Statistics

Netherlands School of Economics

ECONOMETRIC INSTITUTE

Report 7111

MACRO-ECONOMIC MEDIUM-TERM TARGET FUNCTIONS AND FIRST-PERIOD DECISIONS

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August 10, 1971

Preliminary and Confidential

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Contents

1. INTRODUCTION

Generally, a policy-maker will be interested mainly in how to behave in the immediate future. Decisions with respect to later periods can be postponed, and can be based on new information as it becomes available. On the other hand, the policy-maker must take into account the effect of his first-period decision upon the following periods. He has to estimate this influence and decide what significance it has for his first-period decision.

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[1] The author wishes to thank Prof.Ir. J. Sandee for his critical comments and useful suggestions, and Mr. A.S. Louter for his valuable assistance in writing the needed computer programs.

Defining the decision model over only a few periods entails the risk that a great deal of the future effects is neglected. If, on the other hand, a great number of periods is taken into account, the model may no longer be manageable. It would be better to truncate the decision model in such a way that only a limited number of periods is included, while approximately the same first-period optimal policy results as in the case of an infinite horizon.

In the present paper a model, defined over two subsequent periods, is truncated at the end of the first period. In the solution, the effect of a decision upon the second period is represented by the partial derivatives with respect to the variables "linking" two periods, of the part of the decision model to be replaced. These partial derivatives are called "valuations" of link variables. Link variables are both endogenous variables and instruments, which appear in a lagged form in the second period part of the decision model, including the second period part of the target function.

After a presentation of some results published earlier attention is given to the case of a model, consisting of a quadratic target function and linear relations between endogenous and exogenous variables. First, a number of results is derived for a model defined over two periods. Subsequently, a model is introduced defined over infinitely many periods; the truncation method is extended to this case.

Finally, the method is applied to a decision model that is non-linear in the relations between instruments and target variables. After a brief summary of some previously published results, the results of additional experiments are given.

2. THE MATHEMATICAL FORMULATION OF THE TRUNCATION PROBLEM

2.1. The General Problem

In a previous publication [2] a number of results with respect to the truncation of decision models was derived. The most important result is that a deterministic decision model, defined over two subsequent periods, and having a unique optimum, can be truncated at the end of the first period in such a way that the optimum of the original problem, as far as the first period is concerned, satisfies the first-order optimal conditions of the truncated problem.

We formulate the following deterministic decision problem:

(2.1.1) $$\max_{(p_1,p_2)} \phi = \{\phi_1(q_1, p_1) + \phi_2(q_1, q_2, p_1, p_2)\}$$

subject to

(2.1.2) $$q_1 = G_1(p_1)$$

and

(2.1.3) $$q_2 = G_2(q_1, p_1, p_2)$$

where

q_1 : vector of endogenous variables of period 1
q_2 : vector of endogenous variables of period 2
p_1 : vector of instruments of period 1
p_2 : vector of instruments of period 2
ϕ, ϕ_1, ϕ_2: scalar functions
G_1, G_2 : vector functions.

In the following we further restrict ourselves to the class of problems P for which there is an isolated optimum with the optimal solution (p_1^*, p_2^*), where all first-order derivatives with respect to p_1, p_2, q_1, and q_2 exist. By substitution of (2.1.2) and (2.1.3) into ϕ we define:

(2.1.4) $$\bar{\bar{\phi}}(p_1, p_2) = \{\phi_1(G_1(p_1), p_1) + \\ + \phi_2(G_1(p_1), G_2(G_1(p_1), p_1, p_2), p_1, p_2)\}$$

A necessary condition for an optimum of P is that the derivatives of $\bar{\bar{\phi}}$ with respect to p_1 and p_2 vanish:

(2.1.5) $$\frac{\partial\bar{\bar{\phi}}}{\partial p_1} = \frac{\partial\phi_1}{\partial p_1} + \frac{\partial G_1}{\partial p_1} \cdot \frac{\partial\phi_1}{\partial q_1} + \frac{\partial\phi_2}{\partial p_1} + \frac{\partial G_1}{\partial p_1} \cdot \frac{\partial\phi_2}{\partial q_1} + \frac{\partial G_2}{\partial p_1} \cdot \frac{\partial\phi_2}{\partial q_2} + \frac{\partial G_1}{\partial p_1} \cdot \frac{\partial G_2}{\partial q_1} \cdot \frac{\partial\phi_2}{\partial q_2} = 0$$

