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On the Optimal Depletion of Old-Growth Forests and the Preservation of Wilderness

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Theorem 4 (p. 18) should indicate that increases in b, k and r all act to decrease the size of the wilderness system.

The Proof of Theorem 4 should be modified as follows:

$$w'(x^*(r,b,k)) - r\left[\frac{b + ke^{rR}}{f(R)}\right] \equiv 0.$$

Then,

$$x_b^* = r/f(R)w''(\cdot) < 0$$

$$x_k^* = re^{rR}/f(R)w''(\cdot) < 0$$

$$x_r^* = \left[K + \frac{rRke^{rR}}{f(R)}\right]/w''(\cdot) < 0$$

ON THE OPTIMAL DEPLETION OF OLD-GROWTH FORESTS AND THE PRESERVATION OF WILDERNESS

by

Theodore Graham-Tomasi[1/]

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ABSTRACT

The problem of optimal depletion of old-growth forests and establishment of plantations is explored. The old-growth has value for the timber products it contains and the wilderness services it provides if left unharvested. The optimal amount of wilderness preserved in the steady-state is characterized and found to increase with increased interest rates and decreased cost of production. The wilderness preserve is largest when silvicultural effort is utilized optimally in plantations. The problem of small initial stock sizes is investigated.

1. INTRODUCTION

Conflicts between the harvest of old-growth forests for the timber they contain and the preservation of these forests for the environmental services they provide are long-standing and widespread. One set of interests argues that excessive preservation severely will limit capacity for production of wood products and, therefore, implies excessive current or future prices for these products, reliance on undesireable substitutes, and retardation of economic development. Other interests hold that the stock of wilderness land is a highly valuable, irreplaceable asset and insist that further incursion into this stock is unjustified and even immoral.

In this paper, I study the problem of the optimal harvest of an old-growth forest which has value for both the timber it contains and the wilderness services it provides. This stock of old-growth acts like an exhaustible resource; its harvest is an irreversible action. In addition to this old-growth, the economy can establish plantations which supply wood products on a renewable basis. These plantations are a renewable substitute for the exhaustible old-growth and serve here as the "backstop technology" posited to exist in much of the exhaustible resource literature.[2/] The problem is to determine the optimal rate of depletion of the old-growth, the timing of a transition to the use of plantations for the provision of wood products, and the size of the stock of old-growth optimally preserved as wilderness.

This paper has many points of tangency with existing work. Several authors have considered old-growth forests as an exhaustible resource. The first, no doubt, was Hotelling (1931), who mentioned both the exhaustible and renewable components of forests. Binkley (1979) and Hyde (1980) both offer Hotelling's Rule (with costless extraction, price must rise at the rate of interest in competitive equilibrium) as an explanation for historically rising timber stumpage prices and conjecture that a transition to a steady-state with constant prices will ensue. Lyon (1981) studied this issue in detail and derived conditions concerning the time path of stumpage prices in which prices behave much as in standard exhaustible resource theory.

The reason for this price change is well known. In equilibrium, positive amounts of the resource must be supplied at adjacent dates. For this to occur, resource owners must be indifferent between extracting and selling the resource today and extracting and selling it tomorrow.[3/] Thus, to ensure positive future supply, the return to holding the resource, i.e., capital gains, must be at least as great as the return to holding alternative assets. However, if current supply is to be positive, then these capital gains must not be greater than the return on alternatives, and Hotelling's conclusion follows.

Regarding forests, possible forest growth and regeneration of the site must be taken into account. If forests grow, there is an added benefit to holding the resource and rent need rise slower than the interest rate for this intertemporal indifference condition to obtain. However, if forest land is to be regenerated after harvest, postponement of harvest implies, in addition to delay of net revenues, delay of the present value of subsequent management efforts. This implies, in turn, that capital gains from holding the resource must be relatively higher for intertemporal indifference to occur.

One would suppose that, after the old-growth forests are exhausted, a steady-state would arise in which society relies on plantations for its wood products. Much work has been done on the economics of forestry in the steady-state. The Faustmann model of forest management is a well-known and widely-used vehicle for this analysis.[4/] The Faustmann model, however, is not appropriate to analysis of the transitional period preceeding the steady-state.

The transition itself has been studied fairly extensively in the forest management literature as a general problem of converting a forest with some initial distribution of age classes into a forest with a different distribution. To a large extent, this work has been designed to solve specific harvest scheduling problems on relatively small areas. Thus, researchers typically have used an assumption that the output price is fixed and have adopted linear programming approaches.[5/] A notable exception to this, however, is the research of Lyon and Sedjo (1981). Using an optimal control model, they study the time paths of old-growth harvest and second-growth regeneration and harvest which maximize the present value of consumer's surplus from forest management.[6/] This approach is adopted here as well.

One basic difference between most of the forest management literature and this paper concerns the question of wilderness values. In the above analyses, it has most often been assumed that the forest has value only when it is harvested. Thus, the conflict between timber harvest and preservation, with which this paper is concerned, has not been addressed directly.

Several authors have incorporated a valuable stock into analyses of resource problems. Hartman (1976) reconsidered the problem of the growth period in forestry in this case; but his analysis does not deal with the initial stock-adjustment aspect of the problem that is crucial in this paper. Vousden (1974) studied the use of an exhaustible resource when the stock has value, but did not have a backstop technology, formed here by the possible use of plantations. Berck (1981) analyzed the problem of optimal harvest of a renewable resource with stock externalities. Not surprisingly, he found that it is optimal to carry a larger stock of the resource in the steady-state. His analysis of the approach to the steady-state has a close correspondence to the problem studied in this paper. However, due to the wilderness/plantation dichotomy analyzed here, the two investigations are distinct.

Finally, Bowes (1983) investigates a very rich forestry model which incorporates a non-linear objective function, stock adjustment, and multiple uses of the standing forest. The model, while complete, is sufficiently complicated that an algorithmic approach must be taken and solutions are provided only for representative cases.

The model explored here is extremely simple relative to much of the work described above. By studying a continuous time control model and employing several simplifying assumptions, I am able to describe the qualitative characteristics of the solution to the timber/wilderness conflict and to obtain standard comparative statics results. The attempt here is to provide some insight into the solution to the problem. Hence, this paper is complementary to the Lyon and Sedjo (1981) and Bowes (1983) analyses described above.

The paper is organized as follows. In the next section, a basic model of the optimal depletion of an old-growth forest is studied when the stock does not have value as wilderness. In the third section, the wilderness question is explored. In the fourth section, the use of silvicultural effort as an additional control variable in management of second-growth stands is added to the model, and the relationship between the level of effort and the preservation of wilderness is established. Prospects for technological change are studied in the fifth section. The sixth section elaborates on the relationship between this paper and the well-known steady-state Faustmann theory. The final section is a discussion.

2. OLD-GROWTH DEPLETION

Let $x(t)$ be the stock of old-growth at date t. The initial stock is $x(0) = x^{o} < \infty$; it is assumed to be homogeneous in all regards and to exhibit **no** net growth. The rate of old-growth harvest at t is denoted by $q(t)$ and is constrained to be non-negative for all t. Then,

$$\frac{dx(t)}{dt} \equiv \dot{x}_t = -q_t \ . \tag{1}$$

By choice of units, the volume of old-growth per unit of area is equal to one.

Regarding second-growth stands, let $s(t)$ be the area established at t. I assume that second-growth stands are managed on a fixed rotation of duration R. The yield function in this section is of the form $f(m)$; i.e., $f(m)$ is the volume of wood per unit of area in a stand of age m. No silvicultural inputs are used to enhance growth. Letting h_t be the volume of wood harvested from second-growth at t, we have

$$h(t) = s(t-R)\, f(R) \ , \tag{2}$$

so that total harvest is

$$y(t) = q(t) + h(t). \tag{3}$$

The inverse demand function for harvested material is

$$p(t) = D(y(t)) .$$

The gross benefit of harvest is given by the area under this curve, i.e.,

$$B(y(t)) \equiv \int_0^{y(t)} D(w)dw .$$

It is assumed that the demand function $D(\cdot)$ is stationary, continuously differentiable, and strictly decreasing. Hence, $B(\cdot)$ is twice continuously differentiable, strictly increasing, and strictly concave. Further, I assume that the demand curve is asymptotic to the vertical axis, i.e.,

$$\lim_{y \to 0} B'(y) = \infty . \qquad (4)$$

I assume that old-growth may be harvested costlessly and that second-growth stands may be established at a cost of k and harvested at a cost of b, both on a per unit of area basis.[7/] For convenience, the rate of time preference is assumed to be a constant, r.

The forester's problem is to maximize the present value of consumer's surplus.[8/] That is, the problem is

$$P: \max_{q_t, s_t} \int_0^\infty [B(y_t) - ks_t - bs_{t-R}] \, e^{-rt} \, dt$$

$$\text{s.t. } (1), (2), (3)$$
$$s_t \geq 0, \; q_t \geq 0, \; x_t \geq 0 \quad x(0) = x^0 < \infty \, ; \; s(t) = 0 \;\; t \in [-R, 0].$$

The problem P is not a standard control problem in that the control variable s_t affects the objective function after a lag of duration R. However, one can adapt a derivation of necessary conditions with delayed response in Kamien and Schwartz (1981) and other standard results from control theory (see, e.g. Long and Vousden, 1977) to prove the following theorem.

