and, thus, the definition of economies of scope is not generally identified in a stochastic production condition.

Given the identification problem, the study explores restrictions on stochastic technologies that allow for both common random effects and identification of economies of scope. For example, structure from multiplicative risk and a restricted form of multi-output Just-Pope production function are utilized to identify economies of scope in an ex ante cost function. More generally, if input usage depends on both observed output and the own moments of the joint distribution of output, then economies of scope can be identified. However, if input usage is not sufficiently represented by this information (e.g., covariances are also needed) then identification in an ex ante cost function fails.

II. General Definitions of Joint and Non-joint Stochas Technologies

In a deterministic production environment, a multi-output technology is often defined by (see Chambers 1988)

$$V(y)=\{x \mid T(y,x)\le 0\} \qquad \text{(1a)}$$

where $V(y)$ is the set of inputs that fulfills the feasibility criterion defined by the transformation function, $T(.)$. In particular, if $x\in V(y)$, then the output vector $y=(y_1,\ldots,y_m)\in \mathbb{R}_+^m$ can be produced by the input vector, $x=(x_{11},\ldots,x_{1n},\ldots x_{m1},\ldots,x_{mn})$ where x denotes a total input vector and x_{ik} denotes the k-th input used in the production of the i-th output.

A technology is said to be input non-joint if total input usage across outputs is additive, i.e., $x_k=\sum_{i=1}^{m} x_{ik}$ for all $k=1,2,\ldots,n$ inputs. That is, (1a) can be re-expressed as

(1b) $$V(y) = \sum_{j=1}^{m} V(y_j)$$

where $V(y_k) = \{x_k \mid T(y_k, x_k) \leq 0\}$ and $x_k = (x_{k1}, \ldots, x_{kn})$.

Following Chambers and Quiggin (1998), the multi-output stochastic technology corollary to (1) can be represented for each realization of the state of nature, S, as

(2a) $$V(y_s) = \{x \mid T_s(y_s, x) \leq 0\}$$

and

(2b) $$V(y_s) = \sum_{j=1}^{m} V(y_{js})$$

where $V(y_{js}) = \{x_j : T_s(y_s, x_j) \leq 0\}$ where $s = \{1, 2, \ldots, r\}$ are states of nature, $y_s = (y_{1s}, \ldots, y_{ms})$ is the vector of output produced in state S and other variables are defined as above. Hence, as in the certainty case, the technology in (2b) is input non-joint if total deterministic input usage across outputs is additive for all states of nature, i.e., $x_k = \sum_{j=1}^{m} x_{jk}$ for all $k = 1, 2, \ldots, n$ inputs.

Each state of nature in (2) has an associated probability function conditional on the deterministic input usage, $P(y_s \mid x)$, that satisfies the properties of a probability space, i.e.,

(3) $$P(y_s \mid x) \geq 0 \quad \text{and} \quad \sum_{s=1}^{r} P(y_s \mid x) = 1.$$

The corresponding marginal probability of y_{js} is captured in the probability space

(4) $$P_j(y_{js} \mid x) \geq 0 \qquad \text{and} \quad \sum_{s=1}^{r} P_j(y_{js} \mid x) = 1.$$

Consistent with (2b), the marginal probability of y_{js} in a stochastic input non-joint technology is

(5) $$P_j(y_{js} \mid x) = P_j(y_{js} \mid x_j).$$

This condition implies that inputs allocated to output j, where $j \neq i$, do not affect the distribution function of output i.

Conserving the property of stochastic dependence in the definition of input non-jointness is relevant in agricultural applications and other resource based industries. For example, stochastic factors such as rainfall are not allocable across outputs but, importantly, may affect both the input joint and input non-joint technology in a similar manner. Alternatively stated, the effect of the stochastic factor is correlated across outputs for both joint and non-joint technologies. Mathematically, if $y_{is} = y_{is}(x_i)$ and $y_{js} = y_{js}(x_j)$ and s is a state of nature that affects both y_{is} and y_{js}, then the covaraince between y_{is} and y_{js}, is not zero, i.e., $Cov(y_{is}, y_{js}) \neq 0$, regardless of whether (5) holds. Indeed, the definition of an input non-joint technology in (2b) and (5) is sufficiently general to allow for common random effects, i.e., $P(y_s|x) \neq \Pi P_i(y_{is}|x_i)$.

