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Netherlands school of economics

ECONOMETRIC INSTITUTE

AN ABEL-TAUBER THEOREM ON
CONVOLUTIONS WITH THE MÖBIUS FUNCTIONS

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AN ABEL-TAUBER THEOREM ON CONVOLUTIONS WITH THE MÖBIUS FUNCTION

by

J.L. Geluk

ABSTRACT

Suppose $n: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $\frac{n(x)}{x}$ is integrable on $(0, \infty)$.

Then $\tilde{n}(s) = s \int_0^{\infty} \frac{e^{-us}}{1-e^{-us}} n(u) du$ exists for $s > 0$.

In this paper an Abel-Tauber theorem is proved concerning this transform.

Moreover the relation between $\tilde{n}(s)$ and $\sum_{m \leq s} \frac{1}{m} n(\frac{s}{m})$ is studied.

Contents

	<u>page</u>
Introduction	1
Results	3
References	13

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Introduction

Following earlier work by Landau [1] and others, Ingham proved (in [2]) the following theorem:

Theorem A

Suppose that

- (i) $f(x)$ is positive and non-decreasing for $x \geq 1$
(ii) $F(x) \equiv \sum_{n \leq x} f\left(\frac{x}{n}\right) = ax \log x + bx + o(x) \quad (x \rightarrow \infty)$

where a and b are constants. Then

(a) $f(x) \sim ax \quad (x \rightarrow \infty)$

(b) $\int_1^x \frac{f(x) - ax}{x^2} dx = b - a\gamma$

where γ is Euler's constant.

Moreover, for any function $f(x)$, bounded and integrable over every finite interval $(1, X)$ hypothesis (ii) implies conclusion (b). More recently Jukes [14] extended results of Segal (see [3], [4]) and proved the following theorem which can be considered as a generalization of theorem A. For a proof of this theorem see also [5].

Theorem B

Let $f(x)$ be bounded and integrable on every finite subinterval of $[1, \infty)$ and suppose that

$$\sum_{n \leq x} f\left(\frac{x}{n}\right) = xg(x) + o(x^2g'(x)) \quad (x \rightarrow \infty)$$

where $g(x) \in C^2 [1, \infty)$ is positive and satisfies

- (i) $g'(x) > 0$ for all $x \in [1, \infty)$
(ii) for some real r , $xg'(x)(\log x)^{-r}$ is non-increasing for $x > x_0$
(iii) for some real s , $xg'(x)(\log x)^s \equiv q(x)$ is non-decreasing for $x > x_1$ and
 $q(x) \rightarrow \infty \quad (x \rightarrow \infty)$

$$\text{Then } \int_1^x \frac{f(t)}{t^2} dt = g(x) - \gamma xg'(x) + o(xg'(x)) \quad (x \rightarrow \infty)$$

If we suppose $f(x)$ to be positive and non-decreasing we also have $f(x) \sim x^2 g'(x)$ ($x \rightarrow \infty$)

In this note we consider the situation in theorem B where $g(x)$ is allowed to grow faster than in (ii).

We use the concept of regular variation as introduced by Karamata to formulate natural conditions on $g(x)$ and prove a theorem similar to theorem B. Moreover, an application is given which gives a second order condition in a relation considered by Parameswaran in [6]. See our corollary 1 and the remark after theorem 4.

Parameswaran proves the following theorem ([6], theorem II).

Theorem C

If $\int_0^R \frac{n(u)}{u} du$ exists in the Lebesgue sense for every positive R

$$f(s) \equiv \exp \left\{ s \int_0^{\infty} \frac{e^{-su}}{1 - e^{-su}} n(u) du \right\}$$

for all positive s and $n(u)$ non-decreasing, then the relation

$$\log f(s) \sim \int_a^{1/s} \frac{L(u)}{u} du \text{ as } s \rightarrow 0^+ \text{ implies the relation}$$

$n(u) \sim L(u)$ as $u \rightarrow \infty$, provided $L(x)$ is a slowly oscillating function defined for $x \geq a$ such that

$$L(x) \sim K \exp \left\{ \int_a^x \frac{\delta(u)}{u} du \right\} \text{ as } x \rightarrow \infty \text{ where } \delta \text{ is a non-increasing function and}$$

K is a positive real number.

We use our theorem 3 to get a theorem similar to theorem C.

Results

First we define the subclass of slowly varying functions we want to consider.