(2.1.6) $$\frac{\partial\bar{\bar{\phi}}}{\partial p_2} = \frac{\partial\phi_2}{\partial p_2} + \frac{\partial G_2}{\partial p_2} \cdot \frac{\partial\phi_2}{\partial q_2} = 0$$

Substitution of (2.1.3) into ϕ_2 provides the function $\bar{\phi}_2$ defined by

(2.1.7) $$\bar{\phi}_2(q_1, p_1, p_2) = \phi_2(q_1, G_2(q_1, p_1, p_2), p_1, p_2)$$

In the optimum (p_1^*, p_2^*) define vectors u and v as follows:

(2.1.8) $$u = \frac{\partial\bar{\phi}_2}{\partial q_1}(q_1^*, p_1^*, p_2^*) = \frac{\partial\phi_2}{\partial q_1}(q_1^*, q_2^*, p_1^*, p_2^*) + \frac{\partial G_2}{\partial q_1} \cdot \frac{\partial\phi_2}{\partial q_2}(q_1^*, q_2^*, p_1^*, p_2^*)$$

(2.1.9) $$v = \frac{\partial\bar{\phi}_2}{\partial p_1}(q_1^*, p_1^*, p_2^*) = \frac{\partial\phi_2}{\partial p_1}(q_1^*, q_2^*, p_1^*, p_2^*) + \frac{\partial G_2}{\partial p_1} \cdot \frac{\partial\phi_2}{\partial q_2}(q_1^*, q_2^*, p_1^*, p_2^*)$$

where

$$q_1^* = G_1(p_1^*)$$

$$q_2^* = G_2(q_1^*, p_1^*, p_2^*)$$

Consider now the truncated problem Q:

(2.1.10) $$\max_{(p_1)} \mu = \{\phi_1(q_1, p_1) + u'q_1 + v'p_1\}$$

subject to

(2.1.11) $$q_1 = G_1(p_1)$$

and let this problem, like P, have an isolated optimum where all first-order derivatives with respect to p_1 and q_1 exist.

Substitution of (2.1.11) into (2.1.10) provides the definition:

$$\bar{\mu} = \{\phi_1(G_1(p_1), p_1) + u'G_1(p_1) + v'p_1\} \tag{2.1.12}$$

A necessary condition for an optimum of Q is that the first-order derivatives of $\bar{\mu}$ with respect to p_1 vanish:

$$\frac{\partial\bar{\mu}}{\partial p_1} = \frac{\partial\phi_1}{\partial p_1} + \frac{\partial G_1}{\partial p_1} \cdot \frac{\partial\phi_1}{\partial q_1} + \frac{\partial G_1}{\partial p_1} \cdot u + v = 0 \tag{2.1.13}$$

Upon substitution of the definitions (2.1.8) and (2.1.9), equation (2.1.5) reappears, i.e. the vector p_1^* occurring in the optimum solution to P satisfies the necessary condition for an extremum of Q. Although this does not imply that p_1^* is the optimum solution to Q, this was always true in the practical problems considered in [2].

2.2. A Quadratic Target Function with Linear Constraints

In the previous subsection we indicated that the optimum solution to the original problem satisfies the first-order optimum conditions for the truncated problem. For the more restricted case of a quadratic target function with linear constraints it can be proved that the optimum solution to the original problem is also the optimum solution to the truncated problem.

Consider the following decision model:

$$\max_{(p_1,p_2)} \phi = \{\phi_1(q_1, p_1) + \phi_2(q_1, q_2, p_1, p_2)\} \tag{2.2.1}$$

where

$$\phi_1(q_1, p_1) = w_1'q_1 - \tfrac{1}{2}(p_1 - \Lambda_1 p_0)'Q_1(p_1 - \Lambda_1 p_0) \tag{2.2.2}$$

and

$$\phi_2(q_1, q_2, p_1, p_2) = w_2'q_2 - \tfrac{1}{2}(p_2 - \Lambda_2 p_1)'Q_2(p_2 - \Lambda_2 p_1) \tag{2.2.3}$$

subject to

(2.2.4)
$$q_1 = s_1 + R_1 p_1$$

and

(2.2.5)
$$q_2 = s_2 + R_2 p_2 + S_1 p_1 + S_2 q_1$$

where

q_1 : vector of endogenous variables of period 1
q_2 : vector of endogenous variables of period 2
p_1 : vector of instruments of period 1
p_2 : vector of instruments of period 2
Q_1, Q_2: symmetric, non-singular matrices
s_1, s_2, w_1, w_2, Λ_1, Λ_2, R_1, R_2, S_1, S_2: vectors and matrices of parameters of appropriate order.