Theorem 1. Let (q_t^*, s_t^*) solve P. Then there exists a non-zero vector $\{z_0, z_1(t)\}$ and functions u_t, m_t, and n_t which satify

(i) z_0 is a non-negative constant

(ii) $u_t \geq 0$, $u_t s_t^* = 0$, u_t is continuous when s_t^* is,

(iii) $m_t \geq 0$, $m_t q_t^* = 0$, m_t is continuous when q_t^* is,

(iv) $n_t \geq 0$, $n_t x_t^* = 0$, n_t is continuous when q_t^* is,

(v) $z_1(t)$ is continuous when y_t is continuous except (possibly) for dates when the constraint $x_t \geqslant 0$ becomes binding. If T is a date when x_t hits zero, then $z_1^+(T) = z_1^-(T) - \bar{z}$ for $\bar{z}$ a non-negative constant,

(vi) $\dot{z}_1(t) = r z_1(t) - \partial L^*/\partial x_t$

(vii) $\partial L^*/\partial s_t + \partial L^*/\partial s_{t-R}\Big|_{t+R} = 0$

(viii) $\partial L^*/\partial q_t = 0$

(ix) $H(x_t^*, s_t^*, q_t^*, z_0, z_1(t)) \geqslant H(x_t^*, s_t, q_t, z_0, z_1(t))$ for all (x_t^*, q_t, s_t) satisfying the constraints of P,

where[9/]

$$H(\cdot) \equiv z_0[B(y_t) - bs_{t-R} - ks_t] - z_1(t)\, q_t \tag{5}$$

$$L^*(\cdot) \equiv H(x_t^*, s_t^*, q_t^*, z_0, z_1(t)) + u_t s_t + m_t q_t + n_t x_t\,. \tag{6}$$

Further, the transversality condition

$$\lim_{t\to\infty} z_1(t)\, x(t)\, e^{-rt} = 0 \tag{7}$$

is necessary, as well as sufficient, for the problem P.[10/]

The multiplier z_0 is bothersome. However, it easily is shown to be non-zero. Thus, I state

Lemma 1. $z_0 \neq 0$.

The proof of this Lemma and all other propositions in this paper appear in the appendix.

Using Lemma 1, and defining $z(t) \equiv z_1(t)/z_0$, Theorem 1 implies that the following conditions must hold along an optimal path:

$$B'(y_t^*) - z_t + m_t = 0 \tag{8}$$

$$B'(y_t^*) - b - ke^{rR} + u_{t-R}\Big|_{t+R} = 0 \tag{9}$$

$$\dot{z}_t = r z_t - n_t\,. \tag{10}$$

The analysis of the solution to P is facilitated by the following definitions:

$$T_1 \equiv \{t : q_t^* > 0\}$$

$$T_2 \equiv \{t : h_t^* > 0\}$$

$$K \equiv (b + ke^{rR})/f(R) .$$

The quantity K is the unit cost, evaluated at time of harvest, of growing and harvesting second-growth stands. The establishment cost k must be carried with interest for R units of time. At the time of harvest, total cost is $b + ke^{rR}$, so that K is the unit cost.

One complication that arises in the analysis of the solution to P concerns the fact that, for R units of time, second-growth stands are not available for harvest. Thus, if the economy wishes to make use of second-growth stands at some $t<R$, it cannot. Intuitively, if the initial stock of old-growth is large, this problem does not arise. I will discuss this issue in more detail below. However, in the interim, I state results which hold for this (plausible) "large initial stock" case.

The solution path to P is partially characterized by the following theorem.

Theorem 2. (i) $T_1 \cup T_2 = [0,\infty)$, (ii) $T_1 \cap T_2 = \emptyset$, (iii) $B'(y) \leq K$, (iv) if $\hat{t} \in T_2$, then $t \in T_2$ for all $t > \hat{t}$, (v) $t \in T_2 \Rightarrow x(t) = 0$, and (vi) $T_2 \neq \emptyset$.

Theorem 2, in conjunction with the necessary conditions (8), (9) and (10), give rise to the following intuitive description of the solution path.

During an interval [0,T], old-growth forests are harvested and plantations are held in abeyance. During this era, the multipliers m_t and n_t both are zero, so that the stumpage price of old-growth, $B'(q_t)$, is equal to the shadow price of the stock of old-growth, z_t. For the usual arbitrage-based reasons, this stumpage price rises at the rate of interest as long as old-growth is being harvested. Since no second-growth stands are being established, the multiplier u_t is non-zero. This allows the simultaneous satisfaction of (8), (9) and (10).

When the price of old-growth reaches K, the unit cost of growing and harvesting trees in plantations, the harvest of old-growth ceases. At this instant, date T, the stock of old-growth is exhausted, and the economy switches to the use of managed stands. Naturally, for this to be possible, planting must have begun at T-R. The amount of land planted from T-R to infinity is given by

$$s^* = \frac{B'^{-1}(K)}{f(R)} \quad . \tag{11}$$

The price of wood products is equal to the unit cost of their production. Regarding the conditions (8), (9) and (10), the multipliers m_t and n_t "switch on" at T; while the multiplier u become zero at T-R, this is not felt until T due to the way that the delayed response affects the necessary conditions.

What remains to be determined are the optimal "switch-date" T* and the initial rent z_0^* (or, alternatively, the initial rate of old-growth harvest, q_0^*). These are determined by the following equations:

$$\int_0^T q_t \, dt = x^0 \tag{12}$$

$$p(T) = K. \tag{13}$$

The above discussion indicates that, for $t \in T_1$,

$$\dot{P}_t = \lambda p_t , \tag{14}$$

whence

$$p_t = p_0 e^{\lambda t} \quad .$$

Using the demand curve $y_t = D^{-1}(p_t) \equiv d(p_t)$, (12) and (13) become

$$\int_0^T d(p_0 e^{r t})\, dt = x^0 \tag{15}$$

$$p_0 e^{r T} = K. \tag{16}$$

The comparative statics of the solution path easily are obtained from (15) and (16) and are summarized in the following theorem.

Theorem 3. $\partial T^*/\partial x^0 > 0,\ \partial T^*/\partial K > 0\ \ \partial T^*/\partial r < 0$

$\partial p_0^*/\partial x^0 < 0,\ \partial p_0^*/\partial K > 0\ \ \partial p_0^*/\partial r < 0.$

These results are not surprising. The solution path is depicted in Figure 1.

I now return to the problem concerning the desired switch-date relative to the date of first availability of managed stands. Suppose the solution to the problem $\mathcal{P}$ is such that T^* is equal to R, i.e., the economy wishes to switch to plantations at the instant that they first are available. Then

$$T^*(x^0, K, r) - R = 0.$$

Since $\partial T^*/\partial x^0 \neq 0$, there exists a critical initial stock level x(R) such that

$$T^*\{x(R,K,r), K, r\} - R \equiv 0. \tag{17}$$

Clearly, if the initial stock x^0 is greater (less than) $x(R,K,r)$, the desired switch-date T* is greater than (less than) R. Thus, if the initial stock is large enough, the economy can switch to managed stands without difficulty and all of the above analysis holds. However, if $x^0 < x(R,K,r)$, then when the economy wishes to switch, it cannot. Of course, in this instance, it switches at the first date that it is feasible to do so, i.e., $T = R$. Now, the problem is determined, since (14) must hold along an optimal path for $t \in T_1$. Then (15) becomes

$$\int_0^R d(p_0 e^{r t})\, dt = x^0 ,$$

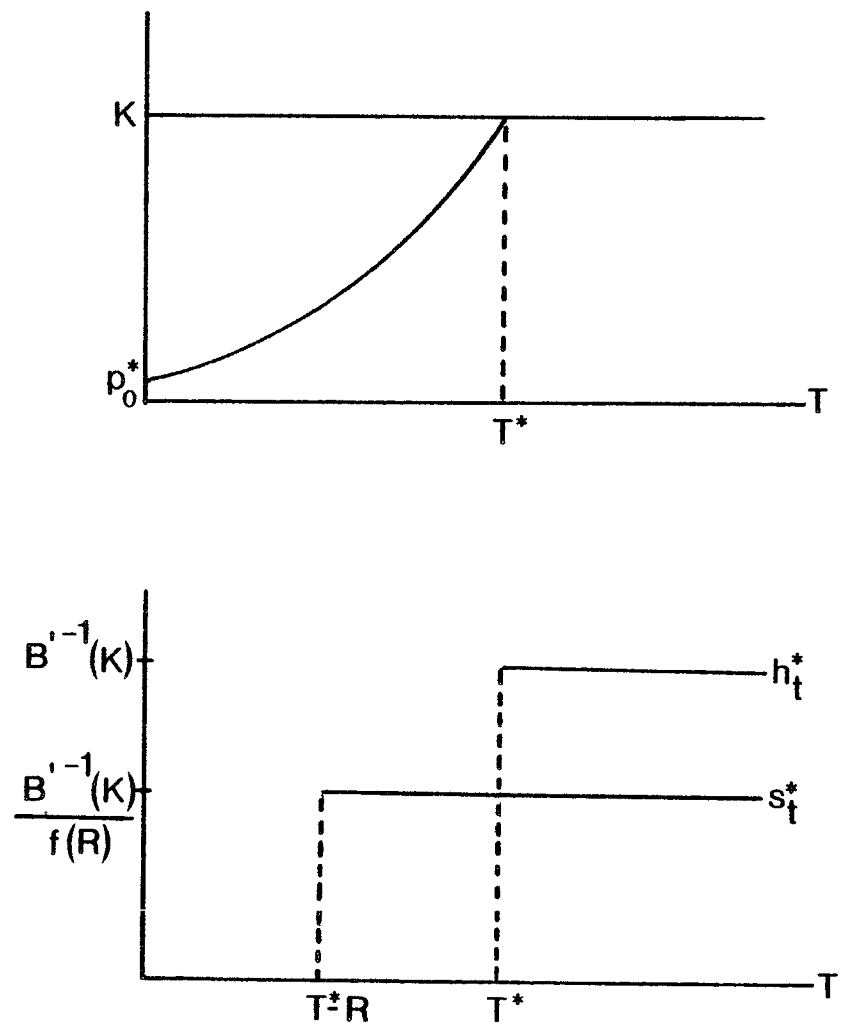

Figure 1. The solution to P, $x^o > x(R)$.

which implies an initial price $\hat{p}_0(x^0, \hbar)$. If the path corresponding to $p_0^*(x^0)$ is followed, the stock would be exhausted before the second-growth is available, and by Theorem 2, such a path cannot be optimal. Thus, a slower rate of depletion is needed, i.e., a higher initial price is needed, and one concludes $\hat{p}_0 > p_0^*$. Since (14) holds, the price of old-growth must rise above K at some point, and then fall discontinuously to K at date R. This is depicted in Figure 2. Note that this is consistent with condition (v) of Theorem 1.[11/]

This "small initial stock" case is more than a curiosity. Obviously, at any point $t \in [0,T]$ the problem can be reconsidered and, as long as the new "initial" date has initial conditions corresponding to the date along the solution to the problem above, the analysis is not altered. Thus, the lag of duration R in availability of plantations becomes a lag of a lesser duration for $t \in [T-R,T]$. This latter situation is one where plantations have been started, but there is an "age gap." This version of the small initial stock case may exist in some areas of the western United States, where old-growth inventories exist, and immature stands exist, but old-growth inventories are insufficient to allow a smooth transition.

