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SENSITIVITY OF BARGAINING SOLUTIONS TO RISK AVERSIONS

by

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1. Introduction

The influence of risk aversion on the outcomes in a two-person Nash bargaining solution has been considered in several studies. It was shown, when all outcomes are riskless, that risk aversion is disadvantageous; Namely, becoming more risk averse improves the outcome of the opponent. This result appears in Kannai (1977) for the Nash solution in a distribution problem, and in Kihlstrom, Roth, and Schmeidler (1981) for other solutions as well (including the Kalai-Smorodinsky solution). Other related works are those of Peters and Tijs (1985) and Wakker, Peters and van Riel (1986). The case where risky outcomes are allowed yields a different result under some conditions. It was shown by Roth and Rothblum (1982), for the Nash solution (when the **disagreement outcome** is riskless), that increasing risk aversion might be advantageous. This occurs when the Nash outcome involves the risk of being worse off than the disagreement outcome. This result was extended by Safra and Zilcha (1988b) to the case where the disagreement outcome is risky as well.

In all these works it was assumed that the bargaining individuals are expected utility maximizers. However, due to recent developments in the theory of decision making under risk, it is clear that this is not always the case (see Machina (1982) for a survey). In this work we investigate the relations between risk aversion and the outcomes in the Nash bargaining solution and the Kalai-Smorodinsky solution when general utility functionals are allowed. We show that in this case there is **no relationship** between risk aversion and the outcomes in these bargaining solutions, and bring examples to demonstrate it. It is shown that the parameter that influences the outcome of those solutions is not risk aversion, but rather, the degree of concavity of the certainty utility functions of the bargainers (where this utility function is obtained by restricting the utility functional to certain outcomes). In the case of expected utility the certainty utilities are the von Neumann-Morgenstern utility functions and their concavity is clearly equivalent to risk aversion. In general, however, there need not be any relation between the concavity of the certainty utility and the degree of risk aversion of the given functional (this is demonstrated by an example).

Section 2 contains some definitions. In section 3 we present the bargaining model and the solutions that we investigate in the sequel. The results for the Nash bargaining solution are described in section 4. Section 5 explains how these results can be extended to the Kalai-Smorodinski solution.

2. Definitions

Let C be a given set of certain alternatives which is assumed to be equal to an interval $[0,M]$ in R. The restriction $C \subset R$ is made only for the sake of facilitating the presentation; our analysis still holds for more general cases. We denote by L the set of cumulative distribution functions on $[0,M]$. An element F of L represents a lottery on C and the degenerate distribution that gives $c \in C$ with probability 1 is denoted by δ_c.

We consider utility functionals $V: L \rightarrow R$ that are monotonically increasing with respect to the relation of first-order stochastic dominance, (i.e., $V(F) \geq V(G)$ if for all x $F(x) \leq G(x)$) and are differentiable in the sense that there exists a function $\xi: R \times L \rightarrow R$ such that for all F and G

$$\frac{d}{d\alpha}V(F + \alpha(G - F))\Big|_{\alpha=0} = \int \xi(x;F)d(G-F)(x).$$

This notion of differentiability is called **Gateaux** differentiability. The function $\xi(\cdot,F)$ is called the **local utility function** at F (see Machina (1982)). In the case of expected utility $\xi(\cdot,F)$ is always equal to the von Neumann-Morgenstern (VNM) utility function.

With every utility functional $V:L \rightarrow R$ we associate a real function $u:C \rightarrow R$ which is the restriction of V to certain outcomes. We call it the **certainty utility** (CU) of V and it is defined by $u(c) = V(\delta_c)$ for all c in C. In the case of expected utility the certainty utility is the VNM utility function.

Examples

(a) **Weighted utility** (WU, see Chew (1983) and Fishburn (1983)).

A weighted utility functional is given by

$$V(F) = \frac{\int w(x)v(x)dF(x)}{\int w(x)dF(x)} \tag{1}$$

where w is the **weight** function and v is the **value** function. The **local utility function** at F is

$$\xi(x;F) = \frac{w(x)[v(x) - V(F)]}{\int w(x)dF(x)} \qquad (2)$$

and the CU is $u(c) = v(c)$.