Suppose this decision problem has a unique optimum solution (p_1^*, p_2^*). A necessary and sufficient condition for such a unique optimum is that the quadratic form

$$-\tfrac{1}{2}(p_1 - \Lambda_1 p_0)' Q_1 (p_1 - \Lambda_1 p_0) - \tfrac{1}{2}(p_2 - \Lambda_2 p_1)' Q_2 (p_2 - \Lambda_2 p_1)$$

is negative-definite. This implies that both Q_1 and Q_2 are positive-definite matrices.

It can easily be verified that the optimum solution to the problem (2.2.1)-(2.2.5) is:

(2.2.6) $$p_1^* = \Lambda_1 p_0 + Q_1^{-1}\{R_1' w_1 + S_1' w_2 + R_1' S_2' w_2 + \Lambda_2' R_2' w_2\}$$

(2.2.7) $$p_2^* = Q_2^{-1} R_2' w_2 + \Lambda_2 p_1^*$$

Define in accordance with (2.1.7)

(2.2.8)
$$\bar{\Phi}_2(q_1, p_1, p_2) = w_2' s_2 + w_2' R_2 p_2 + w_2' S_1 p_1 + w_2' S_2 q_1 - \tfrac{1}{2}(p_2 - \Lambda_2 p_1)' Q_2 (p_2 - \Lambda_2 p_1)$$

and truncate the decision model as follows:

(2.2.9) $$\max_{(p_1)} \mu = \{\phi_1(q_1, p_1) + u'q_1 + v'p_1\}$$

subject to

(2.2.10) $$q_1 = s_1 + R_1p_1$$

where

(2.2.11) $$u = \frac{\partial\bar{\phi}_2}{\partial q_1}(q_1^*, p_1^*, p_2^*) = S_2'w_2$$

and

(2.2.12) $$v = \frac{\partial\bar{\phi}_2}{\partial p_1}(q_1^*, p_1^*, p_2^*) = S_1'w_2 + \Lambda_2'Q_2(p_2^* - \Lambda_2p_1^*)$$

Substitution of (2.2.7) into (2.2.12) gives:

(2.2.13) $$v = S_1'w_2 + \Lambda_2'R_2'w_2$$

A necessary and sufficient condition for the existence of a unique optimum to (2.2.10)-(2.2.11) is that the matrix of the quadratic form, Q_1, is positive-definite. This condition is satisfied because of the uniqueness of the solution to the original problem.

Consider now the first-order necessary condition for an optimum of (2.2.9)-(2.2.10):

(2.2.14) $$\frac{\partial\mu}{\partial p_1} = R_1'w_1 - Q_1p_1 + Q_1\Lambda_1p_0 + R_1'u + v = 0$$

Substitution of (2.2.11) and (2.2.13) into (2.2.14), and rearranging terms gives the following optimum solution:

(2.2.15) $$p_1^* = \Lambda_1p_0 + Q_1^{-1}\{R_1'w_1 + R_1'S_2'w_2 + S_1'w_2 + \Lambda_2'R_2'w_2\}$$

which is identical to (2.2.6).

It follows that for a model consisting of a quadratic target function with linear constraints:

(a) uniqueness of the optimum solution to the original problem implies a unique optimum for the truncated problem;

(b) the solution to the truncated problem is identical to the solution to the original problem as far as the first period is concerned.

2.3. An Extension to Infinity

In the previous subsections we derived formulas for the case of two periods. In this subsection these results are extended to a situation in which there are T subperiods, with the intention of passing to the case of infinitely many subperiods.

In the literature, target function values are often defined as a discounted sum of all future one-period target function values. By passing to infinity we shall tie our formulations to current theory. It will be seen that for a certain class of models with an infinite horizon, truncation leads to very simple formulas.

Consider the decision model, defined over two subsequent periods, where both periods consist of a number of subperiods.

$$(2.3.1) \qquad \max_{(x_1,\ldots,x_T)} \phi = \phi_1(y_1, \ldots, y_\tau; x_1, \ldots, x_\tau) + \phi_2(y_1, \ldots, y_T; x_1, \ldots, x_T) \qquad \tau > 0,\ T > \tau$$

subject to

$$(2.3.2) \qquad y_t = g_t(y_1, \ldots, y_{t-1}; x_1, \ldots, x_t) \quad t = 1, \ldots, \tau$$

and

$$(2.3.3) \qquad y_t = a_t + Ax_t + Cy_{t-1} \qquad t = \tau + 1, \ldots, T$$

where

y_t : vector of endogenous variables of subperiod t
x_t : vector of instruments of subperiod t
ϕ, ϕ_1, ϕ_2 : scalar functions
g_t : vector functions
a_t, A, C : vectors and matrices of parameters of appropriate order.

