UPDATING PARAMETER ESTIMATES: A LEAST SQUARES APPROACH WITH AN APPLICATION TO THE INVENTORY OF BEEF COWS°

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A least squares estimator for utilizing forthcoming information to update or revise estimates of the parameters of an econometric model is derived, some properties of the updated estimates are discussed, and an illustrative application is provided. The forthcoming information includes new sample observations and *a priori* probabilistic information about changes, if any, occurring in the parameters of the estimated model over time. Recursive formulae for sequential updating of the parameter estimates are derived. The procedure is used to update estimates of the parameters of an equation describing annual net changes in the inventory of Australian beef cows.

1 INTRODUCTION

A variant of the mixed least squares estimator derived by Theil and Goldberger [13] is reported for utilizing forthcoming information to update or revise estimates of the parameters of a single equation linear regression model. The forthcoming information refers to new time series sample observations and to independent probabilistic information specifying changes in the parameters over time. The updated estimates provide a more accurate and reliable basis for hypothesis testing, for forecasting and for policy analyses in the current or updated period.

The paper outline is as follows. The next section provides a background discussion of potential reasons for anticipating changes in the parameters of a time series regression model with the passage of time, and hence the need for updating parameter estimates. In section 3 the components of the updating estimation problem are described and the enabling assumptions specified. Section 4 develops the sequential updating formulae. Some properties of the updated parameter estimates are discussed in section 5. In section 6 the updating procedure is applied to a regression equation describing the net change in the inventory of Australian beef cows. The final section provides a brief summary of the major results.

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2 PARAMETER CHANGE IN REGRESSION MODELS

Available empirical evidence and some theoretical considerations described below suggest that the parameters of commonly specified time series econometric models change over time. For example, it is not unusual to observe that models of demand and of supply for commodities and models of the economy fitted with prewar data give significantly different parameter estimates when fitted with postwar data.

Given the simplifying assumptions employed in specifying econometric models and the constraints imposed by available data it is not surprising that the parameters of estimated models change over time. While in a conceptual sense models could be specified so as to allow for the myriad of potential causal relationships, in practice, for reasons of ignorance, unavailability of data, and simplicity we omit many potential explanatory variables. For example, usually we do not include variables to reflect changes in consumer tastes, changes in technology, changes in institutional relationships or all the factors influencing decision-makers' expectations. In econometric models the effects of these excluded variables are assumed to be constant or, at most, random about a stationary level. When the excluded variables exhibit non-stationary behaviour, i.e., when the *ceteris paribus* assumption of no structural change breaks down, changes in the levels of these variables may induce changes in the parameters of the included variables.

Due to data limitations some of the included variables will be proxy variables only. As examples we note the approximations implicit in measures of capital and the measures of expected prices. The proxy variables may detect only partial changes in the economic stimuli they purport to measure. In such cases changes in the real economic stimuli could induce changes in the parameters of the proxy variables. Often estimated econometric models represent an aggregate (macro relation) weighting of the behavioural responses for heterogeneous sets of micro units. Here, changes in the relative importance of the different sets may be expected to influence the weighted average parameters of the relationship being estimated.

Another potential cause of change in the parameters of an estimated regression equation results from the use of approximate mathematical forms. Often the causal effect of variables is assumed to be linear and independent of the levels of other variables. In such cases changes in the levels of the explanatory variables may result in parameter changes in the specified model. For example, if under the pretext of a Taylor series expansion a linear relation is estimated as an approximation to a non-linear specification, the assumption of constant parameters for the estimated relation constitutes a reasonable approximation only if the observed explanatory variables remain within some narrow range.

The cumulative implication of these observations is that with the progress of time we may reasonably expect the specified parameters of estimated econometric models to change. An investigation of the simplifying

assumptions implicit in the maintained hypothesis should provide a starting basis for isolating parameter changes, if any[1].

It is in this context that our estimator for updating estimates of the parameters of an econometric model is developed. Based on the above arguments for parameter change, one set of additional information is specified as a prior probability distribution function of changes, if any, in the regression parameters over time[2]. The second set of additional information refers to more sample observations[3]. We proceed next to specify in detail the updating estimation problem.