(b) **Rank dependent utility** (RDU, see Quiggin (1982) and Yaari (1987)). The rank dependent utility (also known as **Anticipated utility** and **Expected utility with rank dependent probabilities**) is given by

$$V(F) = \int v(x)dg(F(x)) \qquad (3)$$

where the function $g:[0,1]\to[0,1]$ is increasing and onto. The local utility function at F is (see Chew, Karni and Safra (1987))

$$\xi(x,F) = \int_0^x v'(z)g'(F(z))dz \qquad (4)$$

and the CU is $u(c) = v(c)$.

Risk aversion is defined with respect to mean preserving spreads (see Safra and Zilcha (1988a) for other possible definitions). The comparison of the risk aversion of any two utility functionals V and $\hat{V}$ uses the idea of a simple compensated spread. Specifically, we say that $\hat{V}$ is **more risk averse** than V if for any F and G that satisfy (a) for some x^* $F(x) < G(x)$ if $x < x^*$ and $F(x) > G(x)$ if $x > x^*$, and (b) $V(F) = V(G)$, we have $\hat{V}(F) > \hat{V}(G)$. It was shown by Machina (1982) that this definition is equivalent (assuming a stronger notion of differentiability) to the condition that each local utility function of $\hat{V}$ is a concave transformation of some local utility function of V.

3. The Bargaining Model

A two-person bargaining game takes place where the two decision makers bargain in order to reach an agreed outcome in a given set of possible outcomes. We assume throughout this paper that this set is L. If they do not reach an agreement then they receive the disagreement outcome $\bar{c}$, which is an element of C.

We denote the utility functionals of the two bargainers by V_1 and V_2, and define

$$S = \{(V_1(F), V_2(F)) \mid F \in L\}. \tag{5}$$

S is the **feasible set** of utility payoffs. The element $d \in S$ defined by $d = (V_1(\delta_{\bar{c}}), V_2(\delta_{\bar{c}}))$ is called the **threat point** of the game and we assume that there exists $x \in S$ such that $x > d$. The Pareto-optimal subset of S is denoted by $P(S)$.

Hence, a bargaining game is described by $(S, d, \bar{c}, (V_1, V_2))$ where $\bar{c}$ is fixed and S and d depend on (V_1, V_2). The game is **deterministic** if every payoff can be achieved by a deterministic outcome, i.e., if $S = \{(V_1(\delta_c), V_2(\delta_c)) \mid c \in C\}$.

The **Nash bargaining solution** is denoted by $F(S, d)$ and it is the element in S that maximizes the product $(x_1 - d_1)(x_2 - d_2)$ over $\{x \in S \mid x \geq d\}$ (see Nash (1950)).

The **Kalai-Smorodinski (KS) solution** denoted by $G(S,d)$, is the element of $P(S)$ that satisfies $(x_1 - d_1)/(x_2 - d_2) = (x_1^I - d_1)/(x_2^I - d_2)$, where $x_j^I = \max\{x_j \mid x \in S \text{ and } x \geq d\}$, $j = 1,2$ (see Kalai and Smorodinski (1975)). The point x^I is called the **ideal point.**

4. Increasing Risk Aversion and Nash Bargaining Solution

4.1. Deterministic model

Let us compare the Nash solution for the two bargaining games $(S, d, \bar{c}, (V_1, V_2))$ and $(\hat{S}, \hat{d}, \bar{c}, (V_1, \hat{V}_2))$ where $\hat{V}_2$ is more risk averse than V_2 and $(\hat{S}, \hat{d})$ is attained from $(V_1, \hat{V}_2)$.

Under the expected utility assumption it was shown by Khilstrom, Roth and Schmeidler (KRS, (1981)) that player 1 is better off when player 2 becomes more risk averse, i.e., $F_1(S,d) \leq F_1(\hat{S}, \hat{d})$. In our case we show that this result holds if the CU of $\hat{V}_2$ is a concave transformation of the CU of V_2, i.e., $\hat{u}_2(c) = k(u_2(c))$, for some concave function k. We also show that this property **does not imply** that $\hat{V}_2$ is more risk averse than V_2.

Proposition 1: Let $(\hat{S}, \hat{d}, \bar{c}, (V_1, \hat{V}_2))$ be a deterministic bargaining game obtained from $(S, d, \bar{c}, (V_1, V_2))$ by replacing V_2 with $\hat{V}_2$. Let the certainty utilities of V_2 and $\hat{V}_2$ satisfy $\hat{u}_2 = k(u_2)$, for some concave function k, then $F_1(\hat{S}, \hat{d}) \geq F_1(S, d)$.