The second part of the target function is assumed to have a special quadratic form:

(2.3.4)
$$\phi_2(y_1, \ldots, y_T; x_1, \ldots, x_T) = \sum_{t=\tau+1}^{T} \rho^{t-1}\{d'y_t - \tfrac{1}{2}(x_t - \Pi x_{t-1})'B(x_t - \Pi x_{t-1})\}$$

where

B : positive-definite matrix
d, Π : vector and matrix of parameters of appropriate order
$\rho\{0 < \rho \leq 1\}$: discount factor, indicating the time-preference of the policy-maker

In the optimum the following relations should hold:

(2.3.5)
$$\frac{\partial\phi_2}{\partial x_t} + \frac{\partial \sum_{s=t}^{T} \rho^{s-1} d'y_s}{dx_t} = 0 \qquad t = \tau + 1, \ldots, T$$

For period T this implies:

(2.3.6)
$$\frac{\partial\phi_2}{\partial x_T} + \frac{\partial\rho^{T-1} d'y_T}{\partial x_T} = \rho^{T-1}\{A'd - B(x_T - \Pi x_{T-1})\} = 0$$

resulting in

(2.3.7)
$$B(x_T^* - \Pi x_{T-1}^*) = A'd$$

By induction it will be proved that the following relation holds good for all T and all s, $\tau \leq s < T$:

$$(2.3.8) \qquad B(x^*_{s+1} - \Pi x^*_s) = \sum_{t'=s+1}^{T} (\rho\Pi')^{t'-s-1}\{\sum_{t=t'}^{T} A'(\rho C')^{t-t'}\}d$$

First we prove that (2.3.8) holds good for all T and s = T - 1. Let s = T - 1, then (2.3.8) reduces to

$$(2.3.9) \qquad B(x^*_T - \Pi x^*_{T-1}) = A'd$$

which is identical to (2.3.7) and therefore holds good for all T. It follows that (2.3.8) holds good for all T and s = T - 1.
Suppose now (2.3.8) holds good for some s, $\tau < s < T$. In the optimum we have by differentiation of (2.3.4)

$$(2.3.10) \qquad \frac{\partial\phi_2}{\partial x_s} + \frac{\partial \sum_{t'=s}^{T} \rho^{t'-1} d'y_{t'}}{\partial x_s} = 0$$

From (2.3.3) we obtain

$$(2.3.11) \qquad \frac{\partial \rho^{t'-1} d'y_{t'}}{\partial x_s} = \rho^{t'-1}A'(C')^{t'-s}d$$

Substitution of (2.3.11) into (2.3.10), and differentiating ϕ_2 with respect to x_s leads to

$$(2.3.12) \qquad \sum_{t'=s}^{T} \rho^{t'-1}A'(C')^{t'-s}d - \rho^{s-1}B(x^*_s - \Pi x^*_{s-1}) + \rho^s\Pi'B(x^*_{s+1} - \Pi x^*_s) = 0$$

where x^*_s denotes the optimum solution for period s. Dividing by ρ^{s-1} we find

$$(2.3.13) \qquad \sum_{t'=s}^{T} \rho^{t'-s}A'(C')^{t'-s}d - B(x^*_s - \Pi x^*_{s-1}) + \rho\Pi'B(x^*_{s+1} - \Pi x^*_s) = 0$$

Substitute (2.3.8) in (2.3.13):

$$(2.3.14) \qquad \sum_{t'=s}^{T} \rho^{t'-s} A'(C')^{t'-s} d - B(x^*_s - \Pi x^*_{s-1})$$

$$+ \rho\Pi' \sum_{t'=s+1}^{T} (\rho\Pi')^{t'-s-1} \{ \sum_{t=t'}^{T} A'(\rho C')^{t-t'} \} d = 0$$

or

$$(2.3.15) \qquad B(x^*_s - \Pi x^*_{s-1}) = \sum_{t'=s}^{T} \rho^{t'-s} A'(C')^{t'-s} d$$

$$+ \sum_{t'=s+1}^{T} (\rho\Pi')^{t'-s} \{ \sum_{t=t'}^{T} A'(\rho C')^{t-t'} \} d$$

$$= \sum_{t'=s}^{T} (\rho\Pi')^{t'-s} \{ \sum_{t=t'}^{T} A'(\rho C')^{t-t'} \} d$$

which is the same as relation (2.3.8) for $s - 1$. Hence, if (2.3.8) holds good for some s, $\tau < s < T$, it holds good for $s - 1$, which completes the proof.

