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A COST FUNCTION APPROACH TO THE MEASUREMENT OF FACTOR DEMAND ELASTICITIES AND ELASTICITITES OF SUBSTITUTION

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A Cost Function Approach to the Measurement of
Elasticities of Factor Demand and
Elasticities of Substitution

I. Introduction

The use of a cost function rather than a production function for estimating production parameters is shown to have several major advantages.

1) Homogeneity of degree one does not have to be imposed on the production process to arrive at estimation equations. Cost functions are homogeneous in prices regardless of the homogeneity of the production function, because a doubling of all prices will double the costs but not affect factor ratios.

2) In general, the estimation equations have prices as independent variables rather than factor quantities, which, at the firm or industry level, are not proper exogenous variables. Enterpreneurs make decisions on factor use according to exogenous prices, which makes factor levels endogenous decision variables.

3) To derive estimates in many-factor cases of elasticities of substitution or of factor demand, no matrix of estimates of the production function coefficients has to be invested, a procedure which has a strong tendency to exaggerate estimation errors.

4) In the special case of the translog cost function (Christensen, Jorgensen and Lau, 1970) to which the method is applied, problems of neutral of non-neutral efficiency differences among observational units[1/] (firmsor states in a cross section,

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[1/] A non-neutral efficiency difference in the Hicksian sense is one in which the isoquant does not shift inwards homothetically. The factor ratio does not stay constant at constant factor price ratio. If the capital-labor ratio increases, the efficiency gain is labor saving. This implies that the labor share declines at a constant factor price ratio. Efficiency gain biases can therefore be defined as follows:

$$B_i \Big|_{\text{constant factor prices}} = \frac{\partial \alpha_i}{\partial t} \cdot \frac{1}{\alpha_i} \lesseqgtr 0 \rightarrow \text{Hicks} \begin{cases} \text{factor } i\text{-saving} \\ \text{factor } i\text{-neutral} \\ \text{factor } i\text{-using} \end{cases}$$

This definition is more easily handled in the many-factors case than the usual definition in terms of marginal rates of substitution.

years in a time series) or of neutral and non-neutral economies of scale can be handled conveniently. Therefore, these problems will not result in biased estimates of the production parameters. As will be discussed, such differences can result from a variety of sources. Most methods of estimating production parameters cannot handle this problem properly.

5) In the case of the translog cost function (as well as of the translog production function) all estimation equations are linear in logarithms.

6) In production function estimation high multicollinearity among the input variables often causes problems. Since there is usually little multicollinearity among factor prices this problem does not arise in cost function estimation.

The plan of this paper is as follows. The second section is devoted to a derivation of the Allen partial elasticity of substitution in terms of the cross derivatives of the cost function. The results is applied to the case of the translog cost of function in the third section. The fourth section is devoted to a discussion of methods to avoid problems of neutral and non-neutral efficiency differences. In section five the translog method is used to derive estimates of elasticities of derived demand and of elasticities of substitition for the agricultural sector using U.S. cross section data of the states for the years 1949, 1954, 1959 and 1964.

2. Partial elasticities of substitution in terms of cost function parameters

Corresponding to the following cost minimization problems

$$\min \qquad C = \sum_{i=1}^{n} X_i P_i \qquad (i = 1,2,\ldots n) \tag{1}$$

$$\text{subject to} \qquad Y = f(X_1, X_2, \ldots X_n). \tag{2}$$

X_i = input levels,

P_i = factor prices,

Y = output,

there exist a dual minimum cost function[2]

$$C^* = g(Y, P_1, \ldots P_n). \tag{3}$$

This function (also called factor price frontier) assigns to every combination of input prices the minimum cost corresponding to the cost minimizing input levels X_i^*. C^* is homogeneous of degree of one in prices. If all factor prices double the cost will double while leaving input quantities unaffected.

Shephard's lemma (Diewert, 1971) holds for the cost function:

$$\frac{\partial C^*}{\partial P_r} = X_r. \tag{4}$$

Let the bordered Hessian matrix of (2) be

$$f = \begin{bmatrix} 0 & f_1 & \cdots\cdots & f_n \\ f_1 & f_{11} & \cdots\cdots & f_{1n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_n & f_{n1} & \cdots\cdots & f_{nn} \end{bmatrix},$$

where $f_i = \dfrac{\partial Y}{\partial X_i}$ and $f_{ij} = \dfrac{\partial^2 Y}{\partial X_i \partial X_j}$.

Partial elasticities of substitution are defined by Allen as follows:[3]

$$\sigma_{kr} = \frac{\sum_{i=1}^{n} X_i f_i}{X_k X_r} (f^{-1})_{rk}, \tag{5}$$

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[2] While C of (1) is the cost of production under any feasible factor combination, C^* refers to the cost of production when the cost minimizing input combination is used. Since the optimal input combination is a function of the factor prices, so is the minimum cost.

[3] In the two factor case a different definition of the elasticity of substitution is usually used, which is as follows:

$$\sigma'_{kr} = \frac{d \log \left(\frac{X_k}{X_r}\right)}{d \log \left(\frac{P_k}{P_r}\right)}$$

This definition is very cumbersome in the many-factors case (Mundlak, 1967). In the two-factor case

$$\sigma_{kr} = \sigma'_{kr}.$$

Therefore the partial concept (5) is used here.

where $(f^{-1})_{rk}$ is the rk-th element of f^{-1}. From (5) it is apparent that

$$\sigma_{kr} = \sigma_{rk} \tag{6}$$

If estimates of the coefficients of a particular functional form of (2) are available the bordered Hessian can be computed, inverted and the σ's found according to (5) for specific input levels.[4/] The inversion of a matrix of estimates has, however, the tendency to blow up estimation errors to an unknown extent and, because inversion is a nonlinear transformation, econometric properties of $\hat{\sigma}_{kr}$ cannot be found even if such properties of the production function parameters are known.