In the above analysis, I assumed that the stock of old-growth has no value unless it is harvested. This assumption is dropped in the next section of the paper, where the value of old-growth for wilderness is introduced.

3. OPTIMAL DEPLETION WHEN THE STOCK HAS VALUE

There are several ways that one might introduce a valuable stock into the analysis of the previous section. Here, I continue to take a benefit-cost approach and assume that a demand curve for wilderness has been estimated and that the forester seeks to maximize the present value of the consumer's

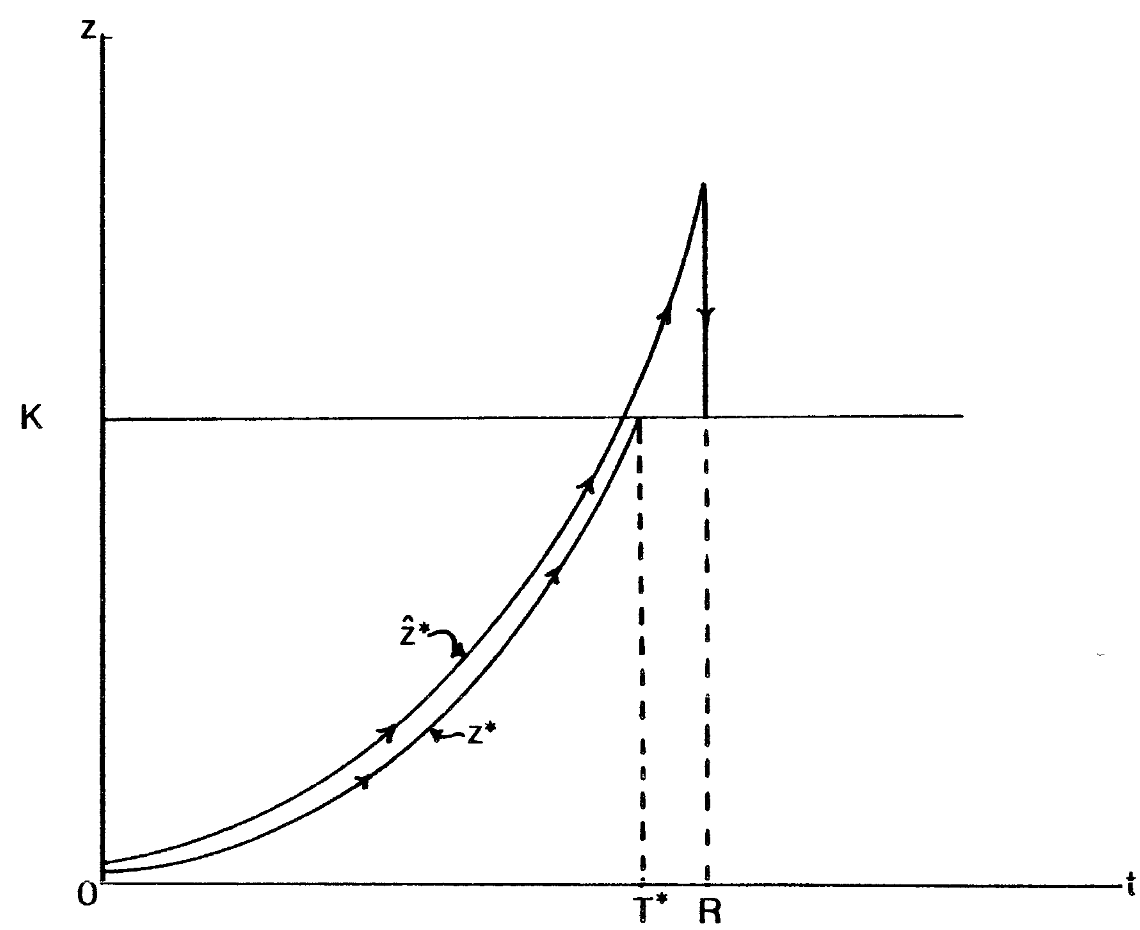

Figure 2. Price path for $T^* < R$, $x^o < x(R)$.

surplus generated by the harvest of trees and the consumption of wilderness services.[12/]

Let the inverse demand curve for wilderness be given by the known, stationary function $d_w(x_t)$, and define

$$w(x_t) \equiv \int_0^{x_t} d_w(\eta)d\eta \, . \tag{18}$$

I assume that wilderness is provided costlessly and that use of wilderness is sufficiently low that congestion and quality deterioration problems do not arise. It is natural to assume that $w(\cdot)$ is strictly increasing and strictly concave.

The forester's problem is

$$P': \quad \max \int_0^{\infty} [B(y_t) + w(x_t) - bs_{t-R} - ks_t] \, e^{-rt} \, dt$$

s.t. (1), (2), (3) non-negativity, and initial conditions.

The conclusions of Theorem 1 apply to the problem P' with the obvious modifications of the definition of the Hamiltonian (5) to account for the modified objective function. In the present context, Theorem 1 implies that the following conditions must hold if the path y_t^* is optimal:

$$B'(y_t^*) - z_t + m_t = 0 \tag{19}$$

$$B'(y_t^*) \, f(R) - b - ke^{rR} + u_{t-R}\Big|_{t+R} = 0 \tag{20}$$

$$\dot{z}_t = rz_t - w'(x_t) - \eta_t \tag{21}$$

Several features of the solution path to P carry over to the solution both to P'. In particular, it still may be concluded that T_1 and T_2 form a partition of R_+ and that only two phases exist, i.e., that $T_1 = [0,T]$ and $T_2 = (T,\infty)$. The conclusions that must be modified concern the rate of increase of the stumpage price of old-growth for $t \varepsilon T_1$ and that the stock of old-growth is exhausted at T.

Regarding the former issue, from (19), (21), and complementary slackness

$$\dot{p}_t/p_t = \dot{z}_t/z_t = r - w'(x_t)/z_t \,. \qquad (22)$$

It is immediate that when the stock of old-growth has value, rents rise slower than the rate of interest while old-growth is being harvested.

This accords well with economic intuition. The reason why old-growth rent must rise at the rate of interest in the previous section is to make the forester indifferent between holding old-growth and selling it and purchasing an alternative asset. If old-growth yields no return other than capital gains when it is held, its price must rise at the percentage rate r. In this section, holding old-growth yields two returns: capital gains as well as a flow of wilderness services. Hence, the capital gain in terms of stumpage price must be less than previously if holding old-growth and holding the alternative asset are to yield the same rate of return.

To discover the size of the wilderness system that is preserved in the steady-state, I exploit the two-phased nature of the solution path. Consider the payoff for $t \varepsilon T_2$ defined by

$$\pi(x^*, K) = B(h^*) - Kh^* + w(x^*) \,, \qquad (23)$$

where

$$h^* = B'^{-1}(K) \,.$$

The present-value of receiving $\pi(\cdot)$ beginning at the switch-date T is just

$$e^{-rT} \int_T^\infty \pi(x^*, k)\, e^{-r(t-T)}\, dt = \frac{\pi(x^*, K)}{r} e^{-rT} . \qquad (24)$$

Then the optimal depletion problem P' may be rewritten as one of optimal depletion over a free horizon with a "terminal value function" given by (24); i.e.,

P' is formulated as

$$P'' : \max_{q_t} \int_0^{\infty} [B(q_t) + w(x_t)]e^{-rt}dt + \frac{\pi(x^*,K)}{r} e^{-rT}.$$

The necessary conditions for this problem are, of course, exactly those for the problem P'. However, there also exist endpoint conditions that must hold at time T. These are (Kamien and Schwartz, 1981):

$$x(T) \geq 0 \quad z(T) \geq \frac{\partial}{\partial x}\left[\frac{\pi(x,K)}{r}\right]$$

$$x(T)\left[z(T) - \frac{\partial}{\partial x}\left(\frac{\pi(x,K)}{r}\right)\right] = 0 \tag{25}$$

$$e^{-rT} H(\cdot)\Big|_T + \frac{\partial}{\partial T}\left[e^{-rT}\frac{\pi(x,K)}{r}\right] = 0 \tag{26}$$

Suppose that $x(T) > 0$, i.e., some wilderness is preserved in the steady-state that holds for $t \in T_2$. Then by (24) and (25),

$$z(T) = \frac{w'(x^*)}{r} \tag{27}$$

Further, from (26)

$$B(q_T^*) + w(x_T) - z_T q_T^* = B(h^*) - kh^* + w(x^*).$$

Since for $T > R$ (I am assuming this here), prices must be continuous at T, this simplifies to

$$z_T = K.$$

Using this in (27), completes the proof of

Theorem 4. If the initial stock of old-growth is large, the optimal wilderness system is given by $x^* = w'^{-1}(rk)$. If $w'(0)/r \leq k$, then $x^* = 0$. Thus, increases in b or k reduce the size of the wilderness system, while an increase in r increases it.

The comparative static aspects of Theorem 4 are demonstrated in the appendix.

The intuition behind the analysis is clear. Suppose that the forester was given a small increment to the stock of old-growth at the switch-date T. If this was harvested and sold, the amount of revenue generated would be $B'(h^*)$ per unit. But, this is just equal to K. Alternatively, if this old-growth is preserved instead of harvested, the addition to the wilderness system is equal to $w'(x^*)$, and this is received forever. Thus, the increase in the objective function is $w'(x^*)/r$.

