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# An Alternative Method for Deriving Optimal Fertilizer Rates: Comment and Extension

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## 1 Introduction

There has been recently renewed interest in decision rules for fertilizer application which take account of the carryover of some fertilizer beyond the period of application (see, *e.g.*, Stauber *et al.* 1975; Dillon 1977; Godden and Helyar 1980; Lanzer and Paris 1981; Maling and McKinlay 1981). As with any time-dependent decision problem, two issues arise in the derivation of optimal rules. One concerns the length of the planning horizon, and the other the process of determining the optimal decision. This note shows how inductive reasoning based on dynamic programming can be used to address both issues. The results are helpful in two ways. First, they may help to resolve an apparent misunderstanding on the part of Godden and Helyar (1980) of the dynamic programming results obtained by Kennedy *et al.* (1973). Godden and Helyar, in commenting on the results, state (p. 86):

. . . the dynamic programming formulation avoids the issue of the ecosystem's movement towards a steady state by specifying the existence of a final period in the fertilizer decision horizon beyond which fertilizer residuals can be ignored. However, specification of the appropriate period beyond which these residuals are negligible requires a model by which that time period may be determined.

In this note further results are presented which emphasize that, given the assumptions of the model described by Kennedy *et al.*, the rule for optimal fertilizer application is not dependent on parameters relating to periods beyond the next period. In other words, the relevant decision horizon beyond which further fertilizer carryover effects and price information can be ignored is known, and is the end of the next decision period.

Secondly, the results indicate the ease with which a computer programme can be devised for solving the fertilizer application problem for the fertilizer-carryover function hypothesized by Godden and Helyar. Such a programme would be more efficient than the heuristic algorithm used by Godden and Helyar. Computational efficiency is an important consideration when on-farm applications are being contemplated, as discussed by Maling and McKinlay (1981). In addition, the results provide an insight into the structure of the problem, without the need for experimentation with an heuristic algorithm. They help to answer the questions posed by Godden and Helyar in footnote 12 (1980, p.93), which they answered themselves but, from their comment, only after significant computational effort.

## 2 Alternative Carryover Hypotheses

Let us consider two hypotheses (H1 and H2) for the fertilizer-carryover function. The first was suggested by Kennedy *et al.* (1973) and the second (in slightly more restricted form) by Godden and Helyar (1980).

(i) *Carryover proportional to fertilizer available: H1*

Fertilizer carryover to period  $t + 1$  is a fixed proportion ( $V$ ) of total fertilizer available in period  $t$  only, for all  $t$ . Total fertilizer available in period  $t$  is defined as fertilizer carryover to period  $t$  plus fertilizer applied in period  $t$ .

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(ii) *Carryover proportional to fertilizer applied: H2*

Fertilizer carryover to period  $t + 1$  is a fixed proportion ( $V_1$ ) of fertilizer applied in period  $t$ , plus a fixed proportion ( $V_2$ ) of fertilizer applied in period  $t - 1, \dots$ , plus a fixed proportion ( $V_m$ ) of fertilizer applied in period  $t - m + 1$ .

**H1 is a special case of H2, with  $V_1 = V, V_2 = V^2, \dots$  and in general  $V_i = V^i$ .** Godden and Helyar assumed  $V_i = b/(i+b)$ , where  $b$  is a constant specific to the production situation.

We can use inductive reasoning to obtain an analytical solution if H2 applies, which also provides the solution if H1 applies. First, let us write the optimal condition for period  $t$  if the most commonly used hypothesis (H0) applies — that there are zero carryover effects:

$$(1) \alpha p_{y_t} dY_t/dQ_t = p_{f_t}$$

where  $\alpha$  is the discount factor;  $p_{y_t}$  is the price of the crop produced;  $Y_t$  is the yield of crop produced which is a function of  $Q_t$ , the fertilizer available; and  $p_{f_t}$  is the price of fertilizer net of per-unit application costs.

