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A NOTE ON THE ESTIMATION OF A SAMPLE OF THE DISTURBANCES IN THE LINEAR MODEL

by C. Dubbelman

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Consider the linear model

$$y = X\beta + u$$

where y is $n \times 1$, X is $n \times k'$, β is $k' \times 1$, and u is $n \times 1$. The non-observable disturbance vector u is assumed to have zero mean and non-singular covariance matrix $\Gamma = SS'$. The data matrix X is assumed to have rank k and is nonstochastic. The parameter vector β consists of k' unknown constants. The vector y is regarded as being stochastic, with given elements.

The purpose of this paper is to provide a generalization of the disturbance estimator given in Dubbelman, Abrahamse and Koerts [1970], for the sake of statistical inference on the distribution of the disturbances. The general estimator w is linear in y, has zero mean and given covariance matrix KK', where K has order $p \times r$ and rank r, $r \leq n - k$, and it minimizes the objective function $E[(w - J'u)'Q(w - J'u)]$, where Q is some symmetric and nonsingular $p \times p$ matrix, and J is some $n \times p$ matrix with minimal rank r.

The conditions on w are formalized as follows. Write

(i) $$w = B'y$$

where the $p \times n$ matrix B' is independent of y. From $E(w) = 0$ we have $E(B'y) = E(B'X\beta + B'u) = B'X\beta = 0$ for all β, so that

(ii) $$B'X = 0$$

Since the rank of X is k, the rank of B cannot exceed n - k. Also we find $B'y = B'u$, so that $\text{var}(w) = E(ww') = E(B'uu'B) = B'\Gamma B$. Hence

(iii) $$B'\Gamma B = KK'$$

and B has rank r. The objective function can be written as $\mathrm{tr}\,[(B - J)Q(B - J)'\Gamma] = \mathrm{tr}\,[QKK'] + \mathrm{tr}\,[QJ'\Gamma J] - 2\mathrm{tr}\,[BQJ'\Gamma]$. Thus B must be chosen in such a way that

(iv) $\quad \mathrm{tr}\,[BQJ'\Gamma]$ is maximized.

Let R be an $n \times k$ matrix spanning $\mathcal{M}(X)$ and define

$$\bar{M} \equiv I_{(n)} - (S^{-1}R)(R'\Gamma^{-1}R)^{-1}(S^{-1}R)' \equiv \bar{N}\bar{N}' \qquad \bar{N}'\bar{N} = I_{(n-k)}$$

$$W \equiv K'QJ'S\bar{N}$$

$$H \equiv (WW')^{-\frac{1}{2}}W$$

then (i), (ii), (iii), and (iv) are fulfilled by

$$w = B'y = KH\bar{N}'S^{-1}y = K(K'QJ'S\bar{M}S'JQK)^{-\frac{1}{2}}K'QJ'S\bar{M}S^{-1}y$$

The estimator w is defined and is unique if and only if the $r \times (n - k)$ W has rank r.

Proof: Obviously (i) is fulfilled. By writing $X = RG$ for some unique $k \times k$ G we find $\bar{N}'S^{-1}X = \bar{N}'(S^{-1}R)G = 0$, so that (ii) is satisfied. Using $B' = KH\bar{N}'S^{-1}$ we find that (iii) is satisfied if and only if $HH' = I_{(r)}$. Substituting the same expression into (iv) we find that H must be chosen in such a way that $\mathrm{tr}\,[H'K'QJ'S\bar{N}] = \mathrm{tr}\,[H'W]$ is maximized. To prove that $H_0 = (WW')^{-\frac{1}{2}}W$ uniquely maximizes $\mathrm{tr}\,[H'W]$ subject to $HH' = I_{(r)}$, we use the following argument. First observe that the $r \times r$ WW' is positive definite if and only if W has rank r. In that case $(WW')^{-\frac{1}{2}}$ exists. Second, $H_0H_0' = (WW')^{-\frac{1}{2}}WW'(WW')^{-\frac{1}{2}} = I_{(r)}$. Third, any other $r \times (n - k)$ H_1 can be written as $H_0 + F$, $F \neq 0$. Substituting $H_0 = (WW')^{-\frac{1}{2}}W$ into $I_{(r)} = H_1H_1' = (H_0 + F)(H_0 + F)'$, premultiplying the result by $(WW')^{\frac{1}{2}}$, and taking traces, we get

$$\mathrm{tr}\,[F'W] = -\tfrac{1}{2}\mathrm{tr}\,[F'(WW')^{\frac{1}{2}}F] = -\tfrac{1}{2}\sum_{i=1}^{n-k}[f_i'(WW')^{\frac{1}{2}}f_i]$$

where f_i is the i-th column of F. Since $f_i'(WW')^{\frac{1}{2}}f_i \geq 0$ for all i, while the equality sign only holds for $f_i = 0$, we have tr $[F'W] < 0$, and hence tr $[H_0'W] >$ tr $[H_1'W]$.

For particular choices of K, Q, J, and Γ some familiar estimators appear. If $J = I_{(n)}$, $Q = \Gamma^{-1}$, and $KK' = S\bar{M}S'$, then

$$w = K(K'\Gamma^{-1}KK'\Gamma^{-1}K)^{-\frac{1}{2}}K'\Gamma^{-1}KK'\Gamma^{-1}y = KK'\Gamma^{-1}y = S\bar{M}S^{-1}y$$

which is the generalized least-squares estimator for $k' = k$. If J is an $n \times (n - k)$ submatrix of $I_{(n)}$, $Q = I_{(n)}$, $\Gamma = \sigma^2 I_{(n)}$, $K = \sigma I_{(n-k)}$, and $k' = k$, so that $\bar{M}$ becomes $M = I - X(X'X)^{-1}X'$, then

$$w = (J'MJ)^{-\frac{1}{2}}J'My$$

which is a BLUS estimator. Finally, if $J = Q = I_{(n)}$, $\Gamma = \sigma^2 I_{(n)}$, $k' = k$, and the $n \times (n - k)$ K is replaced by σK_1 with $K_1'K_1 = I_{(n-k)}$, then

$$w = K_1(K_1'MK_1)^{-\frac{1}{2}}K_1'My$$

which is the estimator developed by Abrahamse and Koerts [1971].

REFERENCES

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