If x^* is optimal, the forester must be indifferent between these options, i.e., the marginal benefit of wilderness (in present value terms) must equal its marginal opportunity cost. Now, if no wilderness is to be preserved in the steady-state, it must be that this increase in endowment is harvested, i.e., that $w'(0)/r < K$.

The endogenous switch-date T^* and initial price $z^*(0) = p^*(0) = B'(q^*(0))$ remain to be determined. As in the previous section, these are given by the solution to

$$\int_0^T d(pt)\,dt = x^o - x^* \qquad (33)$$

$$p(T) = K, \qquad (34)$$

where the evolution of p is determined by (22). In the appendix, the following analog of Theorem 3 is established.

Theorem 5. $\partial T^*/\partial x^o > 0$, $\partial T^*/\partial K > 0$ $\partial T^*/\partial r$ indeterminate.

$\partial z_0^*/\partial x^o < 0$, $\partial z^*/\partial K > 0$ $\partial z_0^*/\partial r < 0$.

All of the signs of these comparative static derivatives are the same as in the simple case, except for the dependence of the switch-date on the rate of discount, which now is indeterminate. What is of concern here is the sign of $\partial T^*/\partial x^o$. It is this that allows me to use above the "large initial stock" language. Before discussing what happens if initial stocks are "small," it is convenient to undertake an analysis of the problem $\mathcal{P}'$ using a phase diagram.

The solution to the optimal depletion problem is governed by the pair of non-linear differential equations

$$\dot{z}_t = r z_t - w'(x_t) \tag{35a}$$

$$\dot{x}_t = -q_t^* \tag{35b}$$

where

$$q_t^* = B'^{-1}(z_t) .$$

Consider the rest point of this system where $\dot{x}_t = \dot{z}_t = 0$. From (29a), it must be that

$$r z_t - w'(x_t) = 0.$$

By the Implicit Function Theorem,

$$z'(x_t)\Big|_{\dot{z}=0} = w''(x_t)/r < 0. \tag{36}$$

Further,

$$z(0)\Big|_{\dot{z}=0} = w'(0)/r. \tag{37}$$

Regarding x_t, define the set

$$X = \{(x,z) \mid \dot{x}_t = 0\}.$$

Naturally, since q_t must be non-negative, x_t is falling, and so if $x_t = 0$, $\dot{x}_t = 0$, since $x_t \geq 0$ is required. Further, if $z_t > K$, $B'(y_t) > K$, which contradicts part (iii) of Theorem 2. Thus,

$$X = \{(x,z) \mid z \geq K\} \cup \{(x,z) \mid x = 0\}. \tag{38}$$

Combining (36), (37) and (38), the phase diagram appears in Figure 3. This diagram is drawn under the assumption that $w'(0)/r > k$, i.e., that the vertical intercept of the $\dot{z} = 0$ isocline is strictly greater than K. Then, the optimal path converges to the steady-state with $x^* > 0$ preserved as wilderness. It is straightforward to demonstrate that the dynamic system given by a linear approximation of (29) around the steady-state values (x^*, z^*) has eigenvalues of the coefficient matrix which are real and of opposite sign. Thus, the linear approximation system exhibits saddle-point instability, as shown in Figure 3.[13/] This is done formally in the appendix as the proof of

Theorem 6. The steady-state (x^*, z^*) exhibits linear-approximation saddle-point instability.

The derivative $\partial T^*/\partial x^0 > 0$ informs us that if the initial stock is large enough, the constraint $T^* \geqslant R$ is of no consequence, but that it might become problematical if the initial stock is small. Clearly, the above questions concerning the amount of wilderness preserved in the steady-state need to be modified in this case.

In particular, it is clear from (33) that a reduction in the wilderness preserve implies a relaxation of the binding time constraint. To what extent should a substitution of wilderness for initial stock take place?

The endpoint condition when a constraint of the form $T \geqslant R$ exists is (Kamien and Schwartz, 1981)

$$e^{-rR} H(\cdot)\Big|_R + \frac{\partial}{\partial R}\left[e^{-rR} \frac{\pi(x, R)}{r}\right] \leqslant 0,$$

or

$$B(q_R) - z_R q_R \leqslant B(h) - Kh. \tag{39}$$

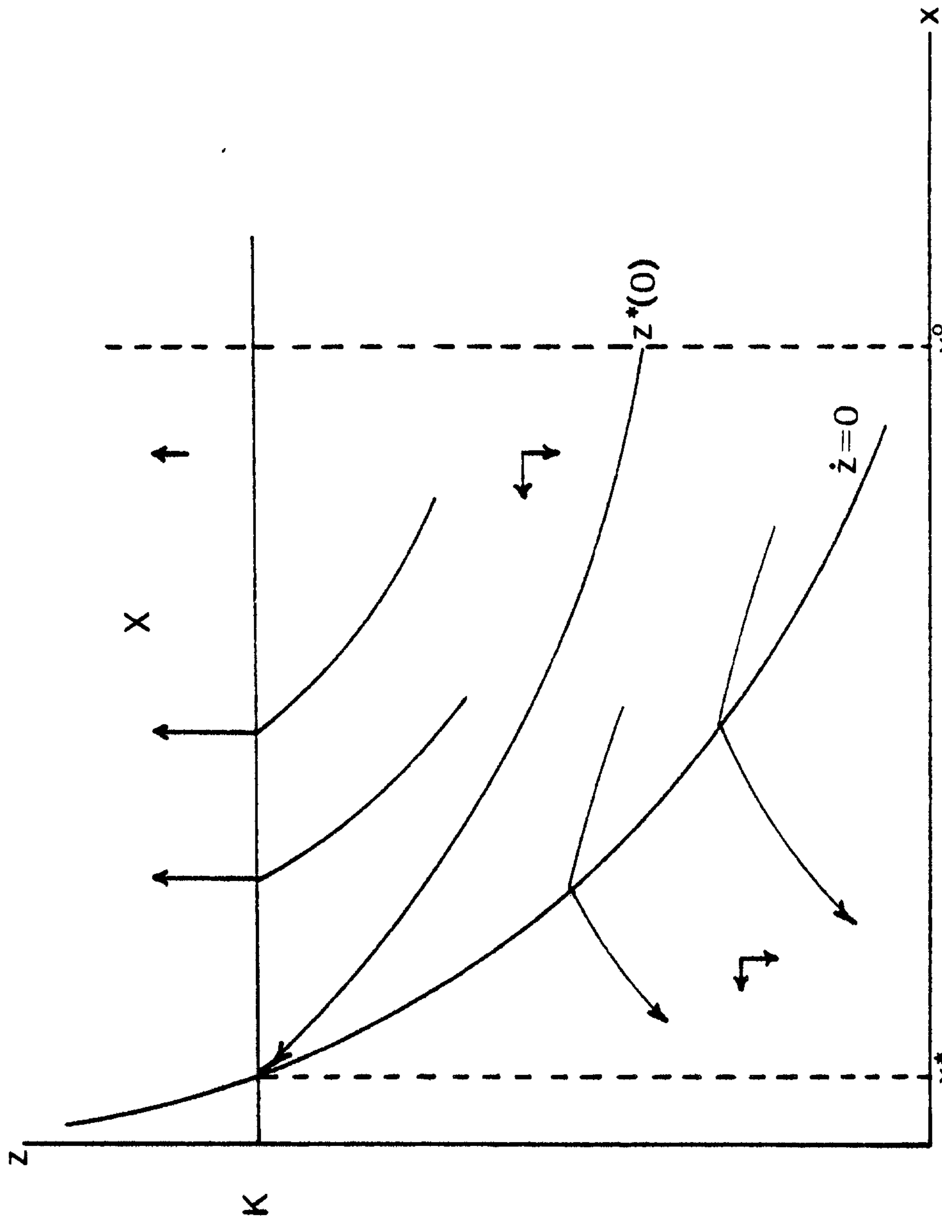

Figure 3. Phase diagram for B".

Since

$$z_t = B'(q_t),$$

the LHS of (39) is the consumer surplus from harvests of old-growth, while the RHS is the surplus from managing plantations. Then, (39) implies that $q_R \leqslant h^*$, whence $z_t \geqslant K$. Since (27) is operative here, it follows that less wilderness is preserved in this instance than otherwise. More formally, the above is the proof of

Theorem 7. Generally, the optimal size of the wilderness system is given by

$$x^* = w'^{-1}(r \max z(t)),$$

with $x^* = 0$ if this has no solution.

This result has the same intuitive interpretation as Theorem 4. The costate variable z(t) represents the shadow value of the stock of old-growth destined for timber harvest. As such, it is the marginal opportunity cost of wilderness preservation. The present value (at time t) of the marginal benefit of preservation is $w'(x)/r$. Naturally, at the optimum, these are equated. More importantly, however, is the realization that marginal wilderness values need to be compared to the <u>largest</u> value that the opportunity cost of preservation will take on as old-growth is depleted and converted to managed stands.

In this case of a small initial stock, all that needs to be determined is the initial price z_0^*. As usual, this is given by the solution to

$$\int_0^R d(z(t))\,dt - x^o - x^*(r\,z(R)) = 0$$

Using this equation and standard comparative statics methodology, it is straightforward to establish

Theorem 8. In the small initial stock case, if a positive solution to $w'(x) = \hbar z(R)$ exists, then $\partial x^*/\partial x^o \in \text{Int}\,[0,1]$, $\partial x^*/\partial \hbar > 0$
(0,1)

4. SILVICULTURAL EFFORT

In the previous two sections, I have assumed that the level of silvicultural effort applied in the growing of managed stands is fixed. Naturally, undertaking various silvicultural activities such as site preparation and planting, use of improved stock, thinning, fertilization, etc., would increase volume on the site for any harvest age. Preservation interests have argued that silvicultural effort used in managed stands should be increased and the resulting increase in harvested volume substituted for harvests of old-growth forests as a source of wood products.[14/]

There are two aspects of this argument that I will analyze in this section of the paper. First, suppose that the level of silvicultural effort actually used is different than the efficient level of effort. In particular, suppose, as the preservationists claim, that it is less than optimal. Then the question becomes "does this lead to preservation of too small a wilderness system in the steady-state?"