The proof of this proposition follows the same argument as in KRS (1981), hence we do not bring it.

Let us show now that more concave CU does not imply more risk aversion. We shall show this claim by the following example. Consider the WU functional V with the following weight and value functions:

$$w(x) = e^{-tx} \quad , \quad v(x) = \lambda x, \quad \lambda > 0 \text{ is constant,}$$

and the expected utility functional $\hat{V}$, with the VNM utility function

$$\hat{v}(x) = -e^{-tx}.$$

In this case the CU are $\hat{u} \equiv \hat{v}$ and $u \equiv v$, hence $\hat{u} = k(u)$ for some strictly concave k.

Let us show that any local utility function of V, $\xi(x,F)$, is more concave than the local utility function of $\hat{V}$ $\hat{\xi}(x,F)$ hence, by Machina (1982), V is more risk averse than $\hat{V}$. Let us compare the absolute risk aversion measures R_F and $\hat{R}_F$ of the local utility functions where $R_F(x) = -\xi_{11}(x;F)/\xi_1(x;F)$ and $\hat{R}_F(x) = -\hat{\xi}_{11}(x;F)/\hat{\xi}_1(x;F)$. For $\hat{V}$ any local utility is $\hat{v}$ hence $\hat{R}_F(x) \equiv t$. For V the absolute risk aversion measure is

$$R_F(x) = - \frac{w''(x)[v(x) - V(F)] + 2w'(x)v'(x)}{w'(x)(v(x) - V(F)) + w(x)v'(x)} = \tag{6}$$

$$= t\left[\frac{- t(\lambda x - V(F)) + 2\lambda}{- t(\lambda x - V(F)) + \lambda}\right]$$

Since V is continuous and all the distributions have support in $[0,M]$ then, by choosing t sufficiently small, we can be assured that $R_F(x) > \hat{R}_F(x)$ for all x. Thus V_2 is more risk averse than $\hat{V}_2$.

Let us also note that for RDU case Proposition 1 is closely related to risk aversion since the condition about the certainty utilities $\hat{u}_2 = k(u_2)$, for concave k, is implied if $\hat{V}_2$ is more risk averse than V_2. Thus the reult, in the deterministic model, that becoming more risk averse implies that your opponent is better off in the bargaining solution (Theorem 1, KRS (1981))holds in the RDU case. However, in our WU example we have actually derived the opposite result since V_2 is more risk averse than $\hat{V}_2$, while (by Proposition 1) $F_1(\hat{S},\hat{d}) > F_1(S,d)$.

4.2. **Risky outcomes**

We would like to show now that in the bargaining models with risky outcomes, when we relax the expected utility hypothesis, the results of Roth-Rothblum (RR) do not hold. Moreover, increasing risk aversion on the part of player 2 may result in an ambiguous change in the utility of player 1. Therefore we shall consider the simplest possible model to show our examples. For simplcity we assume that the Nash product attains a unique maximum in S.

Let C contain exactly 3 elements $\{c_1, c_2, \bar{c}\}$ where c_i is the outcome most preferred by player i. We shall see that Theorem 3 in RR(1982) does not hold if player 2 is a RDU maximizer whi-> player 1 is an expected utility maximizer. First let us indicate that if we increase the risk aversion of player 2 by making v (see (3)) more concave, leaving g unchanged, then RR's Theorem 3 still holds. Namely, if player 2 becomes more risk-averse in this manner the Nash bargaining solution for player 1 improves if the disagreement outcome $\bar{c}$ is worse (for player 2) than c_2 (for player 2). However, if $\bar{c}$ is preferred to c_2 (by player 2) player 1 is worse off. This is summarized in the following proposition which generalizes Theorem 3 in RR (1982).

Proposition 2: Let $(\hat{S}, \hat{d}, \bar{c}, (V_1, \hat{V}_2))$ be a bargaining game obtained from $(S, d, \bar{c}, (V_1, V_2))$ by replacing V_2 with V_2 and let u_2 and $\hat{u}_2$ be the certainty utilities of V_2 and $\hat{V}_2$ respectively. If

(a) The CU satisfy $\hat{u}_2 = k(u_2)$ with k concave, and

(b) $\hat{V}_2(F) = V_2(G)$ whenever $F = (x_1, p;x_2, 1\text{-}p)$ and $G = (y_1, p;y_2,1\text{-}p)$ satisy $\hat{u}_2(x_i) = u_2(y_i)$ $(i = 1,2)$,

Then, $u_2(\bar{c}) \geq u_2(c_1)$ implies $F_1(\hat{S}, \hat{d}) \leq F_1(S,d)$ and $u_2(\bar{c}) \leq u_2(c_1)$ implies $F_1(\hat{S},\hat{d}) \geq F_1(S,d)$.

