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~~A NOTE ON THE ESTIMATION OF A SAMPLE OF~~  
~~THE DISTURBANCES IN THE LINEAR MODEL~~

by C. Dubbelman

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Preliminary and Confidential

A NOTE ON THE ESTIMATION OF A SAMPLE OF  
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Consider the linear model

$$y = X\beta + u$$

where  $y$  is  $n \times 1$ ,  $X$  is  $n \times k'$ ,  $\beta$  is  $k' \times 1$ , and  $u$  is  $n \times 1$ . The non-observable disturbance vector  $u$  is assumed to have zero mean and non-singular covariance matrix  $\Gamma = SS'$ . The data matrix  $X$  is assumed to have rank  $k$  and is nonstochastic. The parameter vector  $\beta$  consists of  $k'$  unknown constants. The vector  $y$  is regarded as being stochastic, with given elements.

The purpose of this paper is to provide a generalization of the disturbance estimator given in Dubbelman, Abrahamse and Koerts [1970], for the sake of statistical inference on the distribution of the disturbances. The general estimator  $w$  is linear in  $y$ , has zero mean and given covariance matrix  $KK'$ , where  $K$  has order  $p \times r$  and rank  $r$ ,  $r \leq n - k$ , and it minimizes the objective function  $E[(w - J'u)'Q(w - J'u)]$ , where  $Q$  is some symmetric and nonsingular  $p \times p$  matrix, and  $J$  is some  $n \times p$  matrix with minimal rank  $r$ .

The conditions on  $w$  are formalized as follows. Write

$$(i) \quad w = B'y$$

where the  $p \times n$  matrix  $B'$  is independent of  $y$ . From  $E(w) = 0$  we have  $E(B'y) = E(B'X\beta + B'u) = B'X\beta = 0$  for all  $\beta$ , so that

$$(ii) \quad B'X = 0$$

Since the rank of  $X$  is  $k$ , the rank of  $B$  cannot exceed  $n - k$ . Also we find  $B'y = B'u$ , so that  $\text{var}(w) = E(ww') = E(B'uu'B) = B'\Gamma B$ . Hence

$$(iii) \quad B'\Gamma B = KK'$$

and B has rank r. The objective function can be written as  $\text{tr} [(B - J)Q(B - J)'\Gamma] = \text{tr} [QKK'] + \text{tr} [QJ'\Gamma J] - 2\text{tr} [BQJ'\Gamma]$ . Thus B must be chosen in such a way that

$$(iv) \quad \text{tr} [BQJ'\Gamma] \text{ is maximized.}$$

Let R be an  $n \times k$  matrix spanning  $M(X)$  and define

$$\bar{M} \equiv I_{(n)} - (S^{-1}R)(R'\Gamma^{-1}R)^{-1}(S^{-1}R)' \equiv \bar{N}\bar{N}' \quad \bar{N}'\bar{N} = I_{(n-k)}$$

$$W \equiv K'QJ'S\bar{N}$$

$$H \equiv (WW')^{-\frac{1}{2}}W$$

then (i), (ii), (iii), and (iv) are fulfilled by

$$w = B'y = KH\bar{N}'S^{-1}y = K(K'QJ'S\bar{M}S'JQK)^{-\frac{1}{2}}K'QJ'S\bar{M}S^{-1}y$$

The estimator  $w$  is defined and is unique if and only if the  $r \times (n - k)$   $W$  has rank  $r$ .

Proof: Obviously (i) is fulfilled. By writing  $X = RG$  for some unique  $k \times k$   $G$  we find  $\bar{N}'S^{-1}X = \bar{N}'(S^{-1}R)G = 0$ , so that (ii) is satisfied. Using  $B' = KH\bar{N}'S^{-1}$  we find that (iii) is satisfied if and only if  $HH' = I_{(r)}$ . Substituting the same expression into (iv) we find that  $H$  must be chosen in such a way that  $\text{tr} [H'K'QJ'S\bar{N}] = \text{tr} [H'W]$  is maximized. To prove that  $H_0 = (WW')^{-\frac{1}{2}}W$  uniquely maximizes  $\text{tr} [H'W]$  subject to  $HH' = I_{(r)}$ , we use the following argument. First observe that the  $r \times r$   $WW'$  is positive definite if and only if  $W$  has rank  $r$ . In that case  $(WW')^{-\frac{1}{2}}$  exists. Second,  $H_0H_0' = (WW')^{-\frac{1}{2}}WW'(WW')^{-\frac{1}{2}} = I_{(r)}$ . Third, any other  $r \times (n - k)$   $H_1$  can be written as  $H_0 + F$ ,  $F \neq 0$ . Substituting  $H_0 = (WW')^{-\frac{1}{2}}W$  into  $I_{(r)} = H_1H_1' = (H_0 + F)(H_0 + F)'$ , premultiplying the result by  $(WW')^{\frac{1}{2}}$ , and taking traces, we get

$$\text{tr} [F'W] = -\frac{1}{2}\text{tr} [F'(WW')^{\frac{1}{2}}F] = -\frac{1}{2} \sum_{i=1}^{n-k} [f_i'(WW')^{\frac{1}{2}}f_i]$$

where  $f_i$  is the  $i$ -th column of  $F$ . Since  $f_i'(WW')^{-\frac{1}{2}}f_i \geq 0$  for all  $i$ , while the equality sign only holds for  $f_i = 0$ , we have  $\text{tr}[F'W] < 0$ , and hence  $\text{tr}[H_0'W] > \text{tr}[H_1'W]$ .

For particular choices of  $K$ ,  $Q$ ,  $J$ , and  $\Gamma$  some familiar estimators appear. If  $J = I_{(n)}$ ,  $Q = \Gamma^{-1}$ , and  $KK' = \bar{S}\bar{S}'$ , then

$$w = K(K'\Gamma^{-1}KK'\Gamma^{-1}K)^{-\frac{1}{2}}K'\Gamma^{-1}KK'\Gamma^{-1}y = KK'\Gamma^{-1}y = \bar{S}\bar{S}^{-1}y$$

which is the generalized least-squares estimator for  $k' = k$ . If  $J$  is an  $n \times (n - k)$  submatrix of  $I_{(n)}$ ,  $Q = I_{(n)}$ ,  $\Gamma = \sigma^2 I_{(n)}$ ,  $K = \sigma I_{(n-k)}$ , and  $k' = k$ , so that  $\bar{M}$  becomes  $M = I - X(X'X)^{-1}X'$ , then

$$w = (J'MJ)^{-\frac{1}{2}}J'My$$

which is a BLUS estimator. Finally, if  $J = Q = I_{(n)}$ ,  $\Gamma = \sigma^2 I_{(n)}$ ,  $k' = k$ , and the  $n \times (n - k)$   $K$  is replaced by  $\sigma K_1$  with  $K_1'K_1 = I_{(n-k)}$ , then

$$w = K_1(K_1'MK_1)^{-\frac{1}{2}}K_1'My$$

which is the estimator developed by Abrahamse and Koerts [1971].

#### REFERENCES

- Abrahamse, A.P.J., and J. Koerts (1971), "New Estimators of Disturbances in Regression Analysis", Journal of the American Statistical Association 66, pp. 71-74.
- Dubbelman, C., A.P.J. Abrahamse, and J. Koerts (1970), "A New Class of Disturbance Estimators in the General Linear Model". Report 7008 of the Econometric Institute of the Netherlands School of Economics, Rotterdam.