If there is carryover of fertilizer, and fertilizer prices do not change, it will be shown that the optimal condition is:

$$(2) \alpha p_{y_t} dY_t/dQ_t = p_f \gamma_m$$

where  $\gamma_m < 1$  depends on the discount factor and carryover parameters relating to  $m$  periods after period  $t$ , and may be termed a reduction factor for fertilizer prices. If the yield function  $Y_t\{Q_t\}$  shows diminishing returns to  $Q_t$ , then  $\gamma_m < 1$  leads to a higher optimal  $Q_t$  than for  $\gamma_m = 1$ . One interpretation of this rule is that with carryover, higher yields are optimal because carryover effectively reduces the cost of access to one unit of fertilizer in any period.

We now derive the value of  $\gamma_m$  for the most general hypothesis, H2, using induction.

### 3 Derivation of Optimal Rules

We adopt the backward recursion method of dynamic programming. Stage subscripts denote the number of periods remaining in the planning horizon. The state of the system at any stage is defined by carryover from all possible fertilizer applications in previous periods. Let  $m$  denote the number of stages for which carryover persists after an initial application. The state variables at any stage  $n$  are  $V_m a_{m+n}, \dots, V_1 a_{n+1}$ , where  $a_n$  is the application of fertilizer at stage  $n$ . Fertilizer carryover to stage  $n$  is  $\sum_{i=1}^m V_i a_{i+n}$ . Total fertilizer available to the crop at stage  $n$  ( $Q_n$ ) is therefore  $a_n = \sum_{i=1}^m V_i a_{i+n}$ .

The objective is to determine the optimal sequence  $a_1^*$  to  $a_N^*$  or equivalently, the optimal sequence  $Q_1^*$  to  $Q_N^*$  such that the present value of net revenue over the  $N$  stages in the planning horizon is maximised. Let us define  $f_n\{V_m a_{m+n}, \dots, V_1 a_{n+1}\}$  as the optimal return at stage  $n$ . For the final stage in the planning horizon  $n = 0$ , further carryover effects are ignored and

$$(3) f_0\{V_m a_m, \dots, V_1 a_1\} = \max_{Q_0} [\alpha p_{y_0} Y_0\{Q_0\} - p_{f_0} a_0]$$

$$\text{where } a_0 = Q_0 - \sum_{i=1}^m V_i a_i$$

Differentiating the RHS of equation (3) with respect to  $Q_0$  and setting the derivative equal to zero gives an optimal value  $Q_0^*$  such that

$$(4) \alpha p_{y_0} dY_0/dQ_0 = p_{f_0}$$

$f_0\{.\}$  can be rewritten substituting  $Q_0^*$  for  $Q_0$  in equation (3). For  $n = 1, f_1\{.\}$  equals the net returns in the penultimate period, plus the present value of the return from optimal fertilizer management in the final period given the vector of fertilizer carryovers consequent on previous fertilizer decisions.

That is

$$(5) f_1\{V_m a_{m+1}, \dots, V_1 a_2\} = \max_{Q_1} [\alpha p_{y1} Y_1\{Q_1\} - p_{f1} a_1 + \alpha f_0\{V_m a_m, \dots, V_1 a_1\}]$$

where  $a_1 = Q_1 - \sum_{i=1}^m V_i a_{i+1}$ .

Again first-order conditions for a maximum give an optimal value  $Q_1^*$  for which

$$(6) \alpha p_{y1} dY_1/dQ_1 = p_{f1} - \alpha V_1 p_{f0}$$

Optimal return with one stage to go becomes

$$(7) f_1\{\cdot\} = \alpha p_{y1} Y_1\{Q_1^*\} - p_{f1}(Q_1^* - \sum_{i=1}^m V_i a_{i+1}) + \alpha [\alpha p_{y0} Y_0\{Q_0^*\} - p_{f0}(Q_0^* - \sum_{i=2}^m V_i a_i - V_1(Q_1^* - \sum_{i=1}^m V_i a_{i+1}))]$$