Proof: First note that the boundary of $\hat{S}$ is the same as that of S. This follows from the application of condition (b) after normalizing

V_1, V_2 and $\hat{V}_2$ to achieve $u_1(c_1) = u_2(c_2) = \hat{u}_2(c_2) = 1$ and $u_1(c_2) = u_2(c_1) = \hat{u}_2(c_1) = 0$. (Note that the boundary of S, and $\hat{S}$, is composed of the utilities of lotteries over c_1 and c_2).

If $u_2(\bar{c}) \geq u_2(c_1)$ (=0) then, since $\hat{u}_2 = k(u_2)$, we get $\hat{u}_2(\bar{c}) \geq u_2(\bar{c})$ and thus $\hat{d}_2 \geq d_2$ (while $\hat{d}_1 = d_1$). It is now immediate to see that moving d upwards results in an increase in $F_2(S,d)$ and a decrease in $F_1(S,d)$. The case where $u_2(\bar{c}) \leq u_2(c_1)$ is proved similarly. □

Note that in the case of RDU increasing risk aversion by making v more concave satisfies the conditions (a) and (b) of Proposition 2.

Now we increase the risk aversion of player 2 by making g more concave and demonstrate that the R-R result does not hold.

Take $V_2(F) = \int v dF$ and $\hat{V}_2(F) = \int v dg(F)$ with g strictly concave. Assume $V_1(F) = \int v_1 dF$, and assume that $v(\bar{c}) = v(c_1) = 0$, $v(c_2) = 1$, $v_1(c_1) = 1$ $v_1(c_2) = 0$ and $v_1(\bar{c}) = 0$. The boundary of S is given by $x_2 = 1 - x_1$ since, for a typical lottery which yields c_1 with probability p and c_2 with probability 1-p, the utility V_1 is equal to p and the utility V_2 is equal to 1-p. The Nash bargaining solution is clearly $F(S,d) = (\frac{1}{2}, \frac{1}{2})$. Consider now $\hat{V}_2$ and $\hat{S}$. Given a lottery as above the value of $\hat{V}_2$ is $1-g(p)$ and thus the boundary of $\hat{S}$ is given by $x_2 = 1 - g(x_1)$. The Nash solution maximizes the product $x_1(1-g(x_1))$. Take now $g(p) = p^{1/2}$. Differentiating $x_1(1-x_1^{1/2})$ and equating to zero gives $x_1 = \frac{4}{9}$ as the global maximum. Hence the utility of the first player decreased from $\frac{1}{2}$ to $\frac{4}{9}$ while his opponent became more risk averse. Note that the same qualitative result will still hold if $d_2 = v(\bar{c}) < 0$. This is in contrast to the results of Roth and Rothblum.

5. Risk Aversion and the Kalai-Smorodinsky Solution

It has been shown by KRS (1981) that the impact of increasing risk aversion (on the part of player 2) on player 1's payoff in the Kalai-Smorodinsky bargaining solution is (in the deterministic model) similar to that attained for the Nash bargaining solution. Departing from the expected utility assumptions, we can show that, as in the Nash solution, there is ambiguity in the results in this case too. Again the results in KRS will depend upon increasing the **concavity of the CU** rather than increasing **risk aversion**. Our Proposition 1 and the examples we have brought can be generalized to the KS solution as well. Finally, we use an example to show that in the risky outcomes model an increase in player 2 risk aversion, while the threat point remains $d = (0,0)$, may make player 1 worse off.

Consider the example brought in section 4.2. The ideal point is $(1,1)$. The KS solution for (S,d) is $G(S,d) = (\frac{1}{2}, \frac{1}{2})$, while the KS solution for $(\hat{S}, \hat{d})$ is $G(\hat{S}, \hat{d}) = (\frac{1}{4}, \frac{1}{4})$. This clearly violates the spirit of Theorem 3 in RR (1982) and shows that this theorem cannot be extended to the KS solution with general utility functionals.

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