Definition: The non-decreasing function $q : \mathbb{R}^+ \rightarrow \mathbb{R}$ belongs to the class Π if there

$$\text{exists a slowly varying function } L \text{ such that } q(x) = \int_1^x \frac{L(t)}{t} dt + L(x)$$

It can be shown that the class Π consists of all non-decreasing functions $q : \mathbb{R}^+ \rightarrow \mathbb{R}$ for which there exist functions $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $b : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that for all positive x

$$\lim_{t \rightarrow \infty} \frac{q(tx) - b(t)}{a(t)} = \log x$$

If the last relation holds, then it is true with $b(t) = q(t)$ and $a(t) = q(te) - q(t)$.

The function $a(t)$ is of course determined up to asymptotic equivalence and is called the auxiliary function. As a consequence of these facts we mention the following

result: if $q(x) \in \Pi$ with auxiliary function $L(x)$, $q_1(x)$ is non-decreasing and

$$\frac{q(x) - q_1(x)}{L(x)} \rightarrow C \quad (x \rightarrow \infty), \text{ where}$$

$C \in \mathbb{R}$ is a constant, then $q_1(x) \in \Pi$ with auxiliary function $L(x)$.

It is easy to see that if $L(x)$ slowly varying, then $\int_1^x \frac{L(t)}{t} dt$ is an element of Π with auxiliary function $L(x)$.

The class Π is a subclass of the slowly varying functions at infinity. Moreover, for each function $q \in \Pi$ it is possible to find a function L_* such that $q(x) =$

$$\int_1^x \frac{L_*(t)}{t} dt + o(L(x)) \quad (x \rightarrow \infty)$$

where $L_* \sim L$ is slowly varying.

For the proofs of these statements and further properties see [10], [11] and [12].

We start with two Abelian results.

Theorem 1

Suppose $n : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is slowly varying, $\frac{n(x)}{x}$ is (Lebesgue) integrable on finite subintervals of $(0, \infty)$

Then $\tilde{n}(\frac{1}{s}) \in \Pi$ with auxiliary function $n(s)$, where \tilde{n} is defined by

$$\tilde{n}(s) = s \int_0^{\infty} \frac{e^{-us}}{1-e^{-us}} n(u) du$$

Moreover,

$$\frac{\tilde{n}(\frac{1}{s}) - \int_0^s \frac{n(t)}{t} dt}{n(s)} \rightarrow 0 \quad (s \rightarrow \infty)$$

Proof:

$$\frac{\int_0^s \frac{n(t)}{t} dt - \tilde{n}(\frac{1}{s})}{n(s)} = \int_0^1 \left(\frac{1}{u} - \frac{e^{-u}}{1-e^{-u}} \right) \frac{n(su)}{n(s)} du - \int_1^{\infty} \frac{e^{-u}}{1-e^{-u}} \frac{n(su)}{n(s)} du$$

We have $\int_0^1 \frac{n(su)}{n(s)} du = \frac{\int_0^s n(v) dv}{sn(s)} \rightarrow 1 \quad (s \rightarrow \infty)$ by Karamata's theorem on regularly

varying functions and since $\frac{1}{u} - \frac{e^{-u}}{1-e^{-u}}$ is bounded on $(0, 1)$ we can apply Pratt's lemma.

For the second part we have $\frac{(su)^{-\epsilon} n(su)}{s^{-\epsilon} n(s)} \rightarrow u^{-\epsilon} \quad (s \rightarrow \infty)$ uniform on $(1, \infty)$ by corollary

1.2.1.4 of [11].

So we have

$$\frac{\int_0^s \frac{n(t)}{t} dt - \tilde{n}(\frac{1}{s})}{n(s)} \rightarrow \int_0^1 \left(\frac{1}{u} - \frac{e^{-u}}{1-e^{-u}} \right) du - \int_1^{\infty} \frac{e^{-u}}{1-e^{-u}} du \quad (s \rightarrow \infty)$$

The right side is zero, as is shown by elementary integration. This implies that

$\tilde{n}(1/s) \in \Pi$ with auxiliary function $n(s)$, since

$\int_0^s \frac{n(t)}{t} dt \in \Pi$ with auxiliary function $n(s)$ and $\tilde{n}(1/s)$ is non-decreasing.

Remarks: 1. The case $n \in RV_\alpha^{(\infty)}$ ($\alpha > 0$) (regularly varying at infinity with exponent α)

gives analogously $\frac{\tilde{n}(1/s)}{n(s)} \rightarrow \zeta(\alpha+1) \Gamma(\alpha+1)$ ($s \rightarrow \infty$). See [13], theorem 1.