In the case of the cost function estimates of σ_{kr} can be obtained as follows if the parameters of the function are known.

$$\sigma_{rk} = \frac{\sum_{i=1}^{n} P_i X_i}{X_k X_r} \cdot \frac{\partial^2 C^*}{\partial P_r \partial P_k} . \tag{7}$$

This was originally proved for homogeneous production function in Uzawa (1962).

Proof: The first order condition of the cost minimizing problem (1) and (2) are:

$$f(X_1,\ldots,X_n) - Y = 0 \tag{8}$$

$$P_i - \lambda f_i = 0 \qquad i=1,\ldots,n \tag{9}$$

Write the total differential of the first order conditions and rearrange the terms in the following matrix form (Mundlak 1967)

$$\lambda \begin{bmatrix} 0 & f_1 & \cdots\cdots & f_n \\ f_1 & f_{11} & \cdots\cdots & f_{1n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_n & f_{n1} & \cdots\cdots & f_{nn} \end{bmatrix} \begin{bmatrix} d\lambda/\lambda \\ dX_1 \\ \cdot \\ \cdot \\ \cdot \\ dX_n \end{bmatrix} = \begin{bmatrix} \lambda dY \\ dP_1 \\ \cdot \\ \cdot \\ \cdot \\ dP_n \end{bmatrix} \tag{10}$$

Solving for the vector of endogenous variables:

$$\begin{bmatrix} d \ln \lambda \\ dX_1 \\ \cdot \\ \cdot \\ \cdot \\ dX_n \end{bmatrix} = \frac{1}{\lambda} f^{-1} \begin{bmatrix} dY \\ dP_1 \\ \cdot \\ \cdot \\ \cdot \\ dP_n \end{bmatrix} \tag{11}$$

[4/] See Berndt and Christensen (1971) for example.

This implies

$$\frac{\partial X_r}{\partial P_k} = \frac{1}{\lambda}(f^{-1})_{rk}. \tag{12}$$

Substituting from (10) into (5) and substituting $f_i = \frac{P_i}{\lambda}$ from (9)

$$\sigma_{kr} = \frac{\sum_i X_i f_i}{X_k X_r} \cdot \lambda \frac{\partial X_r}{\partial P_k} = \frac{\sum_i X_i P_i}{X_k X_r} \frac{\partial X_r}{\partial P_k}, \tag{13}$$

with respect to P_k.

Taking the derivative of (4) with changed indexes

$$\frac{\partial^2 C^*}{\partial P_r \, \partial P_k} = \frac{\partial X_r}{\partial P_k} \tag{14}$$

Combining (13) and (14) and (6)

$$\sigma_{kr} = \sigma_{rk} = \frac{\Sigma X_i P_i}{X_k X_r} \frac{\partial^2 C^*}{\partial P_r \partial P_k} \qquad \text{Q.E.D.}$$

Multiplying and dividing (11) by $\frac{P_k}{X_r}$

$$\sigma_{kr} = \sigma_{rk} = \frac{\eta_{rk}}{\alpha_k} \tag{15}$$

where $\eta_{rk} = \frac{\partial X_r}{\partial P_k} \cdot \frac{P_k}{X_r}$ and $\alpha_k = \frac{X_k P_k}{\Sigma X_i P_i}$ is the share of factor k in total costs.

If the parameters of a specific functional form of a cost function have been estimated (7) can be used to derive elasticities of substitution for given factor levels and total costs.[5]

3. The Translog Case

The Translog cost function is particularly useful in this context. It is written as a logarithmic Taylor series expansion to the second term of a twice differentiable

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[5] Also if η_{kr} had been estimated in a demand for factors equation one can compute σ_{rk} from (13). This may be particularly useful in the two factors case since then the own elasticity of demand can be used. Because of homogeneity of degree one of C*, $\eta_{11} + \eta_{12} = 0$, and $\eta_{12} = -\eta_{11}$ can be substituted into (13).

analytic cost function around variable levels of 1 (i.e. $\ln Y = 0$, $\ln P_i = 0$, $i = 1,\ldots,n$). Rewrite (3) in natural logarithmes

$$\ln C^* = f(\ln Y, \ln P_1, \ldots \ln P_n). \tag{16}$$

Denoting the first and second order derivatives at $\ln(\cdot) = 0$ as follows:

$$\ln C^*\Big|_{\underline{o}} = \nu_o \tag{17a}$$

$$\left.\frac{\partial \ln C^*}{\partial \ln Y}\right|_{\underline{o}} = \nu_y \tag{17b}$$

$$\left.\frac{\partial \ln C^*}{\partial \ln P_i}\right|_{\underline{o}} = \nu_i \qquad (i = 1,\ldots,n) \tag{17c}$$

$$\left.\frac{\partial^2 \ln C^*}{\partial \ln P_i \; \partial \ln P_j}\right|_{\underline{o}} = \gamma_{ij} \qquad (i,j = 1,\ldots,n) \tag{17d}$$

$$\left.\frac{\partial^2 \ln C^*}{\partial \ln P_i \; \partial \ln Y}\right|_{\underline{o}} = \gamma_{iy} \qquad (i = 1,\ldots,n)\ . \tag{17e}$$

(17d) implies the symmetry constraint

$$\gamma_{ij} = \gamma_{ji}\ . \tag{18}$$

Then the Taylor series expansion is as follows:

$$\ln C^* = \nu_o + \nu_y \ln Y + \sum_i \nu_i \ln P_i$$
$$+ 1/2 \sum_i \sum_j \gamma_{ij} \ln P_i \ln P_j + \sum_i \gamma_{iy} \ln P_i \ln Y$$
$$+ \text{remainder} \tag{19}$$

This function[6] is an approximation to an arbitrary analytic function.[7] It is a functional form in its own right if the remainder is neglected and if we assume all

[6] The first power terms of (19) represent a Cobb Douglas cost function. If all γ_{ij} and γ_{iy} were zero the production function would be Cobb Douglas as well because the production function of a Cobb Douglas cost function is Cobb Douglas and vice versa (Hanoch 1970).