III. Ex ante Cost functions of Joint and Non-joint Stochastic Technologies

To demonstrate the implications of stochastic dependence, consider a direct extension of the Pope and Chavas' approach to separating preferences from the technology, which would define firm's behaviors as

$$(6) \qquad Max \sum_{s=1}^{r} U\left[W + \sum_{i=1}^{m} p_i y_{is} - \sum_{k=1}^{n} w_k x_k \right] P[y_s | x, \eta]$$
$$\text{subject to} \quad \eta = F(x), \quad \text{for} \quad i = 1, \ldots, m,$$

where W is initial wealth; w_k is the price of input k, p_i is the price of output i; U is a Von Neumann-Morgenstern utility function; and $F(x)$ is the vector of conditional expectations for each of the moments encompassed in η.[2] To illustrate the source

[2] As in the certainty case, the unconditional input demands acn be

of failure of identification of the property of non-jointness in an ex ante cost function, it suffices to consider a special case of (6), i.e.,

(7a) $$Max \sum_{s=1}^{n} U\left[W + \sum_{i=1}^{2} p_i y_{is} - \sum_{k=1}^{n} w_k x_k \right] P[y_s | x, \eta]$$

subject to $E(y_i) = f_i(x_i, x_2),\ i = 1, 2$

$Var(y_i) = h_{ii}(x_1, x_2),$

$Cov(y_1, y_2) = h_{12}(x_1, x_2),$

where $\eta = [E(y_1), E(y_2), Var(y_1), Var(y_2), Cov(y_1, y_2)]$, is treated as given;[3] $h_{ij}(x_1, x_2) = \left[\sum_{s=1}^{n} y_{is} y_{js} P(y_s | x_1, x_2) \right] - f_i(x_1, x_2) f_j(x_1, x_2)$ and $f_i(x_1, x_2) = \sum_{s=1}^{n} y_{is} P(y_s | x_1, x_2)$.

The representation of (7a) under an input non-joint stochastic technology is

(7b) $$Max \sum_{s=1}^{n} U\left[W + \sum_{i=1}^{2} p_i y_{is} - \sum_{k=1}^{n} w_k x_k \right] P[y_s | x, \eta]$$

subject to $E(y_i) = f_i(x_i),\ i = 1, 2$

$Var(y_i) = h_{ii}(x_i),$

$Cov(y_1, y_2) = h_{12}(x_1, x_2),$

where the restriction that separates preferences from the technology in an ex ante cost function is

(8) $$P(y_s | x, \eta) = P(y_s | \eta)$$

where $\eta = [E(y_1), E(y_2), Var(y_1), Var(y_2), Cov(y_1, y_2)]$, is fixed in

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determined from (6) by first maximizing over $x = (x_1, \ldots, x_n)$ conditional on the moments on he distribution of output, η, and then maximizing with respect to η.

[3] Analogously to the certainty case, η is given in the optimization.

the optimization. To show this separation, note that the first order conditions of the optimization problem in (7b) given (8) are;

(9a) $\phi w_k = \left(\lambda_1 \frac{\partial h_{11}(x_i)}{\partial x_{ik}} + \lambda_2 \frac{\partial h_{22}(x_i)}{\partial x_{ik}} + \lambda_3 \frac{\partial h_{12}(x_i)}{\partial x_{ik}} \right)$ for all k

(9b) $E(y_i) = f_i(x_i)$, $i = 1, 2$

(9c) $Var(y_i) = h_{ii}(x_i)$,

(9d) $Cov(y_1, y_2) = h_{12}(x_1, x_2)$,

where $\phi = \sum_{s=1}^{t} \frac{\partial U(W_{bs})}{\partial W_{bs}} P[y_s|\eta]$ for $W_{bs} = U\left(W + \sum_{j=1}^{m} p_j y_{js} - \sum_{k=1}^{n} w_k x_k\right)$.