2. The statement of the theorem implies $\tilde{n}(1/s) \sim \int_0^s \frac{n(t)}{t} dt$ since $n(s) =$

$o\left(\int_0^s \frac{n(t)}{t} dt\right)$ by Karamata's theorem.

See theorem I of [6].

Theorem 2. If we define $\Psi(s) = \sum_{m \leq s} \frac{1}{m} n\left(\frac{s}{m}\right)$ where n satisfies the conditions of theorem 1 and $xn(x)$ is of bounded variation on intervals of the form $(1, x_0)$, then

$$\frac{\Psi(s) - \int_0^s \frac{n(t)}{t} dt}{n(s)} \rightarrow \gamma \quad (s \rightarrow \infty)$$

Proof We give the proof by an Euler-Maclaurin kind of argument

$$\int_{v-1}^v \frac{x}{t} n\left(\frac{x}{t}\right) dt = xn\left(\frac{x}{v}\right) - xn\left(\frac{x}{v-1}\right) - \int_{v-1}^v t d\frac{x}{t} n\left(\frac{x}{t}\right) =$$

$$\frac{x}{v} n\left(\frac{x}{v}\right) - \int_{v-1}^v \{t\} d\frac{x}{t} n\left(\frac{x}{t}\right)$$

where we use the notation $\{t\} = t - [t]$

Summing over v gives

$$\sum_{1 \leq v \leq \frac{x}{\nu}} \frac{1}{v} n\left(\frac{x}{v}\right) - \int_1^x \frac{1}{t} n\left(\frac{x}{t}\right) dt = n(x) - \int_{[x]}^x \frac{1}{t} n\left(\frac{x}{t}\right) dt + \int_1^{[x]} \{t\} d\frac{1}{t} n\left(\frac{x}{t}\right)$$

Now with $M = \sup_{x \in (1,2)} n(x)$ we have

$$\left| \frac{\int_{[x]}^x \frac{1}{t} n\left(\frac{x}{t}\right) dt}{n(x)} \right| \leq \frac{M \log \frac{x}{[x]}}{n(x)} \leq \frac{M}{n(x)(x-1)} \rightarrow 0 \quad (x \rightarrow \infty)$$

since $(x-1)n(x)$ is 1-varying at infinity.

Furthermore $\int_1^{[x]} \{t\} d \frac{1}{t} n \left(\frac{x}{t} \right) = - \frac{1}{x} \int_1^{[x]} \frac{x}{t} n \left(\frac{x}{t} \right) d \{t\} \sim -n(x) \int_1^{[x]} \frac{1}{t} d \{t\} \sim (\gamma-1) n(x)$

since $\frac{x}{t} n \left(\frac{x}{t} \right) / xn(x) \rightarrow t^{-1}$ uniform for $t \in (1, \infty)$ (see [11], cor. 1.2.1.4)

and $\int_1^{\infty} \frac{\{t\}}{t^2} dt = 1-\gamma$ (see e.g. [7])

This proves theorem 2.

Remarks: 1. The theorem implies $\frac{\Psi(tx) - \Psi(t)}{\Psi(te) - \Psi(t)} \rightarrow \log x \quad (t \rightarrow \infty)$

where $x > 0$ is fixed. If Ψ is non-decreasing we have $\Psi \in \Pi$.

2. With the additional supposition that Ψ is non-decreasing a simpler proof is possible:

$$\tilde{n}(s) - s \int_0^1 \frac{e^{-us}}{1-e^{-us}} n(u) du = s \int_1^{\infty} \sum_{k=1}^{\infty} e^{-kus} n(u) du = \int_0^{\infty} e^{-us} d\Psi(u)$$

Since $\lim_{s \rightarrow \infty} \frac{1}{s} \int_0^1 \frac{e^{-u/s}}{1-e^{-u/s}} n(u) du = \int_0^1 \frac{n(u)}{u} du$ we have

by theorem 1 $\int_0^{\infty} e^{-u/s} d\Psi(u) \in \Pi$ with auxiliary function n .

Hence $\frac{\tilde{n}(1/s) - \Psi(s)}{n(s)} \rightarrow -\gamma(s \rightarrow \infty)$ (see [10], theorem 1)

Combining this with theorem 1 gives the desired result.