The extension of the model to one defined over infinitely many periods ($T \to \infty$) runs as follows. Formula (2.3.8) is rewritten as

$$(2.3.16) \qquad B(x^*_{s+1} - \Pi x^*_s) = \sum_{t'=1}^{T-s} (\rho\Pi')^{t'-1} \{ \sum_{t=0}^{T-s-t'} A'(\rho C')^{t} \} d$$

This is a partial sum of the double sum

$$(2.3.17) \qquad \sum_{t'=1}^{\infty} (\rho\Pi')^{t'-1} \{ \sum_{t=0}^{\infty} A'(\rho C')^{t} \} d$$

which is absolutely convergent provided

$$(2.3.18) \qquad \lim_{t\to\infty} (\rho C')^t = 0 \quad ^{2)}$$

and

$$(2.3.19) \qquad \lim_{t\to\infty} (\rho\Pi')^t = 0$$

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2 This occurs e.g. when $||\rho C'||_\infty < \frac{1}{n}$, where $||\rho C'||_\infty$ denotes the maximum of the absolute values of the elements of $\rho C'$, and C' is an $n \times n$ matrix.

Then the double sum is equal to

$$(2.3.20)\quad \sum_{t'=1}^{\infty} (\rho\Pi')^{t'-1}\{\sum_{t=0}^{\infty} A'(\rho C')^{t}\}d = \sum_{t'=1}^{\infty} (\rho\Pi')^{t'-1}A'(I - \rho C')^{-1}d =$$

$$= (I - \rho\Pi')^{-1}A'(I - \rho C')^{-1}d$$

Hence, passing to the limit $(T \to \infty)$ we find

$$(2.3.21)\quad B(x^*_{s+1} - \Pi x^*_s) = (I - \rho\Pi')^{-1}A'(I - \rho C')^{-1}d$$

In accordance with section (2.2) the model is truncated as follows:

$$(2.3.22)\quad \max_{(x_1,\ldots,x_\tau)} \mu = \phi_1(y_1, \ldots, y_\tau; x_1, \ldots, x_\tau) + u'y_\tau + v'x_\tau$$

In accordance with the definition of u in (2.1.8) we obtain from (2.3.3) and (2.3.4)

$$(2.3.23)\quad u = \frac{\partial}{\partial y_\tau} \sum_{t=\tau+1}^{T} \rho^{t-1}d'y_t =$$

$$= \rho^\tau C'd + \rho^{\tau+1}(C')^2 d + \ldots + \rho^{T-1}(C')^{T-\tau}d =$$

$$= \rho^\tau C'\{I + \rho C' + \ldots + (\rho C')^{T-\tau-1}\}d$$

If (2.3.18) holds good, and we let T go to ∞ we obtain

$$(2.3.24)\quad u = \rho^\tau C'(I - \rho C')^{-1}d$$

From (2.3.4) we obtain, in accordance with the definition of v in (2.1.9)

$$(2.3.25)\quad v = \rho^\tau \Pi' B(x^*_{\tau+1} - \Pi x^*_\tau)$$

If both (2.3.18) and (2.3.19) hold good we obtain from (2.3.21) in the limit $(T \to \infty)$

$$(2.3.26)\quad v = \rho^\tau \Pi'(I - \rho\Pi')^{-1}A'(I - \rho C')^{-1}d$$

Both (2.3.24) and (2.3.26) seem simple and elegant enough.

3. SOME EXPERIMENTS

3.1. Summary of Previous Results

The question arises whether the truncation method, outlined in the previous section, can be used for policy experiments. More expecially it would be interesting to know whether the method is practically useful in a situation where the model, relating endogenous and explanatory variables, is non-linear.

The results of a first investigation into this question were given in the publication mentioned before [2]. A number of experiments were carried out with the following target function:

$$(3.1.1) \qquad \max_{(x_1,\ldots,x_T)} \phi = \sum_{t=1}^{T} \rho_t^* \{d' y_t - \lambda (x_t - \Pi x_{t-1})' B (x_t - \Pi x_{t-1})\}$$

where

- ϕ : scalar function
- y_t: vector of target variables of period t
- x_t: vector of instruments of period t
- d : vector of weights for the target variables
- B : diagonal positive-definite matrix of the quadratic form
- Π : diagonal matrix of growth-rates
- λ : scalar, indicating relative importance of the quadratic part with respect to the linear part
- ρ_t^*: discount factor

The targets used in our experiments are:

- c : deflated private consumption
- w : unemployment
- $\Delta L_{q(ex)}$:creation of liquidities via external payments

As instruments we used the following ones:

- C_g: government consumption expenditure, excluding wages and salaries
- L_g: wages and salaries paid by government
- ΔT_L^*: change in autonomous taxes on wages

The relations between target variables y_t and instruments x_t are given by the CS-model of Van den Beld, a recently published non-linear yearly model of the Dutch economy [1].