Yet, the first order conditions in (9a) are alternatively represented by normalizing the Lagrange multiplier and input prices as;

(10) $$\frac{w_k}{w_1} = \frac{\left[\frac{\partial h_{11}(x_i)}{\partial x_{ik}} + \frac{\lambda_2}{\lambda_1} \frac{\partial h_{22}(x_i)}{\partial x_{ik}} + \frac{\lambda_3}{\lambda_1} \frac{\partial h_{12}(x_i)}{\partial x_{ik}} \right]}{\left[\frac{\partial h_{11}(x_i)}{\partial x_{i1}} + \frac{\lambda_2}{\lambda_1} \frac{\partial h_{22}(x_i)}{\partial x_{i1}} + \frac{\lambda_3}{\lambda_1} \frac{\partial h_{12}(x_i)}{\partial x_{i1}} \right]}$$

Assuming that the conditional moments for all given input allocations are strictly convex sets in (10), it follows that explicit solutions for the optimal conditional input demands exist and equal to

$$x_{ik} = x_{ik}[w, E(y_1), Var(y_1), E(y_2), Var(y_2), Cov(y_1, y_2)]$$

and

$$x_k = \sum_{i=1}^{m} x_{ik}[w, E(y_1), Var(y_1), E(y_2), Var(y_2), Cov(y_1, y_2)].$$

The associated ex ante cost function is

$$C[w, E(y_1), Var(y_1), E(y_2), Var(y_2), Cov(y_1, y_2)].$$

The information in preference, ϕ, is therefore filtered out by conditioning the optimization problem of maximizing utility if (7) holds.[4] Different from the certainty case, however, equations that solve x_{ik} for all k in the first order conditions in (10) cannot be solved independently for each i because of $Cov(y_1, y_2)$ in (9d). Identification of input non-jointness in an ex ante cost function may thus fail even if preferences are separated from technology. This may occur in the presence of stochastic factors such as rain or frost.

More generally, any ex ante cost function defined in terms of the covaraince matrix of the joint distribution of output (and any moments) does not normally contain structure that identifies an input non-joint stochastic technology. In addition, failure of identification using the Chambers and Quiggin's approach under a general stochastic technology can be easily derived.

IV. Identification of Economies of Scope usin Separability restrictions on Risk

A special case of (8) maps the probability of observed output y_s directly from the expected value of output and the state of nature associated with the observed output. That is,

(11) $P[y_s|x, E(y)] = P[y_s|E(y)]$

The structure in (11) only holds if $y_s = E(y) + \varepsilon = f(x) + \varepsilon$ where $E(\varepsilon) = E(\varepsilon|x) = 0$. Therefore, (11) represents multiplicative risk (see Pope and Chavas 1994). Given (11), solve

$$Max \sum_{s=1}^{r} U\left[W + \sum_{i=1}^{m} p_i y_{is} - \sum_{k=1}^{n} w_k x_k\right] P[y_s|x, E(y)]$$

subject to $E(y_i) = f_i(x_i)$ for all i,

[4] To separate preferences from technology, the variance-covariance matrix of the joint distribution of output needs to be included in an ex ante cost function if given the mean of the joint distribution of output, input affect the variance-covariance matrix of the distribution.

where $P[y_s|x, E(y)] = P[y_s|E(y)]$, then reduces to

$$\frac{w_k}{w_1} = \left[\frac{\partial f_i(x_i)}{\partial x_{ik}}\right] / \left[\frac{\partial f_i(x_i)}{\partial x_{i1}}\right] \text{ for all } i \text{ and } k \neq 1 \text{ and } E(y_i) = f_i(x_i)$$

for all i.