3. The case $n \in RV_{\alpha}^{(\infty)}$ ($\alpha > 0$) in remark 2 gives

$$\frac{\Psi(s)}{\int_0^s \frac{n(t)}{t} dt} \rightarrow \alpha \zeta(\alpha+1) \quad (s \rightarrow \infty)$$

4. The statement of the theorem implies $\Psi(s) \sim \int_0^s \frac{n(t)}{t} dt$ since $n(s) =$

$$o\left(\int_0^s \frac{n(t)}{t} dt\right) \quad (s \rightarrow \infty)$$

As a consequence of remark 2 we mention the following:

Corollary 1. Suppose $n : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfies the conditions $\int_0^R \frac{n(u)}{u} du < \infty$ for $R > 0$

and $\sum_{m \leq s} \frac{1}{m} n\left(\frac{s}{m}\right)$ is non-decreasing for $s > 0$.

The assertions $\sum_{m \leq s} \frac{1}{m} n\left(\frac{s}{m}\right) \in \Pi$ with auxiliary function $L(s) \rightarrow \infty (s \rightarrow \infty)$

and $\tilde{n}(1/s) \in \Pi$ with auxiliary function $L(s) \rightarrow \infty (s \rightarrow \infty)$ are equivalent.

Both imply:
$$\frac{\sum_{m \leq s} \frac{1}{m} n\left(\frac{s}{m}\right) - \tilde{n}(1/s)}{L(s)} \rightarrow \gamma \quad (s \rightarrow \infty)$$

For the Tauberian counterpart of the theorems 1 and 2 we need three lemmas.

Lemma 1: If

- (i) $L(x)$ is non-decreasing for $x > 0$, slowly varying, $L(x) \rightarrow \infty (x \rightarrow \infty)$
- (ii) $\frac{L(x)}{L(x-1)} \leq 1 + x^{-\alpha}$ for some $\alpha > 0$, $x \geq x_0 = x_0(\alpha)$

then

$$\sum_{m \leq x} \frac{\mu(m)}{m} L\left(\frac{x}{m}\right) = o(L(x)) \quad (x \rightarrow \infty)$$

Proof: We define $a_n = L(n) - L(n-1)$ for $n \geq 2$, $a_1 = L(1)$

Then

$$\sum_{m \leq x} \frac{\mu(m)}{m} L\left(\left[\frac{x}{m}\right]\right) = \sum_{m \leq x} \frac{\mu(m)}{m} \sum_{\substack{v \leq x \\ v \equiv m}} a_v = \sum_{m \leq x} a_m N\left(\frac{x}{m}\right)$$

where

$N(x) = \sum_{m \leq x} \frac{\mu(m)}{m}$. Since $|N(x)| < \epsilon$ for $x \geq x_\epsilon$ (see [15]) we have:

$$\left| \sum_{\substack{m \leq x \\ m \equiv x_\epsilon}} a_m N\left(\frac{x}{m}\right) \right| < \epsilon \quad \sum_{\substack{m \leq x \\ m \equiv x_\epsilon}} a_m = \epsilon L\left(\left[\frac{x}{x_\epsilon}\right]\right) \leq \epsilon(1+\epsilon) L(x) \quad (x > x_\epsilon)$$

$$\text{and } \left| \sum_{\frac{x}{x_0} < m \leq x} a_m N\left(\frac{x}{m}\right) \right| \leq c \sum_{\frac{x}{x_0} < m \leq x} a_m \leq c \{L(x) - L\left(\frac{x}{x_0}\right)\} = o(L(x)) \quad (x \rightarrow \infty)$$

For $x \geq x_0(\alpha)$ we have by (ii)

$$L(x) - L([x]) \leq L(x) - L(x-1) \leq \frac{L(x)}{x^\alpha} \text{ since } L \text{ is non-decreasing. Hence:}$$

$$\left| \sum_{1 \leq m \leq \frac{x}{x_0}} \frac{\mu(m)}{m} \{L\left(\frac{x}{m}\right) - L\left(\left[\frac{x}{m}\right]\right)\} \right| \leq \sum_{m \leq \frac{x}{x_0}} \frac{1}{m} \frac{L(x/m)}{x^\alpha/m^\alpha} \leq$$

$$x^{-\alpha} \left\{ \int_1^{x/x_0} \frac{1}{u^{1-\alpha}} L\left(\frac{x}{u}\right) du + L(x) \right\} = \int_{x_0}^x \frac{L(v)}{v^{1+\alpha}} dv + o(L(x)) = o(L(x))$$

if we choose $0 < \alpha < 1$.