Equation (5) is recursive, and may be employed iteratively to determine  $f_n\{\cdot\}$ ,  $Q_n^*$  and  $a_n^*$  for  $n = 1$  to  $N$ . For example,  $Q_3^*$  is found to be the  $Q_3$  for which

$$(8) \alpha p_{y3} dY_3/dQ_3 = p_{f3} - p_{f2} \alpha V_1 - p_{f1} \alpha^2 (V_2 - V_1^2) - p_{f0} \alpha^3 (V_3 - 2V_1 V_2 + V_1^3)$$

which gives  $a_3^* = Q_3^* - \sum_{i=1}^m V_i a_{i+3}$ .

For simplicity, assume the price of fertilizer is the same at all stages. Then equation (8) can be written

$$(9) \alpha p_{y3} dY_3/dQ_3 = p_f (1 - \alpha \beta_1 - \alpha^2 \beta_2 - \alpha^3 \beta_3)$$

where the  $\beta_i$  coefficients take the values shown in Table 1.

Table 1: Recursive derivation of the  $\beta$  coefficients

$i$	$\beta_i$	Derivation
1	$V_1$	$= \beta_1$
2	$V_2 - V_1^2$	$= V_2 - V_1 \beta_1$
3	$V_3 - 2V_2 V_1 + V_1^3$	$= V_3 - V_1 \beta_2 - V_2 \beta_1$
4	$V_4 - 2V_3 V_1 + 3V_2 V_1^2 - V_2^2 - V_1^4$	$= V_4 - V_1 \beta_3 - V_2 \beta_2 - V_3 \beta_1$

Continued application of the recursive equation shows that the general rule for stage  $n$  is to set  $Q_n = Q_n^*$  such that

$$(10) \alpha p_{yn} dY_n/dQ_n = p_{fn} - \sum_{i=1}^m \alpha^i \beta_i p_{fn-i}$$

or, for constant fertilizer prices

$$(11) \alpha p_{yn} dY_n/dQ_n = p_f (1 - \sum_{i=1}^m \alpha^i \beta_i) = p_f \gamma_m$$

As the third column in Table 1 indicates,  $\beta_i$  can be determined recursively given  $\beta_1 = V_1$  using

$$(12) \quad \beta_i = V_i - \sum_{j=1}^{i-1} \beta_{i-j} V_j.$$

This makes for easy programming of the solution process for solving equation (10).

In the results presented above, it has been assumed that solutions give  $a_n^* \geq 0$  for all  $n$ , i.e., that optimal application rates are non-negative. This seems to be a reasonable assumption although it may not always hold. As a referee has pointed out, it may not hold if the initial carryover is large, or if prices change dramatically. For example, a drastic reduction in the price of the crop ( $p_{yn}$ ) would greatly reduce  $a_n^*$  and might even cause  $a_n^*$  to be negative. Clearly, in such cases  $a_n^*$  must be constrained to be non-negative. If  $a_n^*$  did have to be constrained, the optimal rule would be more complex than the one given by equation (10). Boundary and interior solutions for  $a_n^*$  would need to be recorded at each iteration of the solution process, in order to be able to determine the sequence of  $a_n^*$  across all stages.

At the same time, it may be noted that Godden and Helyar (1980) make a more restrictive assumption in their heuristic approach to a solution. They assume (p.93) that all application rates are at least as large as the maintenance fertilizer rates, an assumption they find to be valid for their examples.

## 4 Conclusions

### 4.1 Determination of the planning horizon and the steady-state solution

Godden and Helyar implicitly assume in their examples that  $m$ , the number of periods over which residual fertilizer effects occur, is at least as large as  $n$ , the number of periods in the decision horizon. They determine the decision horizon according to the rule that it should be long enough for the present value of net revenue from an additional period to be negligible.