The second aspect concerns the relationship between the efficient level of silvicultural effort and the size of the wilderness system. That is, if wilderness suddenly becomes relatively more valuable, should silvicultural effort be increased? Or, turning this around, if you would like to see more wilderness preserved, should you expend more effort on growing managed stands of timber?

To investigate these questions in the context of this model, I consider only silvicultural inputs that are used when trees are regenerated. Thus, I write

$$h_{t+R} = s_t f(R, E_t) , \qquad (33)$$

where E_t is an index of silvicultural effort used per unit of area regenerated at date t. Since R is assumed fixed, I suppress it in the notation that follows.[15/] I assume that $f(E_t)$ is strictly increasing and strictly concave.

Letting k be the fixed unit cost of silvicultural effort, the overall problem is to solve

$$P''' : \max_{q,s,E} \int_0^{\infty} [B(y_t) - kE_t s_t - bs_{t-R} + W(x_t)] e^{-rt} dt$$

$$\text{s.t.} \quad \dot{x}_t = -q_t$$

$$y_t = q_t + s_{t-R} f(E_{t-R})$$

$$q_t \geq 0, \; s_t \geq 0, \; x_t \geq 0, \; x(0) \text{ given; } s(t) = 0 \; t\varepsilon[-R,0].$$

The analysis of the previous section can be used to study the solution to P'''. Throughout this section, I study only the "large initial stock" case. The solution path is a two-phased path, with an old-growth era and a plantation era. It also is known that the forester seeks to maximize profits in the steady-state taking x^*, the optimal size of the wilderness system, as given. The cost of harvest in the steady-state is $s[b + ke^{rR} E]$. Thus, the steady-state problem is

$$\max_{s,E} \; B(sf(E)) - s[b + ke^{rR} E] .$$

The first order necessary conditions for this are

$$B'(h^*)f(E^*) - ke^{\pi R}E^* - b = 0 \tag{40}$$

$$[B'(h^*)f'(E^*) - ke^{\pi R}]s^* = 0 \ . \tag{41}$$

Since $h^* > 0$, $s^* > 0$ must hold. Thus, solving these for $B'(h^*)$ and equating, it follows that

$$\frac{ke^{\pi R}}{f'(E^*)} = \frac{b + ke^{\pi R}E^*}{f(E^*)} \tag{42}$$

The LHS of (42) is the marginal cost of silvicultural effort and the RHS is the average cost of effort. Equation (42) can be rearranged to read

$$(b + ke^{\pi R}E^*)\, f'(E^*) - ke^{\pi R}f(E^*) = 0 \ . \tag{43}$$

Since the derivative of (43) with respect to E is non-zero, the Implicit Function Theorem can be applied to assert the existence of a differentiable function $E^*(b, k, \pi)$ such that (43) holds identically. Differentiation of this identity provides the proof of

Theorem 9. $E^*_b > 0$, $E^*_k < 0$ $E^*_\pi < 0$.

The key result of this section are established by defining

$$K(E^*) = [b + ke^{\pi R}E^*]\,/f(E^*),$$

and recognizing that Theorem 4 holds replacing K by $K(E^*)$. Thus,

$$x^* = w'^{-1}\,(\pi K(E^*)).$$

Further, it must be that E^* minimizes the average cost of harvest in the steady-state. That this is the case is established by differentiating $K(E^*)$ and noting that the numerator is just the negative of (43). Since x^* is decreasing in K, I have proven

Theorem 10. The size of the wilderness system is largest for $E = E^*(b, k, \pi)$.

Theorem 9 readily is interpreted. Silvicultural effort is chosen to minimize the cost of obtaining second-growth in the steady-state. Naturally, the price of wood products that holds in the steady-state depends on these costs, and this price constitutes the opportunity cost of holding old-growth as wilderness as was discussed following Theorem 4. Thus, using any level of effort other than the efficient one raises costs and, therefore, the cost of wilderness as well.

This answers the questions posed above. Clearly, if the amount of effort being expended is too small (less than E*), an increase in effort will result in a larger stock of wilderness being preserved in the steady-state, and the preservationists logic is correct. However, if E is efficient (equal to E*), preservationists should not ask for larger E, as a decrease in wilderness is implied.

The reason for this conclusion lies in the nature of the model: I have assumed that land is available for establishing plantations which is not obtained from the harvest of old-growth. Thus, should old-growth become relatively more valuable, the optimal level of effort is applied to more land outside of what originally was covered by old-growth and more old-growth is preserved in the steady-state. In a model in which the entire area of land suitable for forest management at all dates initially is covered by old-growth, this conclusion might not hold.

5. TECHNOLOGICAL CHANGE

The manner in which trees in plantations are grown and harvested clearly is changing, and there is considerable evidence that this change is accelerating. New equipment is being developed, fertilization is used more widely and its effects are better understood, and planting of containerized, genetically improved seedlings holds great promise.[16] If agricultural production may be used roughly as a guide, the potential exists for relatively rapid technological change in the forestry sector.

What is important here is that the rate of harvest of old-growth and the size of the wilderness system to be preserved in the steady-state depend on the unit cost of growing and harvesting second-growth stands, K. This depends in turn on the technology underlying the production function f(R). Thus, current preservation decisions must be based on estimates of future technology.

The impact of technological change on the size of the wilderness system readily is established in a world of certainty. Let the production function be rewritten as f(R, t). Since R is fixed, I suppress it and write f(t). Technological change is captured by assuming that $f'(t) \geqslant 0$. Recall the definition of K. Now, write

$$K(t) = \frac{b + ke^{rR}}{f(t)} \quad ,$$

with

$$K'(t) \leqslant 0 .$$

The impact on preservation decisions of having K(t) instead of K is discerned by the transversality condition for an appropriately modified terminal value function as done in the analysis of the solution to P''. The present value of the payoff in the steady-state becomes

$$J(T, x^*) \equiv \int_T^\infty [B(h^*(t)) - K(t)h^*(t)]\, e^{-rt}\, dt + \frac{w(x^*)}{r}\, e^{-rT} . \qquad (44)$$

Using $J(T, k^*)$ in the conditions (25) and (26) yields, for $\hat{x}^* > 0$,

$$z(T) = \frac{w'(\hat{x}^*)}{r} \qquad (45)$$

and

$$B(q_T^*) + w(x_T) - z_T q_T^* + \frac{\partial J}{\partial T} = 0 ,$$

or

$$B(q_T^*) + w(x_T) - z_T q_T^* = B(h^*(T)) - k(T)\, h^*(T) + w(x^*) .$$

Using the continuity of $B'(y^*)$, it follows that

$$z_T = K(T),$$

and (44) becomes

$$\hat{x}^* = w'^{-1}(\lambda K(T)). \tag{46}$$

It is an immediate consequence of Theorem 4 that technological change, which implies that $k(T) \leqslant K$, leads to more wilderness being preserved in the steady-state, i.e., $\hat{x}^* \geqslant x^*$.

An important policy implication of this result is that decision-makers are required to look into the future when making preservation decisions. If the decision-maker allocates lands to wilderness thinking that technology is static when it is not, too little wilderness will be preserved.

It also is interesting to note that, by Theorem 5, $\partial T^*/\partial K > 0$. Thus, for $\Gamma(t) \leqslant K$, the period when old-growth forests are used is shortened. So, even though less old-growth should be devoted to timber harvest when technology is changing, that old-growth which is harvested should be depleted at a faster pace.

6. FAUSTMANN THEORY AND THE STEADY-STATE.

As mentioned in the discussion, the well-known Faustmann model of forest management can be considered as holding in the steady-state that arises after the "old-growth era" has ended. In this section, the tie between this paper and the Faustmann model is elaborated.

Consider the management of one unit of forestland. Assume that, at a cost of $\bar{k}$, a new stand of trees can be established and grown for R units of time and then harvested. A rental payment of C must be made each period for the right to use the land for forests. It costs b to harvest a unit of area. The price received for harvested material is p. The forester's problem is to choose R so as to solve

$$\max_R \quad [pf(R) - b]\, e^{-rR} - \bar{k} - \int_0^R Ce^{-rt}\, dt.$$

The necessary conditions for this problem may be written

$$pf'(R^*) = r[pf(R^*) - b] + C.$$

This familiar condition says that the forest should be harvested when its rate of value growth (the marginal benefit of delaying harvest) is equal to the interest on the value of standing inventory plus the rental fee (the marginal cost of delaying harvest).

Naturally, the rental payment that must be made is such that the maximal present value of forest management is zero. That is, C* is defined by

$$[pf(R^*) - b]\, e^{-rR^*} - \bar{k} - \frac{C^*}{r}\, [1 - e^{-rR^*}] = 0, \qquad (47)$$

so that

$$C^* = r\left[\frac{[pf(R^*) - b]\, e^{-rR^*} - \bar{k}}{1 - e^{-rR^*}}\right].$$

This just says that the rental payment is equal to the interest charge on the value of the land used in forestry. The term in brackets is the present value of receiving ever R* units of time the income $[pf(R^*) - b]\, e^{-rR^*} - \bar{k}$, and hence is the value of forest land.

The relationship between this paper and the above Faustmann Theory is brought out by rearranging (47) to read

$$p = \frac{b + \bar{k} + \left[\frac{C^*}{r}\,(1 - e^{-rR^*})\right] e^{rR^*}}{f(R^*)}.$$

Defining

$$k = \bar{k} + \frac{C^*}{r} (1 - e^{-rR^*}) , \qquad (48)$$

This becomes

$$p = \frac{b + ke^{rR}}{f(R^*)} .$$

Comparing this to the analysis of section 2 above, since

$$p = B'(s^* f(r^*)) = K,$$

the analyses are compatible. It is as if the rotation period (and effort in the previous section) are chosen optimally on each site taking the price of output as given and then the number of sites is chosen such that the price is equal to the average cost of forest management. This, of course, is a well-known approach in competitive economics. Thus, the fixed rotation R* used in the old-growth depletion problem may be considered the Faustmann age in the steady-state for a price equal to average cost.