We estimate the last sum as follows:

$$\left| \sum_{\frac{x}{x_0} < m \leq x} \frac{\mu(m)}{m} \{L\left(\frac{x}{m}\right) - L\left(\left[\frac{x}{m}\right]\right)\} \right| \leq 2 \sum_{\frac{x}{x_0} < m \leq x} \frac{1}{m} L\left(\frac{x}{m}\right) \leq 2 L(x_0) \sum_{\frac{x}{x_0} < m \leq x} \frac{1}{m}$$

$= O(1) = o(L(x)) \quad (x \rightarrow \infty)$. This proves the lemma.

Remark: The conclusion of the lemma is incorrect for arbitrary slowly varying functions.

There exists a function $L(x) \rightarrow C (\neq 0) \quad (x \rightarrow \infty)$ such that:

$$\sum_{m \leq x} \frac{\mu(m)}{m} L\left(\frac{x}{m}\right) \neq o(1) \quad (x \rightarrow \infty)$$

See e.g. [4] example 2.

Lemma 2 Suppose $\sum_{m \leq x} \frac{1}{m} n\left(\frac{x}{m}\right) = \int_1^x \frac{L(u)}{u} du + L(x)$ where L satisfies the conditions of lemma 1. Then $n(x) \sim L(x) \quad (x \rightarrow \infty)$.

Proof: Möbius inversion and lemma 1 gives:

$$\begin{aligned} n(x) &= \sum_{m \leq x} \frac{\mu(m)}{m} \int_1^{x/m} \frac{L(u)}{u} du + \sum_{m \leq x} \frac{\mu(m)}{m} L\left(\frac{x}{m}\right) \\ &= \int_1^x \frac{L(u)}{u} N\left(\frac{x}{u}\right) du + o(L(x)) = \int_1^x L\left(\frac{x}{v}\right) \frac{N(v)}{v} dv + o(L(x)) \sim L(x) \int_1^\infty \frac{N(v)}{v} dv \sim L(x) \end{aligned}$$

($x \rightarrow \infty$) by dominated convergence, since

$$\int_1^x \frac{1}{v} \sum_{k \leq v} \frac{\mu(k)}{k} dv = \sum_{k \leq x} \frac{\mu(k)}{k} \int_k^x \frac{dv}{v} = \log x \cdot \sum_{k \leq x} \frac{\mu(k)}{k} - \sum_{k \leq x} \frac{\mu(k)}{k} \log k + 1 \quad (x \rightarrow \infty)$$

(see [15]).

Remark: The proof of lemma 2 implies the following:

If $h(x) = \int_1^x \frac{L(u)}{u} du + L(x)$ where L satisfies the conditions of lemma 1, then

$$\sum_{m \leq x} \frac{\mu(m)}{m} h\left(\frac{x}{m}\right) \sim \frac{1}{x} \int_1^x s dh(s), \text{ which is the auxiliary function of } h(x). \text{ For}$$

functions $h(x)$ of the form $\int_1^x \frac{L(u)}{u} du$, L slowly varying the conditions of lemma

1 are not necessary for this result. Compare with theorem 1 of [4].

The following lemma gives the same statement under different conditions on $L(x)$.

Lemma 3. Suppose $\sum_{m \leq x} \frac{1}{m} n\left(\frac{x}{m}\right) = \int_1^x \frac{L(u)}{u} du + L(x)$ where $L(x) \rightarrow \infty$ and $L(x) \in \Pi$

Then $n(x) \sim L(x) \quad (x \rightarrow \infty)$

Proof: We write $L(x) = \int_1^x \frac{\gamma(u)}{u} du + o(\gamma(x))$ where γ is slowly varying.

Then Möbius inversion gives:

$$n(x) = \sum_{m \leq x} \frac{\mu(m)}{m} \int_1^{x/m} \frac{L(u) + \gamma(u)}{u} du + \sum_{m \leq x} \frac{\mu(m)}{m} o\left(\gamma\left(\frac{x}{m}\right)\right)$$

The first part is asymptotic to $L(x)$ as in lemma 2 since $\gamma(x) = o(L(x))$ ($x \rightarrow \infty$).