Multiplicative risk not only separates preferences from the technology but also provides a structure in which input non-jointness can be identified from the ex ante cost function. In particular, provided that $E(y_i) = f_i(x_i)$ is defined on a strictly convex set for all i, the conditional input demands may be uniquely solved from the first order conditions as $x_{ik} = x_{ik}[w, E(y_i)]$ for all i and k, and thus, $x_k = \sum_{i=1}^{m} x_{ik}[w, E(y)]$. The associated ex ante cost function is

$$(12) \qquad \sum_{k=1}^{n} w_k x_k = \sum_{k=1}^{n} \sum_{i=1}^{m} w_k x_{ik}[w, E(y_i)] = \sum_{i=1}^{m} C_i[w, E(y_i)]$$

where C_i is the ex ante cost function of output i. If the technology in (12) is non-joint, $C[w, E(y)] = \sum_{i=1}^{m} C_i[w, E(y_i)]$ and if the technology is joint,

$$(13) \qquad C[w, E(y)] \leq \sum_{i=1}^{m} C_i[w, E(y_i)].$$

Proposition 1. *Given (11), a sufficient condition for identifying economies of scope for multi-output plants is the sub-additivity condition of (13).*

Proof. Under a monotonic utility function, if $C[w, E(y)] \leq \sum_{i=1}^{m} C_i[w, E(y_i)]$ then, $U\left[W + \sum_{i=1}^{m} p_i y_{is} - C(w, E(y))\right] \geq U\left[W + \sum_{i=1}^{m} p_i y_{is} - \sum_{i=1}^{m} C(w, E(y_i))\right]$.

Because a probability function is always non-negative it also follows that

$$U\left[W + \sum_{i=1}^{n} p_i y_{is} - C(w, E(y))\right] P[y_s \mid x, E(y)]$$

$$\geq U\left[W + \sum_{i=1}^{n} p_i y_{is} - \sum_{i=1}^{n} C(w_i, E(y_i))\right] P[y_s \mid x, E(y)]$$

and thus,

$$\sum_{s=1}^{S} U\left[W + \sum_{i=1}^{n} p_i y_{is} - C(w, E(y))\right] P[y_s \mid x, E(y)]$$

$$\geq \sum_{s=1}^{S} U\left[W + \sum_{i=1}^{n} p_i y_{is} - \sum_{i=1}^{n} C(w_i, E(y_i))\right] P[y_s \mid x, E(y)]. \quad \text{(QED)}$$

Proposition 1 states that (13) is a sufficient condition for the existence of multi-output plants. Paralleling the definition obtained by Panzar and Willig (1981) for economies of scope under certainty, economies of scope for the stochastic technology with no risk reducing (increasing) inputs requires subadditivity of the ex ante cost functions associated with each output.

Now consider weaker separability restrictions on how risk affects the technology but still enable economies of scope to be identified from the ex ante cost function in the presence of common random effects. In particular, following Just and Pope [1978], this test allows for the existence of inputs that are risk reducing (increasing). To derive input demands conditional on the mean and variance of output, consider the non-joint case,

(14) $$\max \sum_{s=1}^{S} U\left[W + \sum_{i=1}^{n} p_i y_{is} - \sum_{i=1}^{n} w_i x_i\right] P[y_s \mid x, E(y), diag\Omega]$$

subject to $E(y_i) = f_i(x_i)$ and $Var(y_i) = h_i(x_i)$, for all i

where $diag\Omega$ is the diagonal components of the variance-covariance matrix Ω of the distribution of output, and $Var(y_i)$ denotes the variance of output i conditional on input usage allocated to i.

Following the analysis of the previous section, if the density function of the objective function in (14) is not a function of input usage,

(15) $$P[y_s \mid x, E(y), diag\Omega] = P[y_s \mid E(y), diag\Omega].$$

i.e., the producer cannot affect the level of risk associated with a given level of expected output and its variance, then optimal choices from (14) are

$$x_{ik} = x_{ik}[w, E(y_i), Var(y_i)]$$

and, thus, under input nonjointness

$$x_k = \sum_{i=1}^{m} x_{ik}[w, E(y_i), Var(y_i)]$$

and the associated with ex ante cost function can be defined as

(16) $$\sum_{k=1}^{n} w_k x_k = \sum_{k=1}^{n}\sum_{i=1}^{m} w_k x_{ik}[w, E(y_i), var(y_i)] = \sum_{i=1}^{m} C_i[w, E(y_i), var(y_i)].$$