6. DISCUSSION

As stated in the introduction, the model studied here is relatively simple when compared to those appearing in the literature of forest economics. While the model appears useful as a basis for understanding the timber supply-wilderness preservation conflict, further research could generalize the model in a number of ways.

Clearly, the assumptions here that old-growth is homogeneous and that land for establishment of plantations is available at a constant cost are crucial to obtaining the "crisp" transition from the old-growth era to the age of plantations. If costs of extraction of old-growth rise as it is depleted and high-quality land (with respect to biologic and location) is available for plantations, then some overlap of harvests of old-growth and plantations would occur. Similarly, if the initial conditions include forests in widely varying stages of maturity, the transition to the steady-state would be more complex.

In this latter instance, as well as in the simple "age gap" model studied above as the small initial stock case, the fixed rotation age assumption should be modified. While this does not appear restrictive in the large initial stock case (since the analysis of section 6 shows that R may be set equal to the Faustmann age at the steady-state price) if initial stocks are small the rotation age could be altered to smooth the price discontinuity at the switch-date. However, incorporation of this possibility greatly would complicate the analysis.

If the total land area available for forestry initially is inhabited by old-growth, the transition problem becomes quite difficult. It is important to point out, however, that if this is the case, the conclusions regarding silvicultural effort in section 4 no longer hold since a more direct tie between output in the steady-state and acreage devoted to wilderness would exist. Which of these assumptions is more relevant is unclear. In the context of the United States, where much concern over old-growth and wilderness exists for western public lands, but significant acreage for plantations is available in the South, the assumption here may be realistic. If, on the other hand, the conflict is purely regional or regards disposition of a fixed amount of land with no alternative sites for forestry, the independence assumption imposed here is untenable.

As well, it should be noted that if the total area of land to be allocated between wilderness and plantations is fixed, the analysis of technological change would need to be modified. For, with technological change, the opportunity cost of wilderness preservation would be falling over time and the optimal size of the wilderness system would be rising in the age of plantations. Since old-growth harvests by assumption are irreversible, this would not be possible; a situation very much like that described in the natural environments literature might obtain.[17/] The wilderness decision would have to be more forward-looking; I conjecture that a larger stock of old-growth would be preserved than that based on the myopic opportunity cost rule derived above.

The model also is restrictive in the assumption that plantations provide no utility except when harvested. The possibility of multiple uses of these forests has received attention in the forestry literature.[18/] Incorporation of this concept into the model is not difficult analytically. However, consideration of such a possibility raises questions about differences between wilderness values and the value of stock in plantations and, therefore, about the reversibility of old-growth harvests, which, while policy relevant, are outside the scope of this effort. It suffices to point out here that limited reversibility may be possible and, indeed, the degree of reversibility may be a choice variable. This has little impact on the model outlined above, but would be important for the natural environments literature in general.[19/]

Finally, the natural environments literature has stressed the importance of uncertainty in wilderness decision making. The concepts of option value (see Bishop, 1982; Smith, 1983) and quasi-option value (Arrow and Fisher, 1974; Graham-Tomasi, 1983b) were developed in this context and recent work by Smith (1981) has extended the original Fisher, et. al. (1972) natural environments literature in this direction. Important sources of uncertainty in the model above would include uncertainty concerning future demand for wood products and for wilderness and the future technology for management of plantations.

APPENDIX

Proof of Theorem 1: The proof is a straightforward extension of that in Kamien and Schwartz (1981) concerning delayed response to the case in which the control enters the objective functional with a lag.

To prove the theorem, I change notation somewhat and let the problem be:

$$\max \int_{t_o}^{t_1} f(t, x(t), x(t-R), u(t-R))\, dt$$

s.t.

$$\dot{x}(t) = g(t, x(t), x(t-R), u(t), u(t-R))$$

$$x(t) = x^o \text{ for } t \in [t_o - R, t_o]$$

$$u(t) = u^o \text{ for } t \in [t_o - R, t_o]$$

$$x(t_1) \text{ free}$$

where $x(t)$ is the state variable, and $u(t)$ is the control. For now, I ignore the other constraints.

let $\lambda(t)$ be a C^1 function. By definition,

$$\int_{t_o}^{t_1} f(\cdot)\,dt = \int_{t_o}^{t_1} f(\cdot) + \lambda(t)g(\cdot) - \lambda(t)\dot{x}(t)dt. \qquad \text{(A1)}$$

Integrating by parts the last term in the integral in (A1) provides

$$\int_{t_o}^{t_1} f(\cdot)dt = \int_{t_o}^{t_1} [f(\cdot) + \lambda_t g(\cdot) + x_t \dot{\lambda}_t]\, dt$$

$$- \lambda(t_1)x(t_1) + \lambda(t_o)x(t_o).$$

Let J be the value of the objective functional evaluated when control u_t is used and let J* be its value when u_t^* is used. Define

$$\Delta J = J - J^* = \int_{t_o}^{t_1} [f(\cdot) + \lambda_t g(\cdot) + x_t \dot{\lambda}_t - f^*(\cdot)$$

$$- \lambda_t g^*(\cdot) - x_t^* \dot{\lambda}_t] \, dt ,$$

where $h(y^*) \equiv h^*$ and x and x^* are the states generated by the use of controls u and u^*, respectively. Expanding ΔJ around (t, x^*, u^*) using Taylor's Theorem provides the first variation of J.

$$\delta J = \int_{t_o}^{t_1} [(f_{x_t} + \lambda_t g_{x_t} + \dot{\lambda}_t)\delta x_t + (f_{x_{t-R}} + \lambda_t g_{x_{t-R}})$$

$$\delta x_{t-R} + (f_{u_t} + \lambda_t g_{u_t})\delta u_t + (f_{u_{t-R}} + \lambda_t g_{u_{t-R}})$$

$$\delta u_{t-R}] \, dt + \lambda(t_1)\delta x(t_1), \tag{A2}$$

where $\delta y \equiv (y-y^*)$ and $h_{y_t} \equiv \partial h / \partial y_t$.

This expression may be modified by letting $s = t-\dot{R}$ and changing the lower limit of integration (see Kamien and Schwartz (1981) p. 234 for details). This yields

$$\delta J = \int_{t_o}^{t_1 - R} [f_{x_t} + f_{x_{t-R}}\Big|_{t+R} + \lambda_t g_{x_t} = \lambda_t g_{x_{t-e}}\Big|_{t+R} + \dot{\lambda}_t]\delta x_t$$

$$+ [f_{u_t} + f_{u_{t-R}}\Big|_{t+R} + \lambda_t g_{u_t} + \lambda_t g_{u_{t-R}}\Big|_{t+R}] \, \delta u_t \, dt$$

$$+ \int_{t_o}^{t_1} [f_{x_t} + \lambda_t g_{x_t} + \dot{\lambda}_t] \, \delta x_t + [f_{u_t} + \delta_t g_{u_t}]\delta u_t \, dt \tag{A3}$$

If x_t^* and u_t^* are optimal as supposed, then it must be that $\delta J \leqslant 0$. Now, I incorporate the constraints of the problem. Generally, let there be I constraints of the form $\Psi_i(u_t, u_{t-R}) \geqslant 0$. Then $\delta J \leqslant 0$ must hold for all variations of the control such that the constraints are satisfied, i.e., it must be that

$$d\Psi_i = [\partial\Psi_i/\partial u_t + \partial\Psi_i/\Psi u_{t-R}\Big|_{t+R}]\ \delta u_t \ \geqslant\ 0. \qquad \text{(A4)}$$

When $\Psi_i > 0$ holds, any modification of the control u is allowed.

To assure that $\delta J \leqslant 0$, note that λ_t is an arbitrary function. Hence, it may be chosen such that the coefficients of δx_t are zero in (A3). Thus, all that remains is to deal with modifications in u_t that are feasible and require that $\delta J \leqslant 0$ for all such modifications. This can be done by applying Farka's Lemma (Bazaraa and Shelty, 1976) to (A3) for δu_t satisfying (A4). Then Farka's Lemma says that this is equivalent to the statement that there exists a function $w(t) \geqslant 0$ such that

$$f_{u_t} + f_{u_{t-R}}\Big|_{t+R} + \lambda_t(g_{u_t} + g_{u_{t-R}}\Big|_{t+R})$$

$$+ w_t(\Psi_{u_t} + \Psi_{u_{t-R}}\Big|_{t+R}) = 0.$$

$$w_t \geqslant 0,\ w_t\Psi_i(\cdot) = 0$$

Defining the Hamiltonian in the obvious way and the Lagranian similarly, the conditions of Theorem 1 follow. A comment is in order: I have assumed here that a constraint qualification is satisfied by Ψ_i. Further, I have not dealt with the state variable constraint. See Kamien and Schwartz (1981, p. 215-225) for the rest of Theorem 1 regarding these constraints.

q.e.d.

Proof of Lemma 1. Suppose not. Then (8) is written as $z_0 B'(y^*) - z_1(t) + m_t = 0$. By Theorem 2, $q_t^* \neq 0$ for $t \in T$, and $T_1 \neq \emptyset$. Hence, by complementary slackness, $m_t = 0$ for $t \in T_1$. Then $z_1 = z_0 B'(y)$, and $z_0 = 0$ implies $z_1 = 0$ and (z_0, z_1) is the zero vector, which violates Theorem 1.

q.e.d.

Proof of Theorem 2.

(i) This is immediate in light of (4).

(ii) Suppose not. By complementary slackness $m_t = \eta_t = u_{t-R}\Big|_{t+R} = 0$. Then, from (8) and (10), $\dot{z} = B''(y)\dot{y} \neq 0$. But (9) implies $B'(y) = K$, whence $B''(y)\dot{y} = 0$, a contradiction.