[7] By a similar expansion of a production function the translog production function is found.

$$\ln Y = \omega_o + \sum_i \omega_i \ln X_i + 1/2 \sum_i \sum_i \tau_{ij} \ln X_i \ln X_j.$$

derivatives and cross derivatives to be constant. This latter constraint is imposed if the parameters are estimated in regression equations.

Homogeneity in prices is defined as follows: $\lambda\, g(Y,P_1,\ldots,P_n) = g(Y,\lambda P_1,\ldots,\lambda P_n)$. It implies

$$\begin{gathered}\sum_i \nu_i = 1 \\ \sum_i \gamma_{ij} = 0, \quad \sum_j \gamma_{ij} = 0\end{gathered} \tag{20}$$

Homogeneity in prices does not impose homogeneity of degree one of the production function in inputs. No constraints are imposed on elasticities of substitution or of factor demand, which makes the function more general than other functional forms currently in use.

The function can be estimated directly or in its first derivatives which, by Shepard's lemma (4), are factor shares:

$$\frac{\partial \ln C^*}{\partial \ln P_i} = \frac{\partial C^*}{\partial P_i} \cdot \frac{P_i}{C^*} = X_i\, \frac{P_i}{C^*} = \alpha_i. \qquad (i = 1,\ldots,n.)$$

$$\alpha_i = \nu_i + \sum_j \gamma_{ij} \ln P_j + \gamma_{iy} \ln Y \qquad (i = 1,\ldots,n.) \tag{21}$$

Both sets of estimation equations are linear in logs and have proper exogenous variables on the right hand side if the analysis pertains to firms or an industry.[8]

The γ_{ij} have little economic meaning of their own. We will prove that they are related to variable elasticities of substitution and of factor demand as follows:

$$\sigma_{ij} = \frac{1}{\alpha_i \alpha_j}\gamma_{ij} + 1 \qquad \text{for all } i,j\text{: } i \neq j. \tag{22}$$

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[8] In the case of the translog production function the estimation equations are similar but with factor quantities on the right hand side. For the decision-making firm these are endogenous.

$$\sigma_{ii} = \frac{\gamma_{ii}}{\alpha_i^2}(\gamma_{ij} + \alpha_i^2 - \alpha_i) \qquad \text{for all } i. \tag{23}$$

$$\eta_{ij} = \frac{\gamma_{ij}}{\alpha_i} + \alpha_j \qquad \text{for all } i,j\text{:}\quad i \neq j. \tag{24}$$

$$\eta_{ii} = \frac{\gamma_{ii}}{\alpha_i} + \alpha_i - 1 \qquad \text{for all } i. \tag{25}$$

Proof:

$$\gamma_{ij} = \frac{\partial^2 \ln C^*}{\partial \ln P_i \, \partial \ln P_j} = P_j \frac{\partial}{\partial P_j}\left(\frac{\partial C^*}{\partial P_i} \cdot \frac{P_i}{C^*}\right) =$$

$$= P_j\left(\frac{\partial^2 C^*}{\partial P_i \, \partial P_j} \cdot \frac{P_i}{C^*} - \frac{P_i}{(C^*)^2}\frac{\partial C^*}{\partial P_i}\frac{\partial C^*}{\partial P_j}\right) . \tag{26}$$

Substituting $\frac{\partial C^*}{\partial P_k} = X_k$ from (4).

$$\gamma_{ij} = \frac{P_i P_j}{C^*}\frac{\partial^2 C}{\partial P_i \, \partial P_j} - \frac{P_i P_j}{(C^*)^2} X_i \, X_j \, .$$

Therefore,

$$\frac{\partial^2 C^*}{\partial P_i \, \partial P_j} = \frac{C^*}{P_i \, P_j}(\gamma_{ij} + \alpha_i \, \alpha_j). \tag{27}$$

Substituting (27) into (7)

$$\sigma_{ij} = \frac{\Sigma P_i X_i \, . C^*}{P_i \, P_j \, X_i \, X_j}(\gamma_{ij} + \alpha_i \alpha_j) = \frac{\gamma_{ij}}{\alpha_i \alpha_j} + 1 \qquad \text{Q.E.D.}$$

(24) follows from (15). The proof of (23) is similar except that in (26) $\frac{\partial P_i}{\partial P_i} = 1$ which accounts for $-\alpha_i$ in equation (23). (25) follows again from (15).

If the γ_{ij} have been estimated with equations (19) and/or (21), and if the factor shares are known, all elasticities can be estimated. Since $\hat{\sigma}_{ij}$ and $\hat{\eta}_{ij}$ are linear transformations of the γ_{ij}, whose econometric properties are known, the ecometric properties of the elasticities are known as well. No matrix of estimates has to be inverted.[9]

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[9] For estimating marginal products the cost function has the same disadvantage the production function has in estimating elasticities of substitution. Estimates

4. Treatment of Neutral and Non-Neutral Efficiency Differences

If efficiency differences exist among the observational units (states or firms in cross sections, years in time series) the specification of the estimation equations must take account of the problem to avoid bias in estimation.