3 MODEL SPECIFICATION

For the sake of simplicity, and without any real loss of generality, two periods denoted as period n and period $n + 1$ are considered. For the first period, n, parameter estimates of a regression equation based on a sample of size T are obtained. Information in period $n + 1$ becomes available for updating these parameter estimates.

The model for the first period sample is specified as:

(1) $y_t = x_t \beta_n + u_t \qquad \text{for } t = 1, 2, \ldots, T$

where y is an endogenous variable, x is a $1 \times k$ vector of fixed explanatory variables, β_n is an unknown vector of parameters assumed to be constant over the period n sample, and u is an error term with the properties $E\, u_t = O$, $E\, u_t\, u_s = \delta_{ts}\, \sigma^2$, where δ_{ts} denotes the Kronecker delta, and $E\, u_t\, x_s = O$ for $s \leqslant t = 1, 2, \ldots, T$.

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[1] In passing we note three possible statistical tests of parameter changes. First, Chow's test based on the use of dummy variables. This test requires discrete piecewise parameter change assumptions and is demanding of degrees of freedom. Second, tests of the residuals for autocorrelation, heteroscedasticity and non-normality may reveal specification errors associated with parameter changes. Third, failure of actual values to be within forecast confidence intervals may be interpreted as significant structural changes, which includes parameter changes.

[2] The estimation problem differs in several respects from related problems reported in the literature. Models such as the random coefficient regression model, e.g. Hildreth and Houck [10], and the adaptive regression model, e.g. Cooley and Prescott [5], allow for non-constant parameters but the parameters are restricted to the case of a stationary distribution function in the sense that the expected value of the parameter vector is constant over time. Our problem relaxes this stationarity assumption. Kalman filter problems which have been discussed in an econometric context by Duncan and Horn [7] allow for parameter changes over time, but they assume perfect knowledge about these parameter changes; our problem relaxes this perfect knowledge assumption to an assumption of probabilistic knowledge.

[3] Unfortunately, in most cases the number of additional sample observations will not be sufficient to justify the use of a piecewise constant coefficient change model based on the use of dummy variables. For a recent review of such models *see* Goldfeld and Quandt [9].

The period n sample of T observations is used to obtain estimates of the unknown parameters β_n and σ^2. Applying the least squares estimator, the best, linear, unbiased estimates are given by b_n and s^2, respectively, and the unbiased estimated covariance matrix for b_n is given by $s^2_n\ N_n$. These statistics are given as:

(2a) $N_n = [X'_n\ X_n]^{-1}$.

(2b) $b_n = N_n\ X'_n\ y_n$.

(2c) $s^2_n = (1 / (T - k))\ (y'_n\ y_n - b'_n\ X'_n\ y_n)$.

where X_n is a Txk matrix and y_n is a $T \times 1$ vector.

For the second or updating period, $n + 1$, a regression equation of the form:

(3) $y_t = x_t\ \beta_{n+1} + u_t$ for $t = T + 1, \ldots, T + r$

is specified. The terms in (3) are as defined for (1) with the exception that the unknown parameters in β_{n+1} may differ from the corresponding unknown parameters in β_n. The estimation problem involves using information available from the period n sample and new information coming available in period $n + 1$ to obtain estimates of β_{n+1} and σ^2 ($= E\ u^2_t$) in (3).

The additional information which becomes available in period $n + 1$ will be cast into two categories. The first category concerns changes, if any, in the parameter vector between periods n and $n + 1$. It is assumed that β makes a discrete transition between period n and $n + 1$ according to the linear relation:

(4) $\beta_{n+1} = A_n\ \beta_n + d_n$

where β_{n+1} and β_n are as defined in (3) and (1), A_n is a $k \times k$ matrix and d_n is a $k \times 1$ vector. The A matrix provides for multiplicative changes in the regression parameters while the d vector provides for additive changes in the regression parameters between periods n and $n + 1$.

In the general formulation probabilistic information about the elements of A_n and d_n is assumed. Specifically it is assumed that the $k^2 \times 1$ vector α ($= Vec\ A$, where Vec is the vector stacking operator[4]) and the $k \times 1$ vector d are independently distributed with mean $E\ (\alpha, d) = (\bar{\alpha}, \bar{d})$ and covariance matrix $Cov\ (\alpha, d) = \begin{bmatrix} \Omega & O \\ O & \varnothing \end{bmatrix}$ with $\bar{\alpha}$, $\bar{d}$, Ω and $\varnothing$ known and that α and d are independent of u_t for all t. The "larger" or more positive definite the covariance matrices Ω and $\varnothing$ the greater will be uncertainty regarding the parameter changes between periods n and $n + 1$.[5]

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[4] The vestor stacking operator, Vec, has the following interpretation. Specifying $A = [a_1, a_2, \ldots, a_k]$, where a_i is the i-th column of A, $Vec\ A = \alpha = [a'_1\ a'_2\ \ldots\ a'_k]'$.