Because of the non-linear character of the model it was not possible to work with infinitely many periods. We therefore restricted the model to a time-span of 23 years. Furthermore, we used a discount factor slightly different from the usual one.[3] The first question to be answered was whether the long-term model could be truncated along the lines outlined in Section 2, in such a way that the truncated model generated correct results in the sense of the same first-period decision as the original long-term model showed. This turned out to be the case for the model, truncated at the end of the third and the fifth year.

However, one optimization of the target function will in general not suffice. The policy-maker will desire to be informed about the influence of changes in the initial conditions, and of alternative assumptions about the data during the time-span of the target function. Furthermore, it is not unlikely that the policy-maker will be interested in the results of optimizations with several alternative target functions. If the truncated model is to be of interest for policy purposes, it has to generate correct results (in the sense as defined above) in experiments with changed data or parameters.

To test this, a number of experiments were carried out in which parameters of the target function were changed, namely the elements of d and ρ_t^*, the discount factor. The decision model was truncated by means of the valuations, computed in the "central optimum",[4] where in the case of changes in d these valuations were adapted to these changes.[5]

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3 ρ_t^* is defined as $\rho^{t-1} - \rho^{(2T-t+1)}$. See [2].

4 The central optimum is the optimum computed with the original, unchanged set of parameters and data.

5 See formulas (2.2.12) and (2.2.14) for the way in which this adaptation can be carried out for the case of linear restrictions. Similar formulas were derived for this adaptation in the non-linear case.

From subsection 2.2 it can be seen that for a model with a quadratic target function and linear constraints the truncated model leads to correct results, even if parameters are changed, provided that the valuations are adapted to these changes.[5] This is due to the fact that the formulas (2.2.12) and (2.2.14) only contain vectors and matrices of parameters.

This, however, is no longer true if we are dealing with a decision model with non-linear constraints. In that case the valuations of the link variables depend not only on the parameters of the decision model (as in the case of subsection 2.2), but also on the optimal values of instruments and endogenous variables. We cannot fully adapt the valuations to the changed parameters by means of formulas such as (2.2.12) and (2.2.14), which implies that the use of the truncated model with changed parameters will, in general, lead to a first-period decision, different from that obtained by optimizing the long-term model.

The consequence of the above is that our original criterion for the applicability of the truncated model (the same first-period decision) cannot be maintained. We therefore used a new criterion to indicate whether the first-period decision obtained by the optimization of the truncated model was close enough to that obtained by optimizing the long-term model. The usefulness of the method depends on the range within which the parameters can be changed without exceeding the criterion.

Our experiments learned that changes in d do not do much harm. The results obtained with the truncated model do not differ much from the results of the long-term model, which implies that given the underlying decision model, the truncated model can be used to investigate the effects of changes in d.

The same applies for experiments with changes in ρ_t^*. However, the valuations of the central optimum were not adapted to these changes, which resulted in outcomes less satisfactory than those in the case of changes in d.

Finally, some experiments were carried out with changes in the data. Here, too, the truncated model turned out to be a useful tool for policy experiments.

The next subsections will be devoted to the results of a number of additional experiments.

3.2. Changes in the Matrix of the Quadratic Form

The experiments summarized in the previous subsection were devoted to an investigation of the influence of changes in data or parameters upon the usefulness in practice of the truncated model. However, not all parameters were considered. We restricted ourselves to the parameters of the linear term (d), and the discount factor (ρ_t^*). It is of some interest to know something more about the influence of changes in the elements of B. This matrix indicates the relative importance, attached to the losses due to the "use" of the various instruments. In this connection the use of the instruments is defined as the difference ($x_t - \Pi x_{t-1}$) (cf. formula 3.1.1). The values represented by Πx_{t-1} (to be called the "trend values" of x_t) can be considered to be what Theil calls the "desired values" of x_t.[6] The results of some experiments with changes in B are presented in this subsection.

First, eight optimizations with the long-term model were carried out. In these optimizations the diagonal elements of B were changed simultaneously. The results of the experiments as far as the first-period decision is concerned, together with the result of the central optimum,[7] are summarized in Table 3.1, columns "a".