If a limiting value of  $\gamma_m$  in equation (11) could be determined for  $m \rightarrow \infty$ , then equation (11) could be used to specify the steady-state solution. However, the author has failed to find a limit for  $\sum_{i=1}^m \beta_i$ , and hence has not found one for  $\gamma_m$ . For  $\alpha < 1$ , and  $\beta_i$  reducing as  $i$  increases, there must however be a limit for  $\sum_{i=1}^m \alpha \beta_i$ , and hence for  $\gamma_m$ . In the absence of any known analytical limit for  $\gamma_m$ , resort must be made to numerical procedures.

For the values of  $b$  used by Godden and Helyar for New England pasture and Northern Territory sorghum (1.6 and 0.8),  $\gamma_m$  does not converge rapidly (see Table 2.) Assuming here that  $m = n$ , calculation of a practical steady-state solution requires the determination of  $n$  sufficiently large that  $a_n^*$  for  $\gamma_m$  is not appreciably different from  $a_{n-1}^*$  for  $\gamma_{n-1}$ . Five does not appear to be sufficiently large for  $n$  for the example applications studied by Godden and Helyar for which  $m = 30$ . Values of  $a_5^*$  are 84.9, 54.6 and 61.0 (kg P/ha/year) compared with optimal maintenance rates calculated by Godden and Helyar of 86.2, 57.6 and 66.3 for New England fat lambs (Basalt), New England fat lambs (Granite) and Northern Territory sorghum respectively. This experience suggests that to obtain accurate estimates of the optimal steady-state rates of fertilizer application,  $n$  may have to be considerably larger than 5, and the solution procedure computerized.

### 4.2 The nature of the optimal rule when H2 applies

From equation (10) the following may be deduced:

- (i) The appropriate planning horizon is  $m$  stages.

Table 2: Convergence of the reduction factor for fertilizer price<sup>†</sup>

m	$\gamma_m$	
	b = 1.6	b = 0.8
1	.4406	.5950
2	.3863	.5231
3	.3609	.4898
4	.3461	.4705
5	.3365	.4570

<sup>†</sup> based on  $V_i = b/(i + b)$  and  $a = 1/1.1$ .

- (ii)  $Q_n^*$  and hence  $Y_n \{ Q_n^* \}$  are independent of fertilizer carried over from previous stages (though this is not true of  $a_n^*$ ).
- (iii) For constant crop and fertilizer prices, and constant carryover coefficients  $V_i$ , there exists a steady-state yield, which applies from the first production period onwards, regardless of whether initial carryover is zero.
- (iv) Neither  $Q_n^*$  nor  $a_n^*$  are dependent on the prices of the crop or the crop response functions in periods beyond the current period.

### 4.3 The nature of the optimal rule when H1 applies

The optimal rule turns out to be surprisingly simple when it is hypothesized that fertilizer carryover is a proportion of fertilizer available. Calculation of the  $\beta_i$  coefficients using  $V_i = V^i$  in equation (9) gives  $\beta_i = V^i - VV^{i-1} = 0$  for all  $i > 1$ . The general rule given in equation (10) collapses to

$$(13) \quad \alpha p_{yn} dY_n/dQ_n = p_{fn} - \alpha V p_{fn-1}$$

or, for constant fertilizer prices

$$(14) \quad \alpha p_{yn} dY_n/dQ_n = p_{fn} (1 - \alpha V)$$

which was the rule originally derived by Kennedy *et al.* (1973). The optimal  $Q_n$  depends only on: the current discount factor, price of the crop and the price of fertilizer; the crop response function; the carryover coefficient; and next period's price of fertilizer. Whatever the value of  $n$ , and parameter values for stages  $n - 2$  to  $0$ , the rule remains the same. The rule given in equation (13) differs from that given in equation (10) only by the exclusion of any effect of fertilizer price beyond that at stage  $n - 1$ . Under both H1 and H2 the rule is to top up fertilizer available to the relevant  $Q_n^*$  level, regardless of the residual level at the start of stage  $n$  before application. However, for the special hypothesis H1, the optimal carryover level from stage  $n$  to stage  $n - 1$  is always  $Q_n^* V$ , regardless of the residual level at the start of stage  $n$  (though this also is predetermined if the optimal rule is followed — it is  $Q_{n+1}^* V$ ). This makes the current decision independent of fertilizer prices in periods beyond the next period.<sup>1</sup>