It is best to distinguish two kinds of efficiency differences:

a) Differences which can be functionally related to a variable such as output (scale effects), a technical change index, time (as a proxy for technical change) or education and management (the left out variables problem).

b) Differences among cross-sectional units arising from past differences in technical change, which cannot be functionally related to a variable. If the cross sectional units have had a different past history of technical change, they are no longer on the same isoquant. This is likely to happen in many cross sections.

The first case is easily handled. Let the variable Y in (19) and (21) stand for any of the variables which cause the neutral and non-neutral efficiency differences (output, time, technical change index or education). Then (19) and (21) are immediately correctly specified, provided that the variable Y changes efficiency at constant logarithmic rates, and that data on the variable Y are available. As an example, if time series data are used for the regression and technical change alone causes the efficiency differences at constant logarithmic rates over time, let Y stand for time. The coefficient γ_Y will then be an estimator of the rate of technical change. The coefficients γ_{iY} will be estimators of the rates of bias. If all γ_{iY} were zero, time alone would not affect the factor shares (equation 21). This

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of its bordered Hessian have to be inverted. Since the translog cost function and the translog production function use the same basic data (input quantities and prices) it would be preferable to estimate the σ_{ij} and η_{ij} using the former function while using the latter one for the marginal products.

is the definition of neutrality of footnote 1. If γ_{iY} was greater then zero, the share of factor i would rise at constant factor prices at the logarithmic rate γ_{iY}. This would be factor i- using technical change.

If a variable causes efficiency differences on which no data are available, the γ_{ij} can still be estimated in an unbiased way, provided the left out variable affects efficiency neutrally. In that case all γ_{iY} are zero and (21) is still properly specified without data on the variable Y. But (19) is not correctly specified any more because γ_Y is not zero. Therefore, the γ_{ij} parameters have to be estimated in (21) alone.[10/]

In the next section scale effects will be assumed to be neutral. Output is therefore not included as a variable in (21). On the other hand, technical change over time is assumed to be non-neutral and time included as a variable (Y thus standing for time).

Problem b) of efficiency differences among cross-sectional units can be handled in the same way as the left-out variables problems above, provided the efficiency differences are neutral. The proper variable would be an efficiency index of the cross-sectional units which is generally unknown. However, if the efficiency differences are non-neutral due to biased technical change in different directions in previous periods, it would be necessary to know the efficiency index and include it as a variable in (21). If the index is not available but the cross-sectional units can be grouped into regions, within which no non-neutral differences exist, regional dummies in equation (21) will again insure unbiased estimates of the parameters of the cost function, because they allow the regions to have differing shares at equal factor prices. This again precludes simultaneous estimation of (17) and (19).

[10/] Including education etc. in a Cobb Douglas product in function assumes that these variables affects efficiency of the other factors neutrally because all elasticities of substitution are 1.

The discussion of this entire section applies equally to the translog production function.

5. Estimation, Data and Conclusions

The cost function was estimated with state data from the United States. Four sets of cross-sectional data were obtained for 39 states or groups of states. The cross sections were derived from census data and other agricultural statistics for the years 1949, 1954, 1959, and 1964. The combination of cross sections over time poses problems which are discussed below. In general, Griliches' (1964) definitions of factors were used. He distinguishes the following five factors: land, labor, machinery, fertilizer and all others. Intermediate inputs are included in this list and the function fitted corresponds to a gross output function rather than a value-added function.[11/] For all data pooled the following model was fitted:

$$\alpha_{ikt} = \nu_i + \sum_j \gamma_{ij} \ln P_{jkt} + \gamma_{it} \ln t + \sum_{r=1}^{4} \delta_r d_r + \varepsilon_{ikt} \qquad \begin{array}{l} i = 1,\ldots,n-1, \\ j = 1,\ldots,n. \end{array} \tag{26}$$

where i and j stand for factors of production, k for state, t for time, r for groups of states and

$$d_r = \begin{cases} 1 \text{ if } k \in r \\ 0 \text{ if } k \notin r. \end{cases}$$

δ_r is the coefficient of non-neutral efficiency difference between group r and group 5 (Western States). One share equation has to be dropped from the model because only n-1 equations are linearly independent due to the homogeneity constraint (20). In this form the model allows for neutral efficiency differences of any kind among

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[11/] The data are discussed in more detail in the appendix.

states, non-neutral efficiency differences among groups of states and non-neutral efficience differences over time.

Within each of the four cross sections (time period), the error terms of the n-1 estimation equations are not independent, since for each state the same variables which might affect the shares in addition to the prices were left out of the model. If restrictions across equations ($\gamma_{ij} = \gamma_{ji}$) are imposed, OLS estimators are no longer efficient despite the fact that all equations contain the same explanatory variables on the right hand side (Theil 1971). Therefore, the seemingly unrelated regression problem applies and restricted generalized least squares (RGLS) have to be applied to all equations simultaneously (Zellner 1962, 1963, Theil 1971, Chapter 7).[12/]

If all four cross sections are pooled there is an additional problem of error interdependence over time. The correct way of handling both problems would be to specify an equation for each share in each year, then test and impose the symmetry and homogeneity constraints and the constraints that the γ_{ij} parameters are constant over time. This exceeded the capacity of the TTLS program. The correct procedure would also have required that one impose constraints of equality of the auto-correlation coefficients over time on the estimated variance covariance matrix which was not possible with TTLS. The following procedure was therefore adopted: The search for an exact specification was done in RGLS regressions applied to the data of each cross section separately to avoid any biases in the tests used for this purpose.[13/] Once the decision was made to use a specification including equations

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[12/] The Computer Program used was Triangle Universities Computing Center: Two and Three Stage Least Squares, Research Triangle Partk, N.C., 1972 (TTLS).