Given (15), if the stochastic technology is input nonjoint, then

$$C[w, E(y), diag\Omega] = \sum_{i=1}^{m} C_i[w, E(y_i), var(y_i)],$$

and if the stochastic technology is joint, then

(17) $$C[w, E(y), diag\Omega] \le \sum_{i=1}^{m} C_i[w, E(y_i), var(y_i)]$$

Proposition 2. *Given (15), a sufficient condition for multi-output plants is the subadditivity condition in (17).*

Proof. From the proof of Proposition 1,

if $C[w, E(y), diag\Omega] \le \sum_{i=1}^{m} C_i[w, E(y_i), var(y_i)]$, then

$$\sum_{s=1}^{S} U\left[W + \sum_{i=1}^{m} p_i y_{is} - C(w, E(y), diag\Omega)\right] P[y_s | x, E(y), diag\Omega]$$

$$> \sum_{s=1}^{S} U\left[W + \sum_{i=1}^{m} p_i y_{is} - \sum_{i=1}^{m} C_i[w, E(y_i), var(y_i)]\right] P[y_s | x, E(y), diag\Omega].$$

Once again, like the definition by Panzar and Willig for economies of scope under certainty, proposition 2 says that economies of scope requires sub-additivity of the ex ante cost functions associated with each output. In contrast to (11), however, (15) implies that input allocations to output i for a given level of expected output is able to affect the variance or riskiness of output j for $j \neq i$.

Note that restrictions similar to (11) and (15) but defined using other or additional higher moments of each output are also consistent with the sub-additivity condition of the ex ante cost function. That is, if

$$(18) \quad P[y_s | x, E(y_1), ..E(y_1^h), ..E(y_m), ..., E(y_m^h)] \\ = P[y_s | E(y_1), ..E(y_1^h), ..E(y_m), ..., E(y_m^h)]$$

where $E(y_j^h)$ is the h-order moment of output j, then there are economies of scope if

$$(19) \quad C[w_1, ..w_n, E(y_1), ..E(y_1^h), ..E(y_m), ..E(y_m^h)] \\ \leq \sum_{i=1}^{m} C_i[w_1, ..w_n, E(y_i), ..E(y_i^h)].$$

Consequently, (19) says the most general structure that identifies economies of scope. The inability to identify economies of scope occurs if the separability restriction in (18) fails. This means that if an ex ante cost function requires inclusion of cross moments of the distribution of output, then it is not generally possible to identify economies of scope.

V. Conclusions

While Chambers and Quiggin's assertion that "duality theory applies exactly for stochastic technologies under the same assumptions required for it to apply to non-stochastic technologies" is true, this study reveals an important distinction between stochastic and non-stochastic technologies. While a well-behaved cost function may be defined for a stochastic technology with common random effects, this study demonstrates that it is not generally possible to test for economies of scope using *ex ante* cost function. Instead, separability restrictions must be imposed on the stochastic technology for economies of scope to be identified.

Three restricted specifications of a stochastic technology that enable economies of scope to be identified are defined in this work. For each specification, the results obtained from this study exactly parallel those of Panzar and Willig for defining economies of scope using a non-stochastic technology. Sub-additivity of the individual *ex ante* cost function of each output is a sufficient condition for multi-output firms operating in a stochastic production environment.

Importantly, both the separability restrictions imposed for identification purposes of economies of scope and economies of scope itself are derived as testable hypotheses in this study. This suggests that future applications involving multi-output stochastic technologies will not be a direct extension of the deterministic case. Instead, this study reveals an evidence that the test for economies of scope is properly identified so that empirical tests of joint production can be properly executed. If the separability restrictions are rejected, it may not be possible to test for the economies of scope. Given the importance attached to identifying joint production, this may limit the usefulness of dual cost function analyses in agriculture and other resource based industries subject to stochastic production.

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