We see that the changes in the elements of B lead to very different first-period policies, a result that could be expected. This is, however, no answer yet to the question, whether the truncated model will lead to almost the same first-period policies.

To answer this question the long-term model was truncated at the end of the fifth year, using the valuations of link variables as computed in the central optimum. We did not develop a formula to adapt the valuations to the changes in B, as was done in our experiments with changes in d (cf. subsection 3.1).

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[6] Theil [3] introduces the "desired values" to express the desires of the policy-maker with respect to the future values (or changes in future values) of targets and instruments. The decision problem consists of a quadratic target function to be maximized, subject to linear equality constraints "and this maximization problem is formulated in terms of the minimization of the sum of squares of the discrepancies between actual and desired values" (Theil [3], page 29).
Our desired values of the instruments are equal to the trend values Πx_{t-1}. It has to be noted that the trend values Πx_{t-1} are obtained from the actual values of the previous year (x_{t-1}).

[7] See the definition of "central optimum" in the previous subsection.

The use of "wrong"[8] valuations in the optimization of the truncated model will lead to first-period decisions, that are different from the decisions, obtained by the optimization of the long-term model (both optimizations carried out with the same changes in B). These differences can be seen from Table 3.1, where the first-period decision from the truncated optimization is given next to the results of the long-term optimization (columns "b").

To judge these results we computed the "loss" due to the optimization of the truncated model with "wrong" valuations. This loss is obtained as follows. The value of the long-term target function, given the changed elements of B, is defined as the "true" optimum value. For every experiment we find such a "true" optimum value in Table 3.1, column "c". Furthermore, we define the conditional optimum as the long-term optimum, given the changes in B, and given furthermore the first-period decision resulting from the optimization of the truncated model with the same changes in B. The conditional optimum value will be smaller than the true optimum value, because the first-period decision of the conditional optimum is not equal to that of the true optimum. The values in the conditional optimum can be found in column "d" of Table 3.1. Finally, the loss is defined as the difference between the true and the conditional optimum values. These losses are given in column "c-d" of Table 3.1.

The losses were compared with the loss, resulting from taking the instrument values equal to their trend values (i.e. $x_t = \Pi x_{t-1}$). In our definition given in the beginning of this subsection this implies that the "use" of the instruments is zero. The target function value resulting from this policy would be much lower than the optimum value, and a considerable loss would be incurred. This loss is a relevant standard by which to measure the losses of truncation.

The losses, resulting from taking $x_t = \Pi x_{t-1}$ can be found in column "c-e", whereas column "f" shows the loss due to the conditional optimization as a percentage of the loss due to taking the instrument values equal to their trend values.

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8 The valuations are "wrong" because they are not equal to the valuations belonging to the long-term optimum obtained with changed elements of B. It has to be noted that even if the valuations would be adapted to changes in B (as has been done in the case of changes in d) the valuations still would have been somewhat "wrong". This is due to the fact that the non-linear character of the model prevents a full adaptation.

TABLE 3.1. RESULTS OF EXPERIMENTS WITH CHANGES IN THE ELEMENTS OF THE MATRIX OF THE QUADRATIC FORM

	Elements of B			First-period values of instruments*						Long-term target function values*			Losses*		
	b_{11}	b_{22}	b_{33}	C_g		L_g		ΔT_L^*							
				a	b	a	b	a	b	c	d	e	c-d	c-e	f
Central optimum	1.00	1.60	2.20	1.876		3.572		-.032							
Experiment															
1	1.50	2.35	3.20	1.923	1.923	3.545	3.544	-.025	-.023	191.553	191.552	187.107	0.001	4.446	0.02
2	1.50	2.35	1.20	1.941	1.923	3.552	3.544	-.100	-.060	192.309	192.234	187.107	0.075	5.202	1.44
3	1.50	0.85	3.20	1.946	1.930	3.685	3.671	-.032	-.025	195.837	195.794	187.107	0.043	8.730	0.49
4	1.50	0.85	1.20	1.944	1.930	3.682	3.669	-.083	-.067	196.480	192.442	187.107	0.038	9.373	0.40
5	0.50	2.35	3.20	1.727	1.742	3.538	3.545	-.024	-.024	196.289	192.278	187.107	0.011	9.182	0.12
6	0.50	2.35	1.20	1.720	1.741	3.535	3.544	-.058	-.064	196.602	196.580	187.107	0.022	9.495	0.22
7	0.50	0.85	3.20	1.743	1.741	3.636	3.648	-.021	-.023	198.479	198.473	187.107	0.006	11.372	0.06
8	0.50	0.85	1.20	1.744	1.735	3.636	3.645	-.057	-.058	198.750	198.744	187.107	0.006	11.643	0.06