[13/] No *a priori* information is available to decide which equation to drop and whether or not to include regional dummies. To make these decisions I was looking for the specification in which the imposition of the symmetry constraint $\gamma_{ij} = \gamma_{ji}$ and the homogeneity constraint $\sum_j \gamma_{ij} = 0$ led to the smallest weighted F-ratio according for land, labor, machinery and fertilizer, with regional dummies in all equations, all four cross sections were pooled and the symmetry and homogeneity constraints imposed in the restricted generalized least squares estimation of all four equations simultaneously. Since the error interdependence over time is neglected, the reported t-ratios will be overstated to some extent,[14/] but the estimators are still unbiased.

The results of the regressions are reported in Tables 1 and 2. Table 1 gives the OLS single equations R^2 of the four shares equation with homogeneity imposed on the data.[15/] The R^2 are not very high.

Table 1: OLS Single Equation R^2

Equation	Land	Labor	Machinery	Fertilizer
OLS R^2	.68	.75	.45	.75

to the test static 3.6, p. 314, Theil 1971. Since both of these constraints are "true" constraints, they can be used in this way to eliminate some specifications, although several specifications might satisfy the constraints. A specification in which "other" inputs are excluded and dummies added to all equations satisfied this criterion best for the four cross sections. Homogeneity was accepted at the .05 level in all cross sections. Symmetry was only rejected in the cross section of 1964 with an F-ratio of 4.19 (Critical $F_{.05} = 2.17$).

14/ Despite the 5 year interval between the cross sections, error interdependence over time was still quite large. Correlation coefficients of the OLS errors of individual share equations between the years 1949 and 1964 were between .62 and .87. To check whether the neglect of this interdependence among cross sections had a large impact on the estimated values of the γ_{ij} the estimates of the pooled cross sections were compared with the simple average of the estimates obtained in the four cross sections individually. The differences of the estimates were small.

15/ The residuals of these equations are used to estimate the variance-covariance matrix for the GLS regressions.

Table 2. Restricted estimates of the coefficients of the translog cost function and t-ratios[a]

Equation	Price of Land	Price of Labor	Machinery Price	Fertilizer Price	Ln year	Intercept	MN	GR	SE	GS[c]	Other[b]
	VARIABLE										
Share of Land	.07747 (6.02)	-.03613 (3.25)	.00478 (.47)	.01066 (2.14)	.00847 (1.47)	.2603 (9.96)	-.1021 (10.2)	-.0394 (4.1)	-.1073 (8.9)	-.0577 (4.7)	-.05678
Share of Labor		-.06367 (3.67)	-.00661 (.59)	-.02805 (4.97)	-.05482 (9.08)	.5218 (14.91)	.0194 (1.63)	-.0016 (.15)	.0169 (1.09)	.0246 (1.63)	.13446
Share of Machinery			-.03485 (1.31)	-.00877 (.97)	.02498 (4.66)	.0926 (3.46)	-.0033 (.41)	.0369 (5.08)	-.0186 (1.86)	.0072 (.73)	.04545
	(Symmetric)										
Share of Fertilizer				.00068 (.12)	.00178 (.63)	.0745 (5.6)	.0104 (2.5)	-.0041 (1.10)	.0370 (7.24)	-.00247 (.49)	.02548
Share of Other											-.14861

a/ Critical values with 578 degrees of freedom are $t_{.05} = 1.96$ and $t_{.01} = 1.65$. F-ratios may be overstated due to error interdependence over time.

b/ Computed using the homogeneity constraint, not estimated.

c/ MN, GR, SE, GS are dummies for mixed northern agriculture, grain farming states, Southeast, and Gulf states respectively. The intercept stands for Western States and the coefficients of MN, GR, SE, GS are deviations from this intercept.

From Table 2 the following conclusions emerge:

a) Out of the ten γ_{ij} only 5 are statistically significant. This is not a "bad" result because $\gamma_{ij} = 0$ implies that the elasticity of substitution is equal to the Cobb-Douglas value of 1. Note, however, that when the Cobb-Douglas constraint $\gamma_{ij} = 0$ for all i, j was tested in various single cross section models, it was always rejected. Therefore, the conclusion is that the Cobb-Douglas form is inappropriate for aggregate production or cost function fitting.

b) At the .05 level the coefficients of the time variable is significant in the labor and machinery equations. This means that, at constant factor prices the factor shares would have changed which implies non-neutral technical change during the period 1949 to 1964. The coefficient of time in the labor equation is -.0548. Hence, technical change was labor saving. On the other hand the positive coefficient of time in the machinery equation (+.025) implies machinery using technical change. This is consistent with the findings of Lianos (1971) and my own findings (Binswanger 1972).

c) Six of the regional dummies are significant. At equal factor prices the shares would not be equal among the groups of states. The coefficients of the dummies in the land equation of all four regions is negative. This implies that the technology in all regions is land saving relative to the technology used in the Western states--Washington, Oregon and California. The significant positive coefficients in the fertilizer equation of the dummies for the mixed northern states and the southeastern states would indicate that these regions use a more fertilizer using technology than the western states.