For the second term we proceed as follows:

Suppose $|o(\gamma(x))| < \epsilon \gamma(x)$ for $x \geq x_0$. Then we have:

$$\begin{aligned} \left| \sum_{m \leq x} \frac{\mu(m)}{m} o(\gamma(\frac{x}{m})) \right| &\leq \epsilon \sum_{\substack{m \leq \frac{x}{x_0} \\ m \leq x_0}} \frac{1}{m} \gamma(\frac{x}{m}) + \sum_{\substack{\frac{x}{x_0} < m \leq x \\ m \leq x_0}} \frac{1}{m} |o(\gamma(\frac{x}{m}))| \\ &\leq \epsilon \sum_{\substack{m \leq \frac{x}{x_0} \\ m \leq x_0}} \frac{1}{m} \gamma(\frac{x}{m}) + c \sum_{\substack{\frac{x}{x_0} < m \leq x \\ m \leq x_0}} \frac{1}{m} < \epsilon \sum_{m \leq x} \frac{1}{m} \gamma(\frac{x}{m}) + o(1) \end{aligned}$$

where $c = \sup_{x \in (1, x_0)} |o(\gamma(x))|$

Now we have $\sum_{m \leq x} \frac{1}{m} \gamma(\frac{x}{m}) \sim L(x)$ ($x \rightarrow \infty$) by theorem 2. This proves

the lemma.

Theorem 3. Suppose

$$(i) \quad \int_1^x \frac{L(u)}{u} du + o(L(x)) = \sum_{m \leq x} \frac{1}{m} n(\frac{x}{m})$$

where $L(x) \rightarrow \infty$ ($x \rightarrow \infty$) is slowly varying

$$(ii) \quad R(x) \equiv \sum_{m \leq x} \frac{1}{m} n(\frac{x}{m}) \text{ is non-decreasing}$$

$$(iii) \quad L_*(x) \equiv \frac{1}{x} \int_1^x t dR(t) \text{ is non-decreasing}$$

$$(iv) \quad \frac{L_*(x)}{L_*(x-1)} \leq 1 + x^{-\alpha} \text{ with } \alpha > 0 \text{ for } x > x_0(\alpha)$$

then $n(x) \sim L(x)$ ($x \rightarrow \infty$).

Proof: We have $R(x) = \int_1^x \frac{L_*(t)}{t} dt + L_*(x)$ where $L_*(x)$ is defined as

above (see [11] theorem 1.3.4.) ; $L_*(x) \sim L(x)$ satisfies the

conditions of lemma 2. This proves the theorem.

Theorem 4. Suppose (i) and (ii) of theorem 3 and $L_*(x) \in \Pi$

Then $L(x) \sim n(x) \quad (x \rightarrow \infty)$

Proof: We use now lemma 3. The rest of the proof is as in theorem 3.

Combining corollary 1 with the theorems 3 or 4 gives the following Tauberian theorem.

Theorem 5.

If $\tilde{n}(1/s) = \int_1^s \frac{L(u)}{u} du + o(L(s)) \quad (s \rightarrow \infty)$ with $L(s) \rightarrow \infty$

slowly varying and $n : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfies the conditions

(i) $\frac{n(u)}{u}$ is integrable on $(0, R)$ for every $R > 0$

(ii) $R(x) \equiv \sum_{m \leq x} \frac{1}{m} n\left(\frac{x}{m}\right)$ is non-decreasing

(iii) $L_*(x) \equiv \frac{1}{x} \int_1^x t dR(t)$ is non-decreasing

and (iv) $L_*(x) \in \Pi$

or (v) $\frac{L_*(x)}{L_*(x-1)} \leq 1 + x^{-\alpha}$ with $\alpha > 0$ for $x \geq x_0(\alpha)$

then
$$\frac{\int_1^s \frac{n(t)}{t} dt - \int_1^s \frac{L(t)}{t} dt}{L(s)} \rightarrow 0 \quad (s \rightarrow \infty)$$

Proof: Applying corollary 1 and theorem 3 or 4 we have $n(x) \sim L(x)$

Application of theorem 1 yields the result.

For regularly varying functions the following analogue of theorem 5 is well-known.

Theorem 6. If $\tilde{n}(1/s) \sim \zeta(\alpha+1) \Gamma(\alpha+1) s^\alpha L(s)$ where

$L(s) \in RV_\alpha^{(\infty)}$ with $\alpha > 0$ and $n(u)$ satisfies the following conditions

(i) $\frac{n(u)}{u}$ and $\frac{n(u)}{u} \log u$ are integrable on $(0, R)$ for every $R > 0$

(ii) $n(u)$ is non-decreasing

then $n(u) \sim u^\alpha L(u) \quad (u \rightarrow \infty)$

For a proof see [13] theorem 2.

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