(iii) By hypothesis, there exists a feasible path (q^*, s^*) which satisfies the necessary conditions for P, and is such that $B'(y^*) \leqslant K$. Let y be an alternative path such that $\hat{y}_t = y_t^*$ except for $t \in \hat{T} \subset [0,\infty)$, with $\hat{T} \neq \emptyset$ and $B'(\hat{y}_t) > K$ for $t \in \hat{T}$. Suppose first that $\hat{T} \subset T_2$. But $B'(h^*) = B'(\hat{h}) = K$ is necessary for optimality, so that either $\dot{y}$ is not optimal or $\hat{T} \subset T_1$. Suppose $\hat{T} \subset T_1$. Since $B''(y) < 0$, $q < q^*$. Since $B'(y) > 0$, $B(\hat{q}) < B(q^*)$. But then

$$\int_0^\infty B(q^*)\, e^{-rt}\, dt - \int_0^\infty B(\hat{q})\, e^{-rt}\, dt = \int_{\hat{T}} (B(q^*) - B(q))\, e^{-rt} dt < 0.$$

This contradicts the optimality of y^*.

(iv) Suppose not. Let $[t_0, t_1] \subset T_1$; $[t_1, t_2] \subset T_2$; and $[t_2, t_3] \subset T_1$ for $t_0 < t_1 < t_2 < t_3$. Obviously, $x_t > 0$ for $t \in [t_1, t_3]$. Hence,

$$z(t_3) = z_0 e^{rt_3} > z_0 e^{rt_2} = z(t_2).$$

But $B'(y_t) \leqslant K$ all t and $B'(y_t) = K$ for $t \in T_2$. Further, $z_t = B'(y_t)$ for $t \in T_1$. This is a contradiction.

(v) Let T be such that $t \in T_1$ for $t \leqslant T$ and $t \in T_2$ for $t > T$. Let y* be optimal and suppose that

$$\int_0^T q_t^* \, dt = \hat{x} < x^0$$

Then there exists $\hat{y}$ such that $\hat{q} > q^*$, and $\int_0^T \hat{q} \, dt \leqslant x^0$. Since $B'(y) > 0$, this contradicts the optimality of q^*.

(vi) Suppose $T_2 = \emptyset$. Then by (i), $T_1 = [0,\infty]$. Suppose $z(0) \neq 0$. Then, since $\dot{z}(t) = \hbar z(t)$ for $t \in T_1$, there exists $\bar{t}$ such that $z_t = z(0)e^{\hbar\bar{t}} > K$. By (iii) above, this cannot be optimal. Suppose $z(0) = 0$. Then, from (8), $B'(q) = 0$ must hold for all t. Let $\hat{q}$ be this bliss level of wood production. Clearly, $\hat{q} > 0$, so that

$$\int_0^\infty q_t dt = \infty > x^0,$$

and $\hat{q}$ is not feasible.

q.e.d.

Proof of Theorem 3. Define

$$\Psi(T, p_0; x^o, K, \hbar) \equiv \int_o^T d(p_0 e^{\hbar t}) \, dt - x^o = 0$$

$$\Theta(T, p_0; x^o, K, \hbar) \equiv p_0 d^{\hbar t} - K = 0$$

and let J be the Jacobian matrix of first partial derivatives of Ψ and Θ. J has the following elements:

$$\frac{\partial \Psi}{\partial T} = d(p_o e^{\hbar t}) > 0$$

$$\frac{\partial \Psi}{\partial p_0} = \int_0^T d'(p_0 e^{\hbar t}) e^{\hbar t} \, dt < 0$$

$$\frac{\partial \Theta}{\partial T} = \hbar p_0 e^{\hbar T} > 0$$

$$\frac{\partial \Theta}{\partial p_0} = e^{\hbar T} > 0.$$

Clearly, J has a non-zero determinant: in fact, $|J| > 0$. Thus, there exist C^1 functions $T^*(x^o, K, r)$ and $p_0^*(x^o, K, r)$ such that $\Psi(T^*, p_0^*; x^o, K, r)$ and $\Theta(T^*, p_0^*; x^o, K, r)$ are identically zero. Further,

$$[J]\begin{bmatrix} \nabla T^* \\ \nabla p_0^* \end{bmatrix} = - \begin{bmatrix} \partial\Psi/\partial x^o & \partial\Psi/\partial K & \partial\Psi/\partial r \\ \partial\Theta/\partial x^o & \partial\Theta/\partial K & \partial\Theta/\partial r \end{bmatrix} . \tag{A5}$$

The matrix on the RHS of this equation has the following elements:

$$\frac{\partial\Psi}{\partial x^o} = -1$$

$$\frac{\partial\Psi}{\partial K} = 0$$

$$\frac{\partial\Psi}{\partial r} = \int_0^{T^*} d'(p_t)t\, {}_0^* e^{rt} < 0$$

$$\frac{\partial\Theta}{\partial x^o} = 0$$

$$\frac{\partial\Theta}{\partial K} = -1$$

$$\frac{\partial\Theta}{\partial r} = T^* p_0^* e^{rT^*} > 0 .$$

Using Cramer's Rule, all of the results follow immediately from the signs except $\partial T^*/\partial r$. Writing this out,

$$\begin{aligned} \text{sign } \frac{\partial T^*}{\partial r} &= \text{sign } \frac{\partial\Psi}{\partial p}\frac{\partial\Theta}{\partial r} - \frac{\partial\Psi}{\partial r}\frac{\partial\Theta}{\partial p} \\ &= \text{sign } T^* p_0^* e^{rT^*} \int_0^{T^*} d'(p_t) e^{rt} dt - e^{rT^*} \int_0^{T^*} d'(p_t) t p_0^* e^{rt} dt . \end{aligned}$$

Since T^* and p_0^* are fixed, and $T^* \geq t$, $\frac{\partial T^*}{\partial r} < 0$.

Proof of Theorem 4. Applying the Implicit Function Theorem to equation (28), there exists a function $x^*(r, b, k)$ such that

$$r w'(x^*(r, b, k)) - \frac{b + ke^{rR}}{f(r)} \equiv 0.$$

Differentiation of this with respect to b provides

$$r w''(x^*(\cdot))\, x_b^* - 1/f(R) = 0 \ ,$$

so that

$$x_b^* = \frac{1}{f(R)\, r\, w''(\cdot)} < 0.$$

Similarly, for k

$$x_k^* = \frac{e^{rR}}{f(R)\, r\, w''(\cdot)} < 0.$$

Regarding r, differentiation yields

$$x_r^* = w'(x^*(\cdot)) - rke^{rR}/f(R).$$

Since (28) gives

$$w'(x^*) = rK = \frac{rb}{f(R)} + \frac{rke^{rR}}{f(R)}$$

it follows that x_r^* is positive.

q.e.d.

Proof of Theorem 5. The proof is very similar to that for Theorem 3, except that now $\Psi(\cdot)$ and $\Theta(\cdot)$ are defined by

$$\Psi(T, p_0; x^o, K, r) = \int_0^T d(p_t)dt - x^o + x^*(rK) = 0$$

$$\Theta(T, p_0; x^o, K, r) = p(t) - K = 0 ,$$

and obtaining an expression for p(t) is not as straigtforward. Regarding p(t), recall (22):

$$\dot{p}_t = rp_t - W'(x_t).$$

Taking $r p_t$ to the LHS and multiplying by e^{-rt} gives

$$e^{-rt}(\dot{p}_t - r p_t) = - W'(x_t)\, e^{-rt} .$$

The LHS of this is just $\frac{d}{dt}(p_t e^{-rt})$, whence

$$\int_o^T \frac{d}{dt}(p_t e^{-rt})\, dt = -\int_o^t W'(x_s) e^{-rs}\, ds .$$

Using the Fundamental Theorem of Calculus and rearranging provides

$$p_t = p_0 e^{rt} - e^{rt} \int_0^t W'(x_s) e^{-rs}\, ds.$$

When this expression is used in the definitions of $\Psi(\cdot)$ and $\Theta(\cdot)$ above, these become

$$\Psi(T, p_0; x^o, K, r) = \int_0^T d(p_0 e^{rt} - e^{rt} \int_0^t W'(x_s) e^{-rs} ds)\, dt - x^o + x^*(r,K) = 0$$

$$\Theta(T, p_0; x^o, K, r) = p_0 e^{rt} - e^{rT} \int_0^T W'(x_s) e^{-rs}\, ds - K = 0.$$

The Jacobian of the system now has elements

$$\frac{\partial \Psi}{\partial T} = d(p(T)) > 0$$

$$\frac{\partial \Psi}{\partial p_0} = \int_0^T d'(p_t) e^{rt}\, dt < 0$$

$$\frac{\partial \Theta}{\partial T} = r p_0 e^{rT} - r e^{rT} \int_0^T W'(xs) e^{-rs}\, ds - e^{rT} W'(x(T)) e^{-rT}$$

$$= r p(T) - W'(x(T)) = 0$$

$$\frac{\partial \Theta}{\partial p_0} = e^{rT} > 0.$$

As before, $\det(J) > 0$, and the Implicit Function Theorem can be applied to assert the existence of C^1 functions $T^*(x^o, K, r)$ and $p_0^*(x^o, K, r)$ such that $\Psi(T^*, p_0^*; x^o, K, r)$ and $\Theta(T^*, p_0^*; x^o, K, r)$ are identically zero.