Not too much should be made out of these regional differences because they may be due to different product mixes rather than to true technological differences in each production. If they reflected product mix differences, the dummies will at least correct for possible biases due to these differences.

The γ_{ij} have little economic meaning. They are best evaluated by the values which they imply for elasticities of factor demand and of substitution.[17/] The values are computed for the simple average of factor shares for all 39 states between 1949 and 1964. In Table 3 the elasticities are compared with what they would be at equal factor shares in the Cobb-Douglas case ($\gamma_{ij} = 0$).

Tables 3 and 4 indicate the following conclusions:

d) All own elasticities of factor demand have the correct sign. Land demand seems to be very inelastic. In empirical work with Cogg-Douglas production functions the coefficient of land is usually between .15 and .4. According to equation (25) with $\gamma_{ij} = 0$ (the Cobb-Douglas constraint) these values imply land demand elasticities of -.85 to -.60 which is substantially higher than the elasticity found with the translog cost function.

The values of the other own demand elasticities are close to one and, except for fertilizer, higher than they would be in the Cobb-Douglas case. The fertilizer demand is substantially less elastic than Griliches (1959) estimate of -2.0.[18/]

e) Elasticities of substitution and cross-elasticities of demand are positive for substitutes and negative for complements. These relationships are easier

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[17/] The elasticities were also computed using aggregate factor shares reported in Binswanger (1972) for the years 1912, 1952, 1964 and 1969. While differences exist with the values reported here they seemed not important enough to report these values. The main advantage of using a variable elasticity of substitution function rather than a CES framework is not that, for observed values of shares, the elasticities vary widely, but that this format does not constrain all elasticities of substitution to be equal.

[18/] These estimates are not necessarily in conflict. Griliches estimates a long-run elasticity in a time series. This implies that, if there is an induced fertilizer-using innovation due to a fall in the price of fertilizer, his price response picks up part of the adjustment due to the technical change. This is what happened in U.S. agriculture (Binswanger 1972). Since the inclusion of the time variable in our regression equations picks up the influence of technical change, the estimates presented here are net of any technical change influence.

Table 3. Factor demand and cross demand elasticities[a] implied in the estimated γ_{ij} and the standard errors around their value in the Cobb-Douglas case[b]

	Land[d]	Labor	Machinery	Fertilizer	Other
Estimated Translog values[c]					
Land	-.3356 (.09)	.0613 (.07)	.1792 (.07)	.1062 (.03)	-.0112
Labor	.0308 (.04)	-.9109 (.06)	.1256 (.04)	-.0577 (.02)	.8122
Machinery	.1833 (.07)	.2560 (.08)	-1.0886 (.18)	-.0239 (.06)	.6733
Fertilizer	.4508 (.10)	-.4878 (.20)	-.0991 (.30)	-.9452 (.16)	1.0815
Other	-.0046	.6690	.2720	.1053	-1.0417
Cobb-Douglas values for comparison[e]					
Land	-.8491	.3008	.1475	.0356	.3652
Labor	.1509	-.6992	.1475	.0356	.3652
Fertilizer	.1509	.3008	-.8525	.0356	.3652
Machinery	.1509	.3008	.1475	-.9644	.3652
Other	.1509	.3008	.1475	.0356	-.6348

[a]Each element in the table is the elasticity of demand for the input in the row after a price change of the input in the column. These elasticities are not symmetric.

[b]The shares used are the same as for the Cobb-Douglas η_{ij}.

[c] $\eta_{ij} = \frac{\gamma_{ij}}{\alpha_i} + \alpha_j, \qquad \eta_{ii} = \frac{\gamma_{ii}}{\alpha_i} + \alpha_i - 1.$

[d] $SE(\eta_{ij}) = \frac{SE(\gamma_{ij})}{\alpha_i}.$

[e] $\eta_{ij} = \alpha_j, \quad \eta_{ii} = \alpha_i - 1.$

Table 4. Estimates of the partial elasticities of substitution and standard errors with respect to the value of +1 (Cobb-Douglas case)a/

	Land	Labor	Machin-ery	Fert-ilizer	Other
Land	-2.225 (.57)	.204 (.24)	1.215 (.46)	2.987 (.93)	-.031
Labor		-3.028 (.19)	.351 (.25)	-1.622 (.53)	2.224
Machinery		Symmetric	-7.379 (1.22)	-.672 (1.71)	1.844
Fertilizer				-26.573 (4.61)	2.961
Other					-2.852

a/ $\sigma_{ij} = \frac{\gamma_{ij}}{\alpha_i \alpha_j} + 1$, $\sigma_{ii} = \frac{1}{\alpha_i^2} (\gamma_{ii} + \alpha_i^2 - \alpha_i)$.

The elasticities of substitution are symmetric.
The own elasticity of substitution has little economic meaning except that it has to obey the constraint $\sum_j \alpha_j \cdot \sigma_{ij} = 0$.

evaluated by the elasticities of substitution in table (4) than the cross-elasticities of demand because the latter reflect the relative importance (share) of a factor while the former do not. Complementarity seems to exist between the labor-fertilizer pair, the machinery-fertilizer pair and the land-other inputs pair. That the first two pairs should be bad substitutes comes as no surprise but the significant complementarity of the labor-fertilizer pair was not expected. The elasticity of substitution between machinery and fertilizer is not significantly different from zero. Hence that complementarity might be spurious. The value for the land-other inputs pair is very small and probably not significantly different from zero.