[5] The matrix K is said to be greater than the matrix L if, for all non-zero vectors x, the scalar quantity $x'K\ x > x'L\ x$.

Clearly, upon imposing restrictive assumptions on A and d in (4) a number of special cases regarding parameter changes may be specified. If we assume perfect knowledge about A and d and specify the former to be the identity matrix and the latter to be the null vector we have the textbook constant parameter regression model. Duncan and Horn [7] consider the case where A is known and d has a null vector expectation and known covariance matrix $\varnothing$.

The second category of information that becomes available in period $n + 1$ is the set of new sample observations.

We represent these observations as:

(5) y_{n+1}, X_{n+1}

where y_{n+1} is an $r \times 1$ vector and X_{n+1} is an $r \times k$ matrix.

The updating problem is one of combining the additional information generated during period $n + 1$ as described in (4) and (5) with the parameter estimates available from period n as described in (2) to obta' estimates of β_{n+1} and σ^2.

4 UPDATED PARAMETER ESTIMATES

In obtaining estimates of the unknown parameters β_{n+1} and σ^2 of (3) we proceed in two steps. Step one involves combining information relating to changes in the β parameter vector specified in (4) with the parameter estimates available from the period n sample specified in (2). The resulting estimates are termed updated parameter estimates with independent parameter change information (but without additional sample observations). In step two the mixed least squares estimator is used to combine these estimates with the additional sample observations, (5). The resulting estimates are termed the updated estimates with parameter change information and with sample information.

4.1 UPDATED ESTIMATES WITH PARAMETER CHANGE INFORMATION

Given the period n sample estimates (2) and the parameter change information (3), together with the assumptions of independence of A_n, d_n and β_n, all of which are implied by our probabilistic assumptions, the updated estimate of β_{n+1}, which we denote as b^*_{n+1}, and the covariance matrix of this estimate, which we denote as $\sigma^2 N^*_{n+1}$, are derived as:

$$(6)\quad b^*_{n+1} = E(A_n\beta_n + d_n) = \bar{A}_n b_n + \bar{d}_n$$

and

$$(7)\quad \begin{aligned} \sigma^2 N^*_{n+1} &= E\{(A_n\beta_n + d_n - \bar{A}_n b_n - \bar{d}_n)(A_n\beta_n + d_n - \bar{A}_n b_n - \bar{d}_n)'\} \\ &= E\{(A_n\beta_n - \bar{A}_n b_n)(A_n\beta_n - A_n b_n)'\} + \varnothing \\ &= J + \varnothing \\ &= \sigma^2[\sigma^{-2}(J + \varnothing)] \end{aligned}$$

where, with the exception of J, all terms are as defined above. The ij-element of the $k \times k$ matrix J, $J(ij)$, is given by:

(8) $J(ij) = \sigma^2 \bar{a}_i N_n \bar{a}_j + b'_n \Omega_{ij} b_n + \sigma^2 \text{ trace } (N_n \Omega_{ij})$

where a_i is the i-th row of $\bar{A}$, Ω_{ij} is the ij-submatrix of Ω corresponding to the i-th and j-th rows of A, and all other terms are as defined above.

4.2 UPDATED ESTIMATES WITH PARAMETER CHANGE INFORMATION AND WITH SAMPLE OBSERVATIONS

The next step of the updating procedure requires a variant of the mixed least squares approach[6].