1. It should be stressed again that these conclusions depend on the assumption that solutions to the fertilizer problem with  $a_n^*$  unconstrained give  $a_n^* \geq 0$ . This caveat was also made by Kennedy *et al.* (1973, p. 106.)

An issue raised by Godden and Helyar (1980, p.86) — that the opportunity cost of maintenance fertilizer stocks in the ecosystem is not accounted for by Kennedy *et al.* — can also be dealt with here. The general formulation of the problem encapsulated in the recursive equation (5) accounts for all fertilizer costs and returns across all periods in the decision horizon. In other words, under H2, the opportunity cost of maintaining fertilizer stocks is accounted for. Likewise under H1, being a special case of H2, opportunity costs are not ignored.

#### 4.4 Final comments

Assuming that: a) the fertilizer carryover process suggested by Godden and Helyar is a reasonable representation of reality; b) the  $b$  values they applied are not untypical of values for other crops in other localities; and c) a 10 per cent real discount rate is not too low (probably a very plausible assumption); then the economic implications of carryover for the fertilizer decision need to be recognized more widely. Even for planning horizons extending to only two or three stages, Table 2 indicates that with carryover the appropriate reduction factor to apply to fertilizer price in equation (2) is of the order of 0.5.

Although this note has explored the analytical solution to the fertilizer decision problem if hypotheses H1 or H2 hold, further research may suggest many more complex hypotheses. For example, fertilizer carried over may be a quadratic function of fertilizer applied or available. That is, carryover to stage  $n$  might be specified as

$\sum_{i=1}^m (U_i + V_i a_{i+n} + W_i a_{i+n})$  where  $U_i$ ,  $V_i$  and  $W_i$  are carryover parameters. Another is that fertilizer carryover from any period may be dependent not only on fertilizer applied in that period, but as well on fertilizer applied in other periods.

For example carryover to stage  $n$  might be specified as  $\sum_{i=1}^m V_i (a_{i+n} - a_{i+n+1})$ . In either case, the dynamic programming approach would still be applicable without further conceptual problems. In the latter case the problem would be reformulated with fertilizer applied in each of the previous  $m$  periods instead of fertilizer carried over to period  $n$  from each of the previous  $m$  periods. Given these state variables, the contribution of fertilizer application at stage  $n$  to subsequent stages could be determined. It should be noted that whilst dynamic programming problems to be solved numerically are restricted to a few state variables because of limitations in computing capacity, dynamic programming problems to be solved analytically are not subject to the curse of dimensionality. There is no particular limit of the value of  $m$ , the number of state variables, that can be worked with in the fertilizer-carryover problem.

Further realism could be incorporated by characterizing the carryover process as stochastic rather than deterministic, and by accounting for the application costs of fertilizing. A stochastic dynamic programming approach incorporating application costs has been reported by Stauber *et al.* (1975). They obtained solutions numerically, and assumed carryover to any stage to be dependent only on carryover to the previous stage, application at the previous stage, and rainfall in the previous period. To have allowed for a stochastic version of the more complex carryover process suggested by Godden and Helyar very likely would have made infeasible the numerical solution procedure described by Stauber *et al.*

It may be concluded that if further realism in dynamic programming approaches necessitates numerical instead of analytical solution procedures, alternative solution techniques such as the type of heuristic algorithm suggested by Godden and Helyar should be considered. However, at this stage, a prerequisite for greater realism is more knowledge of the biology of the carryover process.

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