* In billions of guilders

a = long-term optimization
b = optimization with truncation after five years
c = target function value from true optimization
d = target function value from conditional optimization
e = target function value obtained from $x_t = \Pi x_{t-1}$
$f = \frac{100(c - d)}{c - e}$

It can be concluded that if the losses are considered in this way they are in general negligible. Only in the case of experiment 2 the loss is more than one percent of the loss, obtained by putting x_t equal to its trend value Πx_{t-1}. Like the experiments described in Subsection 3.1 it can be concluded that the truncated model can be used as a substitute for the long-term model in the case the policy-maker is interested in the influence of changes in the matrix of the quadratic form.

In this connection it has to be noted that the valuations of the link variables, used in our experiments with changes in B, were not adapted to these changes. It is conceivable that such an adaptation can be carried out just as has been done in our experiments with changes in d (as described in Subsection 3.1). Further research may very well lead to the conclusion that such an adaptation leads to results even better than those, obtained in this subsection.

3.3. Changes in the Growth-rates of the Instruments

In the previous subsection we considered the influence of changes in the elements of B, the matrix indicating the relative importance attached to the losses due to the use of the various instruments. This use was defined as the difference between x_t and Πx_{t-1}, where Π is a diagonal matrix of growth-rates. As in the case of changes in B it is of some interest to know what influence a change in the elements of Π has upon the applicability of the truncated model. The way in which this question has been answered is analogous to that of Subsection 3.2, so our comments can be rather brief.

Four experiments were carried out with changed elements of Π, in which experiments the long-term model was optimized.

The following step was the optimization of the model, truncated after 5 years, with the same changes in the elements of Π, and using the valuations of link variables as computed in the central optimum. As in the case of changes in B we did not adapt the valuations of the link variables to the changed elements of Π.

TABLE 3.2. RESULTS OF EXPERIMENTS WITH CHANGES IN THE GROWTH RATES OF THE INSTRUMENTS

	Elements of Π			First-period values of instruments*						Long-term target function values*		Losses*
	π_{11}	π_{22}	π_{33}	C_g		L_g		ΔT_L^*				
				a	b	a	b	a	b	c	d	c-d
Central optimum	1.06	1.06	0.00	1.876		3.572		-.032				
Experiment												
1	1.04	1.04	0.00	1.847	1.850	3.493	3.506	-.036	-.035	189.774	189.761	0.013
2	1.04	1.08	0.00	1.892	1.858	3.672	3.655	-.048	-.038	203.533	203.414	0.119
3	1.08	1.04	0.00	1.879	1.906	3.497	3.504	-.037	-.033	185.165	185.112	0.053
4	1.08	1.08	0.00	1.932	1.920	3.669	3.657	-.044	-.039	200.184	200.157	0.027

* In billions of guilders

a = long-term optimization

b = optimization with truncation after five years

c = target function value from true optimization

d = target function value from conditional optimization

Finally, the losses due to the use of wrong valuations in the truncated model were computed. These losses were defined in the same way as in Subsection 3.2.

The results of our computations can be found in Table 3.2. All losses except for the case of experiment 2, are of the same order of smallness as those of Table 3.1.[9] Because in the experiments the valuations were not adapted to the changed elements of Π it can be expected that the results can be improved by such an adaptation. Our conclusion therefore is the same as that given at the end of the previous subsection, namely that the truncated model can be used as a substitute for the long-term model in the case the policy-maker is interested in the influence of changes in the growth-rates Π of the instruments upon the first period decision.

4. CONCLUSIONS

In the present study we extended the method of truncating a deterministic decision model, as outlined in [2], in two ways. First, a number of mathematical results was derived for two specific models. Second, a number of experiments were carried out with changes in parameters, not considered in [2].

A number of problems remains to be solved. As already outlined in [2], one of the most important restrictions is the specific form of the target function, which is linear in the target variables only and quadratic in the instruments only. More insight is needed into the consequences for the applicability of the method if more general functions are used.

However, in the experiments carried out with a long-term decision model, consisting of a quadratic target function and the relationships between endogenous and explanatory variables of the Van den Beld model, the truncated model generated the same optimal policy as the long-term model. Experiments with changes in the coefficients of the target function, or in the data did not influence this materially, as can be seen from [2]. This conclusion can be maintained after our experiments with changes in the parameters of the quadratic part of the target function, as described in this paper. These results could now be useful in practice.

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[9] Cf. columns "c-d" in both tables.

REFERENCES

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