Formally, the problem is to estimate β_{n+1} from the augmented model:

(9) $$\begin{bmatrix} b^*_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} I \\ X_{n+1} \end{bmatrix} \beta_{n+1} + \begin{bmatrix} b^*_{n+1} - \beta_{n+1} \\ u_{n+1} \end{bmatrix}$$

or, in compact form,

(10) $\underline{y}_{n+1} = \underline{X}_{n+1} \beta_{n+1} + \underline{u}_{n+1}$

where all terms are defined above. Now, from:

(3) $E u_{n+1} = O$ and $E u_{n+1} u'_{n+1} = \sigma^2 I_r$, and from (6) and (7)

$E(b^*_{n+1} - \beta_{n+1}) = O$ and $E(b^*_{n+1} - \beta_{n+1})(b^*_{n+1} - \beta_{n+1})' = \sigma^2 N^*_{n+1}$, and our independence assumptions imply $E(b^*_{n+1} - \beta_{n+1}) u'_{n+1} = O$.

Then,

(11) $$E \underline{u}_{n+1} = O \text{ and } E \underline{u}_{n+1} u'_{n+1} = \sigma^2 \begin{bmatrix} N^*_{n+1} & O \\ O & I_r \end{bmatrix} = \sigma^2 \Sigma_{n+1}.$$

Given the covariance matrix Σ_{n+1} of (11), generalized least squares may be employed in the context of (10) to obtain updated estimates of β_{n+1}. The mean estimate, b_{n+1}, and its associated error covariance matrix, $\sigma^2 N_{n+1}$ are derived as:

(12) $b_{n+1} = N_{n+1} [N^{*-1}_{n+1} b^*_{n+1} + X'_{n+1} y_{n+1}]$

and

(13) $\sigma^2 N_{n+1} = \sigma^2 [N^{*-1}_{n+1} + X'_{n+1} X_{n+1}]^{-1}$.

Clearly the updating formulae (6), (7), (12) and (13) may be applied sequentially as the process moves from one period to another.

4.3 COMPUTATION OF THE PARAMETER ESTIMATES

In practice formulae for computing b_{n+1} in (12) are not directly applicable when the covariance matrices Ω and $\varnothing$ of the parameter change relation (4) are not null, i.e. when we do not have perfect knowledge of the A_n and d_n parameters of the parameter change relation (4). The problem arises because the matrix N^*_{n+1} of (7) depends on the matrices $\sigma^{-2} \Omega$ and

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[6] For details of this estimator see Theil [12, pp. 347–349].

$\sigma^{-2}\,\varnothing$ and σ^2 generally is unknown. The matrices $\sigma^{-2}\,\Omega$ and $\sigma^{-2}\,\varnothing$ measure the relative importance of the variance of the error term u in the observation equation (3) to the covariance matrices of the independent distribution function for the elements of the multiplicative change matrix A and the additive change vector d, respectively, of the parameter change relation (4). In an operational setting the following approximate procedure might be followed. Replace σ^2 with a consistent estimate, say $\hat{\sigma}^2$, and proceed as if $\hat{\sigma}^2 = \sigma^2$. It can be shown the $s_n{}^2$ of (2c) is a consistent estimate of σ^2 and that the difference $s^2{}_n - \sigma^2$ is of order $T^{-\frac{1}{2}}$ in probability (Theil [11, p. 405]). Furthermore, information about Ω and $\varnothing$ is unlikely to be exact so that even if σ^2 were known, the ratios $\sigma^{-2}\,\Omega$ and $\sigma^{-2}\,\varnothing$ would be approximate only.

For computationally efficient sequential updating, and for the purpose of obtaining further insights into the updated parameter estimate (12) and (13), the estimation formulae for b_{n+1} and N_{n+1} may be specified in the form of recursive equations. Using a matrix inversion lemma reported in Aoki [1, p. 69] the matrix $\hat{N}_{n+1}$, where $\hat{N}_{n+1}$ is N_{n+1} as stated in (13) with $s^2{}_n$ substituted for σ^2, may be restated as[7]:

$$(14)\quad \hat{N}_{n+1} = \hat{N}^*{}_{n+1} - \hat{N}^*{}_{n+1}\, X'_{n+1}\, [I_r + X_{n+1}\, \hat{N}^*{}_{n+1}\, X'_{n+1}]^{-1}\, X_{n+1}\, \hat{N}^*{}_{n+1}$$

where

$$(15)\quad \hat{N}^*{}_{n+1} = s^{-2}{}_n\, (\hat{J} + \hat{\varnothing})$$

where $\hat{J}$ and $\hat{\varnothing}$ are as specified in (7) and (8), respectively, with $s^2{}_n$ replacing σ^2, and all other terms are as previously defined. Substituting (14) for $\hat{N}_{n+1}$ in (12) and after some algebraic manipulations, the updated parameter estimate b_{n+1} may be restated as[8]:

$$(16)\quad b_{n+1} = b^*{}_{n+1} + \hat{N}_{n+1}\, X'_{n+1}\, [y_{n+1} - X_{n+1}\, b^*{}_{n+1}]$$

where $b^*{}_{n+1}$ is as specified in (6).