Naturally, equation (A5) still holds. The matrix on the RHS of (A5) has elements

$$\frac{\partial \Psi}{\partial x^o} = \int_0^{T^*} d'(p_t) e^{rt} W''(x(0))dt - 1 < 0$$

$$\frac{\partial \Psi}{\partial K} = r x^{*\prime}(rK) < 0 \quad \text{(by Theorem 4)}$$

$$\frac{\partial \Psi}{\partial r} = \int_0^{T^*} d'(p_t)\,[t\, p_t + e^{rt} \int_0^t sW'(x_s) e^{-rs}\, ds\,]\, dt + Kx^{*\prime}(rK) < 0$$

$$\frac{\partial \Theta}{\partial x^o} = -e^{rT^*} W''(x(0)) > 0$$

$$\frac{\partial \Theta}{\partial K} = -1$$

$$\frac{\partial \Theta}{\partial r} = T^* p(T^*) + e^{rT^*} \int_0^{T^*} sW'(x_s)^{-rs}\, ds > 0$$

The signs of all the comparative static derivatives except $\partial T^*/\partial x^o$ and $\partial T^*/\partial r$ follow immediately from Cramer's Rule and the signs of the elements of the system. Regarding $\partial T^*/\partial r$, since $\det(J) > 0$,

$$\text{sign}\,\frac{\partial T^*}{\partial x^o} = \text{sign}\ \det\begin{bmatrix} -\partial\Psi/\partial x^o & \partial\Psi/\partial p_0 \\ -\partial\Theta/\partial x^o & \partial\Theta/\partial p_0 \end{bmatrix}$$

$$= \text{sign}\ e^{rT^*}\left(\int_0^{T^*} d'(p_t)e^{rt}\, W''(x^o)\,dt + 1\right) - e^{rT^*} W''(x^o)\int_0^{T^*} d'(p_t)e^{rt}\, dt$$

$$= \text{sign}\ e^{rT^*}.$$

Turning to $\partial T^*/\partial r$, a similar procedure results in

$$\text{sign}\ \partial T^*/\partial r = \text{sign} \int_0^{T^*} [e^{rt} T^* p(T^*) - t\, p(t)\, ^{rT^*}]\, d'(p_t)\, dt$$

$$+ e^{rT^*} \int_0^{T^*} d'(p_t)e^{rt}\, dt \int_0^{T^*} sW'(x_s)e^{-rs}\, ds$$

$$- e^{rT^*} \int_0^{T^*} d'(p_t)e^{rt} \int_0^t sW'(x_s)e^{-rs} ds\,dt - e^{rT^*} Kx^{*\prime}(rK).$$

I have not discovered any means of establishing the sign of this expression.

Proof of Theorem 6. Let $g(y) \equiv B'^{-1}(y)$. Then (29) can be written as

$$\dot{x} = -g(z)$$
$$\dot{z} = -u'(x) + \hbar x.$$

The stability of this non-linear system can be studied in a neighborhood of the steady-state (x*, z*) by studying the stability of a linear approximation of the system, i.e.,

$$\dot{x} \approx -g'(z)\ (z - z^*)$$
$$\dot{z} \approx -u''(x)\ (x - x^*) + \hbar(z - z^*).$$

In matrix form, this becomes

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} \approx \begin{bmatrix} 0 & -g' \\ -u'' & \hbar \end{bmatrix} \begin{bmatrix} (x - x^*) \\ (z - z^*) \end{bmatrix} .$$

Letting C be the above coefficient matrix, the eigenvalues are obtained by solving

$$\det\ (C - I\lambda) = 0\ .$$

Then

$$\lambda_i = \frac{\hbar}{2} \pm (\hbar^2 + \hbar g'u'')^{\frac{1}{2}}/2.$$

Since $g'(\cdot)$ and $u''(\cdot)$ both are negative, the eigenvalues are real. Further, since

$$\lambda_1\lambda_2 = \det\ (C) = -g'u'' < 0,$$

they are distinct. Thus, the steady-state (x*, z*) exhibits saddle point instability.

q.e.d.

Proof of Theorem 8. As in the proof of Theorem 5, the solution of (22) is of the form

$$p(t) = p_0 e^{rt} - e^{rt} \int_0^t W'(x_s) e^{-rs}\, ds ,$$

and the condition establishing p_0^* is

$$\Psi(p_0;\ x^o,\ r,\ R) = \int_0^R d(p_t)\, dt - x^o + W'^{-1}(rp(R)) = 0.$$

As usual, since

$$\Psi p_0 = \int_0^R d'(p_t)\, e^{rt}\, dt + rW'^{-1}(rp(R)) e^{rR}, < 0$$

There exists a C^1 function $p_0^*(x^o, r, R)$ such that

$$\Psi(p_0^*(x^o, r, R);\ x^o, r, R) \equiv 0.$$

Then,

$$\frac{\partial p_0}{\partial x^o} = \frac{-\Psi x^o}{\Psi p_0} \tag{A6}$$

and

$$\frac{\partial p_0^*}{\partial r} = \frac{-\Psi r}{\Psi p_0} .$$

Taking the indicated derivatives,

$$\Psi x^o = -\left[\int_0^R d'(p_t) W''(x^o) e^{rt}\, dt + rW''^{-1}(rp(R)) W''(x^o) e^{rR} + 1 \right] < 0$$

$$\Psi r = \int_0^R d'(p_t) \left[t\, p_t + e^{rt} \int_0^t sW'(x_s) e^{-rs}\, ds \right] dt$$

$$+ W''^{-1}(rp(R)) \left[p(r) + rRp(r) + e^{rR} \int_0^R sW'(x_s) e^{-rs}\, ds \right] < 0$$

Thus,

$$\frac{\partial p_0^*}{\partial x^o} < 0, \quad \frac{\partial p_0^*}{\partial r} < 0$$

Now, written more suggestively,

$$x^* = W'^{-1}(r p(R, p_0^*)).$$

Hence,

$$\frac{\partial x^*}{\partial r} = W''^{-1}(r p(R)) \frac{\partial p(R)}{\partial p_0^*} \frac{\partial p_0^*}{\partial r} > 0$$

$$\frac{\partial x^*}{\partial x^o} = W''^{-1}(r p(R)) \frac{\partial p(R)}{\partial p_0^*} \frac{\partial p_0^*}{\partial x^o} > 0$$

To see that $\partial x^*/\partial x^o \in \text{Int}[0,1]$, note that

$$\frac{\partial x^*}{\partial x^o} = \frac{\partial}{\partial x^o}\left[W'^{-1}(r p(R, p_0^*))\right]$$

$$= r W''^{-1}(\cdot) e^{rR}\left[\frac{\partial z_0^*}{\partial x^o} - W''(x^o)\right] .$$

Substitution from (A6) in this provides, after some rearrangement,

$$\frac{\partial x^*}{\partial x^o} = \frac{r W''^{-1}(\cdot) e^{rR}}{\int_0^R d'(p_t) e^{rt} dt + r W''^{-1}(\cdot) e^{rR}} .$$

Since all terms are strictly negative, the conclusion follows.

q.e.d.

FOOTNOTES

1/ Helpful comments on an earlier draft were made by Richard Porter, Glen Loury, G. Robinson Gregory, Richard Brazee, and Chung-Huang Huang. They, of course, are absolved of responsibility for remaining errors. Financial assistance via a fellowship from Resources for the Future gratefully is acknowledged.

2/ The "backstop" terminology was introduced by Nordhaus (1973). See Dasgupta and Heal (1979) for a discussion of the exhaustible resource literature.

3/ Dasgupta and Heal (1979, Chapter 6) discusses this arbitrage aspect of Hotelling's Rule.

4/ This interpretation is given by Hyde (1980). The Faustmann model has a long history; see Samuelson (1976) for a discussion of this literature. The comparative statics properties of the model and the supply curve implied by it are derived by Graham-Tomasi (1983a) and Jackson (1980).

5/ For an overview of such models, see Johnson and Scheurman (1977). Walker (1975) incorporates a downward-sloping demand curve and optimizes the present value of profits. However, this is not appropriate for deriving socially optimal harvest policies.

6/ Actually, the demand for wood products is a derived demand, and the surplus is not directly that of consumers. Throughout this paper, the distinction between "input market consumer's surplus" and "consumer's surplus" is neglected. The usual caveats concerning use of surplus measures apply.

7/ The assumption that old-growth harvests are costless is not restrictive. All of the analyses to follow are unchanged if a constant unit cost of b^o is assumed as long as this cost is less than the unit cost of growing and harvesting plantations. Then, p is replaced by $p - b^o$. The analysis would be significantly complicated, however, if these costs were assumed to rise with either increased rate of harvest or increased depletion of the stock.

8/ The problem as stated assumes that the entire area of forestland is controlled by a forester (or forest service) who seeks an optimal policy. Alternatively, since competitive outcomes correspond to social optima in this model if private and social discount rates are equal and if complete markets exist, the model may be interpreted as a competitive model. Still another alternative is to assume that a central authority is a "dominant firm" and a private sector exists which is a "competitive fringe." If the fringe takes the actions of the forest service as given and the forest service seeks to maximize surplus taking into account the "response" of the fringe, an identical outcome would ensue. This latter case is not unrealistic in the United States. However, uncertainty about the others' strategies or situations and strategic interaction to manipulate decisions in this case (which also is not unrealistic in the United States) are ruled out.

9/ Note that in writing down the Lagrangian, I have left off the multiplier on the objective function. This is legitimate since a rank constraint qualification obviously is satisfied and I, therefore, can divide the Fritz-John Conditions by this multiplier and interpret remaining multipliers as ratios. On this, see Bazaran and Shelty (1976, pp. 116-120).

10/ This is shown in Sierstaad (1978).

11/ This is very similar to the problem studied by Dasgupta and Stiglitz (1981) where the backstop technology must be inverted before innovation can take place.

12/ How such a demand curve is estimated is not pursued here. On valuation of non-priced goods generally, see Freeman (1979).

13/ The specific nature of the stock depletion equation and the form of objective function underly this result. In different problems where utility is a function of the stock, very complicated paths may arise. See Heal (1979) and Heal and Ryder (1973).

14/ See Kutay (1977) and Fight et al., (1978).

15/ The form of the growth function f(R,E) as it relates to comparative statics properties of the Faustmann model of forest management is discussed by Graham-Tomasi (1983a).

16/ An estimate of the potential for such technologies is provided by Hyde (1980), who incorporates them into a steady-state supply curve.

17/ The _locus classicus_ for this literature is Krutilla (1967). An important work is that by Fisher, Krutilla and Cicchetti (1972). Porter (1982) provides a useful overview of this literature.

18/ See Gregory (1972) and Hartman (1976).

19/ Cummings and Norton (1974) have discussed this possibility in the natural environments literature, as has Porter (1982b).

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