The best substituability relation exists between land and fertilizer, which was expected. It was a surprise, however, to find that machinery is a better substitute for land than for labor (although the machinery-labor elasticity falls within one standard deviation of the land-labor elasticity, so that there is no statistical difference). Even if the machinery-land elasticity were over-estimated to some extent the finding should cast doubt on the notion that one can dichtomize agricultural technology in mechanical technology which acts exclusively as a labor substitute and biological technology which acts exclusively as a land substitute.[12] The small elasticity of substitution between land and labor was expected.

Overall the result seems to be reasonable and show that cost functions in general and the translog cost function in particular lead to valuable methods of production parameter estimation.

[12] This idea is put forward in Hayami and Ruttan (1972).

LIST OF REFERENCES

Berndt, E. R. and L. R. Christensen. 1971. The translog production function and factor substitution in U.S. manufacturing, 1929-1968. Unpublished paper presented at the Winter meetings of the Econometric Society in New Orleans, Louisiana.

Binswanger, H. P. 1972. The measurement of biased technical change in the many factors case: U.S. and Japanese Agriculture. Staff Paper series #P72-28, Department of Agricultural and Applied Economics, University of Minnesota.

_______. 1973. The measurement of biased efficiency gains in U.S. and Japanese Agriculture to test the induced innovation hypothesis. Unpublished Ph.D. thesis. Department of Economics, Nort Carolina State University, Raleigh. University Microfilms, Ann Arbor, Michigan.

Christensen, L. R., D. W. Jorgenson, and J. L. Lau. 1970. Conjugate duality and the transcendental logarithmic production function. Unpublished paper presented at the Second World Congress of the Econometric Society, Cambridge, England, September 1970.

Diewert, W. E. 1971. An application of the Shephard duality theorem: A generalized Leontief production function. Journal of Political Economy 70(3):481-505.

Fishelson, G. 1968. Returns to human and research capital, United States agriculture, 1949-1964. Unpublished Ph.D. thesis, Department of Economics, North Carolina State University, Raleign. University Microfilms, Ann Arbor, Michigan.

Griliches, Z. 1963a. Estimates of the aggregate agricultural production function from cross section data. Journal of Farm Economics 45:419-428.

_______. 1963b. The sources of measured productivity growth: United States agriculture, 1940-1960. Journal of Political Economy, 71:331-346.

_______. 1964. Research expenditures, education and the aggregate agricultural production function. American Economic Review, 54:961-974.

_______. 1968. Notes on the role of education in production functions and growth accounting. Center of Mathematical Studies in Business and Economics, University of Chicago, Report 6839.

_______. 1958. The demand for fertilizer: an economic interpretation of a technical change. Journal of Farm Economics 40(3):591-606. August.

Hayami, Y. and V. W. Ruttan. 1971. Agricultural development: an international perspective. John Hopkins Press, Baltimore and London.

Lianos, T. P. 1971. The relative share of labor in United States agriculture. American Journal of Agricultural Economics 53(3):411-422.

McFadden, D. 1963. Production functions with constant elasticities of substitution. Review of Economic Studies, 30:73-83.

Theil, H. 1971. Principles of econometrics. John Wiley & Sons, Inc., New York.

Timmer, C. P. 1970. On measuring technical efficiency. Food Research Institute Studies in Agricultural Economics, Trade and Development, 9(2).

Triangle Universities Computing Center. 1972. Program manual for two and three stage least squares. Research Triangle Park, North Carolina.

Uzawa, H. 1962. Production functions with constant elasticity of substitution. Review of Economic Studies, 30:291-299.

Wallace, T. D. and V. G. Ashar. 1972. Sequential methods in model construction. Review of Economics and Statistics, 54(2):172-178.

Zellner, A. 1962. An efficient method of estimating seemingly unrelated regressions and tests for aggregation bias. Journal of the American Statistical Association, 57:348-368.

_______. 1963. Estimators for seemingly unrelated regression equations: Some exact finite sample results. Journal of American Statistical Association, 58:977-992.

APPENDIX

VARIABLE CONSTRUCTION AND DATA SOURCES

For the 39 states and groups of states (see Table D1) aggregate input quantity data and expenditure data were derived.

Quantity Data

Except for "other" inputs, the quantity data were taken from Fishelson (1968), who used Griliches (1962) data with some changes. His discussion of the construction of the variables is reproduced here:

"Material Inputs

...Land. In the U. S. Census of Agriculture (U. S. Bureau of the Census, 1952, 1956, 1962 and 1966), the average value of land and buildings per farm in each state was reported. However, the land value represented not only the value of land to agricultural production but also included the site value of land. The value of buildings included both farm structures and dwellings. Hence, census data on value of land and buildings were inadequate for the purposes of this study. To measure land by the number of acres per farm (giving each acre a value of one) is also inadequate because of the diversity of soil quality, fertility and uses.

In this study the weighting procedure for measuring land value was based on a study by Hoover (1961). The value of each acre in each state at each cross section was measured by its 1940 price relative to that of an acre of pasture in the corresponding state. The value of an acre of pasture in each state in 1940 was calculated by dividing the total value of land in 1940 by the number of pasture equivalent units of the land in 1940. This value of an acre of pasture was kept constant over time. Since all prices were deflated to the 1949 price level in this study, the value of an acre of pasture in 1940 was also adjusted to the 1949 price level. The deflator used was total value of land in the United States agriculture sector in 1949, i.e., the value of agricultural land in 1949 measured in 1940 relative land prices ratio. The ratio was 2.2. The use of this method provided a measure of the stock of land in constant prices. According to this method, changes in the stock of land occurred only because of changes in the number of acres or their use. The stock of land was unaffected by changes in prices of agricultural products, site effects, or government programs.