Usually (14), (15) and (16) are more convenient than (12) and (13) for computing since the former involves inversion of an $r \times r$ matrix (where r is the number of additional observations) while the latter involves inversion of a $k \times k$ matrix (where k is the number of regression parameters). Formulae (6), (7), (14), (15) and (16) are readily applicable for the sequential updating of parameter estimates for a sequence of forthcoming sets of new information.

5 PROPERTIES OF THE UPDATED ESTIMATES

In this section we consider some properties of four estimates of the parameter vector β_{n+1} obtained under different information assumptions. While we focus on the bias property of the estimates of β_{n+1} and on the

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[7] The lemma states that the matrix $[G^{-1} + D'\, H^{-1}\, D]^{-1}$ can be expanded to the matrix $[G - G\, D'\, [H + D\, G\, D']^{-1}\, D\, G]$.

[8] Details may be obtained from either of the authors.

precision (i.e. the inverse of the covariance matrix) of the parameter estimates we note that the bias squared plus the variance determine another interesting parameter estimate property, the mean square error. It seems useful to consider properties of the following four estimates of β_{n+1}. The case 1 estimate—the non-updated estimate is given by the estimator in (2). This estimate of β_{n+1} ignores the contributions of new information coming available in period $n + 1$. The case 2 estimate—the updated estimate with parameter change information—is given by the estimator of (6) and (7). The case 3 estimate—updated estimate with additional sample information—utilizes the n period and $n + 1$ period sample sets as an augmented sample to estimate β_{n+1}. In essence this estimator assumes $\beta_n = \beta_{n+1}$, i.e. $A_n = I$ and $d_n = 0$, and ignores independent information about parameter changes. The case 4 estimate —updated estimate with parameter change information and with additional sample information—utilizes all sources of information and the estimator is given by the equations (14), (15), and (16).

To assess the bias property of the different mean estimates of β_{n+1} we apply the expectation operator under the assumptions that $Eu_t = 0$ and that x_t is exogenous for all t, and that the estimate of β_n based on the n period sample, b_n in (2a), is an unbiased estimate. The case 1 or non-updated estimate of β_{n+1}, b_n will provide an unbiased estimate only if the true situation corresponds to one of no parameter change between period n and period $n + 1$; the expected bias, if any, will be given by $\beta_{n+1} - \beta_n$. Given our assumption that information about the elements of the A_n matrix and d_n vector of the parameter change relation (4) are independent of u_t in (1) and (3) for all t, the expected value of the mean updated estimate without sample data b^*_{n+1} as specified in (6) will be:

$$(17)\quad Eb^*_{n+1} = E\bar{A}_n Eb_n + E\bar{d}_n = E\bar{A}_n\beta_n + E\bar{d}_n.$$

Then, b^*_{n+1} provides an unbiased estimate of β_{n+1} if $\bar{A}_n$ and $\bar{d}_n$, which specify expected parameter changes, are unbiased estimates. To assess the expectation of the case 3 and case 4 estimates of β_{n+1} we take the expectation of (16) as follows:

$$(18)\quad E\tilde{b}_{n+1} = E\tilde{A}_n Eb_n + E\tilde{d}_n + \hat{N}_{n+1} X'_{n+1}(X_{n+1}\beta_{n+1} + Eu_{n+1} - X_{n+1}[E\tilde{A}_n Eb_n + E\tilde{d}_n])$$
$$= E\tilde{A}_n\beta_n + E\tilde{d}_n + \hat{N}_{n+1} X'_{n+1} X_{n+1}(\beta_{n+1} - E\tilde{A}_n\beta_n - E\tilde{d}_n)$$

where $\tilde{A}_n$ and $\tilde{d}_n$ correspond to the actual levels of A_n and d_n, respectively, used by the estimator: for the case 3 estimator $\tilde{A}_n = I$ and $\tilde{d}_n = 0$ and for the case 4 estimator $\tilde{A}_n = \bar{A}_n$ and $\tilde{d}_n = \bar{d}_n$. Now $\tilde{b}_{n+1}$ provides an unbiased estimate of β_{n+1} if $E\tilde{A}_n = A_n$ and $E\tilde{d}_n = d_n$, where A_n and d_n refer to the true values. That is, the updated estimate with sample data, case 3, will be an unbiased estimate if the true case is one of no parameter change; otherwise it yields biased estimates. Similarly, the updated estimates with parameter change information and with sample data, case 4,

will be unbiased if $\bar{A}_n$ and $\bar{d}_n$ are unbiased estimates of A_n and d_n, respectively. The direction and magnitude of the parameter estimate bias depends on $(A_n - \bar{A}_n)\ \beta_n + (d_n - \bar{d}_n)$ and the kxk matrix $\hat{N}_{n+1}\ X'_{n+1}\ X_{n+1}$. The latter matrix is peculiar to the particular sample set and this makes it difficult to make general statements about the direction and order of bias in the estimates.

Further inspection of (18) indicates that for cases 3 and 4 the extent of bias in $\tilde{b}_{n+1}$ approaches zero as the matrix $\hat{N}_{n+1}\ X'_{n+1}\ X_{n+1}$ approaches the identity matrix. Expansion of $\hat{N}_{n+1}\ X'_{n+1}\ X_{n+1}$ to $[\hat{N}^{*-1}_{n+1} + X'_{n+1}\ X_{n+1}]^{-1}\ X_{n+1}\ X_{n+1}$ indicates that the bias declines (1) as the additional or $n + 1$ period sample increases and (2) as the matrix $\hat{N}_{n+1}$, which in turn is influenced by the covariance matrices Ω and $\varnothing$, becomes larger. The latter implies that, other things being equal, the greater the uncertainty about parameter changes reflected in Ω and $\varnothing$, the smaller becomes the extent of the bias. The first point puts the case for including forthcoming sample observations in the updated parameter estimates.

Inspection of the recursive form of the updating estimator, equation (16), indicates two aspects of the inclusion of additional sample observations in the parameter estimates. The case 4 updated parameter estimate, b_{n+1}, is given by the case 2 updated parameter estimate, b^*_{n+1}, plus a correction term proportional to the difference between the actual value of the vector of additional observations on the explanatory variable and the mean prediction of this vector using b^*_{n+1}, i.e. on $y_{n+1} - X_{n+1}\ b^*_{n+1}$. Thus, *ceteris paribus*, the more accurately the updated estimate with parameter change information predicts forthcoming observations, the smaller will be the effect of incorporating the additional sample data on the mean updated estimate of β_{n+1}. The relative weight placed on the mean prediction error vector is given by $\hat{N}_{n+1}\ X'_{n+1}$, where $\sigma^2\ \hat{N}_{n+1}$ is the covariance matrix for the updated parameter estimate. Thus, the greater is the precision of the parameter estimates the less weight is placed on the additional sample observation information.

Turning to the covariance matrices for the updated parameter estimates two observations are noted. First, comparison of the covariance matrix $\sigma^2\ N_n$ for the non-updated estimate with the covariance matrix $\sigma^2\ N^*_{n+1}$ for the updated estimate with parameter change information indicates that the latter is augmented by the covariance matrices Ω_n and $\varnothing_n$ of the parameter change relation (4) and it is augmented (decreased) by the $\bar{A}_n$ matrix when the characteristic roots of $\bar{A}_n$ are outside (inside) the unit circle[9]. Thus, while greater uncertainty about the parameters of the parameter change relation, as reflected in larger Ω and $\varnothing$ matrices, tends to result in estimates with a lesser degree of bias at the same time

[9] In general, the incorporation of probabilistic information about parameter changes in the updated estimates has the effect of bounding the parameter covariance matrix $\sigma^2\ \hat{N}_{n+1}$ away from the null matrix.

it results in less precise estimates. Second, incorporating additional sample observations in the parameter estimates increases the precision of the estimates. This result is shown by expanding $\sigma^2 N_{n+1}$ in (13) as

(19) $\sigma^2 N_{n+1} = \sigma^2 [N^*_{n+1} - N^*_{n+1} X'_{n+1} [I_r + X_n N^*_{n+1} X'_n]^{-1} X_{n+1} N^*_{n+1}]$.