Table D.1. Listing of states and their groups

State No.	Listing of states and their groups	Group[a]
1	Maine, New Hampshire, Vermont, Massachusetts, Rhode Island and Connecticut	MN
2	New York	MN
3	New Jersey	MN
4	Pennsylvania	MN
5	Ohio	MN
6	Indiana	MN
7	Illinois	GR
8	Michigan	MN
9	Wisconsin	MN
10	Minnesota	GR
11	Iowa	GR
12	Missouri	GR
13	North Dakota	GR
14	South Dakota	GR
15	Nevada	GR
16	Kansas	GR
17	Delaware, Maryland	SE
18	Virginia	SE
19	West Virginia	MN
20	North Carolina	SE
21	South Carolina	SE
22	Georgia	SE
23	Florida	SE
24	Kentucky	MN
25	Tennessee	MN
26	Alabama	SE
27	Mississippi	GS
28	Arkansas	GS
29	Louisiana	GS
30	Oklahoma	GS
31	Texas	GS
32	Montana	GR
33	Idaho	MW
34	Wyoming, Utah, Nevada	MW
35	Colorado	GR
36	New Mexico, Arizona	MW
37	Washington	MW
38	Oregon	MW
39	California	MW

[a]MN Mixed agriculture, north
GR Grain farming
SE South East
GS Gulf States
MW Mixed agriculture, west

Labor. The labor input was measured in physical flow units defined as the number of days worked per farm per year. The labor input was obtained from three sources, operator's labor, labor of other family members and unpaid workers, and hired labor. Physical labor was adjusted for age (.6 for operators above 65) and for labor supplied by other family members (.65). No adjustments were made for changes in labor's quality.

The computational equation for labor is given in Griliches (1964, p. 974).

Machinery. The machinery variable was a measure, in constant prices, of the cost of the flow of services obtained through the use of farm machinery and equipment. The variable was the sum of deflated expenditures on repairs and operation (1949=100) and 15 percent of the stock value (after adjusting to 1949 prices) of machinery and equipment on farms. The latter item was an attempt to approximate machinery services by the costs of interest and depreciation assuming a constant proportion, over states and time, between the stock value and the flow of services.

Fertilizer. The fertilizer input was defined to be the weighted sum of the quantity of plan nutrients. The nutrients are nitrogen (N), phosphoric acid (P_2O_5) and potash (K_2O). The weights were their 1955 relative prices or 1.62, .93 and .45, respectively (Griliches, 1964, p. 967). Thus, the fertilizer input was measured in equivalent tons per year, i.e., a flow measure. This measure provided a more accurate estimate of the real input than a cost measure because of the declining price per unit of nutrient over time and the changing nutrient content per ton of fertilizer over states."

The only change which was made in these quantity data was that, whenever quantities per farm were used, the farm number was taken from the Census of Agriculture (U.S. Department of Commerce, 1950 - 1964), rather than from Farm Labor (U. S. Department of Agriculture, 1945 - 1972).

Other Inputs: Since expenditure data corresponding to Fishelson's quantity data could not be constructed, new quantity data were defined as follows: They are the sum of the explicit and implicit annual expenditures on all other material inputs used in production. The explicit expenditures were the cash expenditures on purchase of livestock, poultry, feed, seeds, plants and bulbs, operation and repairs of farm structures and other miscellaneous costs. The implicit expenditures were 8 percent interest on livestock and crop inventories, depreciation (4.2%) and interest (5%) on the value of farm structures, and the share of real estate taxes falling on buildings. Each of the expenditures was separately deflated to its 1949 price level to arrive at a quantity measurement (for taxes the agricultural output price index was used.)

Expenditures and Factor Shares:

The expenditure variables were defined, as far as possible, to correspond to the quantity variables. Expenditure shares were obtained by dividing the expenditures through the sum of the expenditures.

Land: Expenditures on land is simply 6 percent of the value of land plus the share of real estate taxes falling on land.

Labor: Expenditures for labor is the number of man days of labor from Fishelson (1968) multiplied by a daily wage rate without room and board (Farm Labor, 1945-1972). This assumes that the opportunity cost of farm operators is the wage rate which they could earn as workers on other farms.

Machinery: Expenditures are assumed to be 15 percent of the value of farm machinery and equipment for interest and depreciation plus the current expenditures for operation and repair of machinery and equipment.

Fertilizer: Fertilizer expenditures are directly reported by the U.S.D.A.

Other Expenditures: These expenditures were computed exactly as the quantity, except that the individual items were not deflated. Aggregate expenditures estimated in this way had a tendency to exceed aggregate income by up to ten percent.

Prices:

Prices were taken to be the expenditures divided by the quantities. They were then deflated to the 1949 price level using the U.S. Agricultural output price index. Note that this procedure implies that the price of other inputs is equal to one for all states in the year 49. Table D2 lists all the data sources.

Table D2. Sources for the cross section data

Variables	Source
Farm income, change in inventories, rental value of dwellings, all explicit current operating expenditures	Farm Income Situation, July supplement, USDA (1954-1972)
Annual average daily wage rate without board or room	Various issues of Farm Labor, USDA (1945=1972)
Farm number	Various Issues of Census of Agriculture. U. S. Department of Commerce, (1950, 1954, 1959, 1964)
Input and output price indexes	Various issues of Agricultural Statistics, USDA, (1936-1972)
Repairs and operation of farm dwellings and service structures, depreciation of dwellings, service buildings, motor vehicles, other machinery and equipment, value of farm machinery and equipment, value of crop inventories	USDA, unpublished