Now $\sigma^2 N^*_{n+1}$ is the covariance matrix for the updated parameter estimate with parameter change information (but without additional sample observations) and the second right hand matrix of (19) is a positive definite matrix.

While the foregoing discussion has been concerned with properties of estimates of β_{n+1} obtained under different information assumptions, analogous properties may be derived for forecasts yielded by the different estimates. Some of these properties may be briefly summarized as: (i) forecasts yielded by the non-updated estimates will be unbiased only in the event of no parameter changes over time; (ii) forecasts yielded by updated estimates incorporating the parameter change information will be unbiased if expected changes in the parameters are unbiased; (iii) the effects of biased estimates of parameter change will be reduced by increasing the covariance matrices for the parameter change relation parameters, but this involves some loss of precision in the forecasts, or by incorporating additional sample observations in the parameter estimates; and, (iv) incorporation of additional sample observations in the parameter estimates leads to more precise forecasts.

6 ILLUSTRATIVE APPLICATION

In this section we apply the least squares updating estimator reported above to update estimates of the parameters of a regression model describing the net change in the annual inventory of Australian beef cows.[10] For an initial sample period 1953–4 through 1968–9 ordinary least squares estimates of the parameters are derived. Assuming we arrive at 1971 and are interested in parameter estimates applicable for an analysis of the early 1970's beef cow inventory levels, we illustrate how the additional information coming available since 1968–9 could be used to update the initial period parameter estimates. The section concludes with a discussion of the parameter estimates and forecasts obtained under different information assumptions.

6.1 BEEF COW INVENTORY RESPONSE MODEL

The underlying economic model of the specified relationship describing net changes in the annual inventory of Australian beef cows is a product-product production economics model. In this model producers allocate a bundle of fixed pasture resources between beef production and the production of alternative grazing activities—wool, lamb and dairy

[10] To date no published study known to the authors has used econometric procedures to analyse the Australian inventory of beef cattle.

production. The available pasture resources are measured in terms of two proxy variables: the net change in the area of sown grasses and clovers and a rainfall index variable. Producers are assumed to allocate the pasture resources so as to maximize expected net returns to the limiting resource. Conceptually the net return variables should be specified as output price less variable input costs, including labour costs, per unit production. In practice the lack of suitable data regarding input costs led us to approximate net returns by output prices[11]. In algebraic terms the model to be estimated is specified as:

(20a) $y_t = \beta_{0n} + \beta_{1n} x_{1t} + \beta_{2n} x_{2t} + \beta_{3n} x_{3t} + u_t$

or, in the concise notation of the conceptual portion of the paper,

(2ob) $y_t = x_t \beta_n + u_t$ for $t = 1953\text{–}4, \ldots, 1968\text{–}9$,

where y denotes annual net change in the inventory of Australian beef cows, x_1 denotes the ratio of expected beef prices to a weighted average of expected wool, lamb and butter prices, x_2 denotes the annual change in the area of sown grasses and clovers, x_3 denotes the index of annual rainfall, and u is the error term[12]. We assume a constant vector of parameters, β_n, for the sample period 1953–4 through 1968–9.

6.2 FIRST PERIOD SAMPLE ESTIMATES

For the sample period 1953–4 through 1968–9 the following least squares estimates of the unknown parameters β_n and σ^2 in (20) were obtained:

$$(21)\quad y = \underset{(204\cdot9)}{-\,481\cdot3} + \underset{(373\cdot9)}{1\,493\cdot9}\, x_1 + \underset{(99\cdot91)}{67\cdot24}\, x_2 + \underset{(92\cdot76)}{330\cdot90}\, x_3$$

$$s^2{}_n = 36312.$$

The estimated standard errors of the parameter estimates are given in parentheses[13].

The estimated function indicates that the net change in the annual inventory of beef cows is positively related to the expected profitability of beef production relative to that of alternative grazing activities and to the availability of pasture resources.

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[11] J. Street has suggested that a more appropriate specification would include the labour index variable as a separate explanatory variable. This suggestion had to be rejected on statistical grounds because the high level of collinearity between this variable and the x_1 variable of (20a) resulted in unsatisfactory estimates for the sample period of interest.

[12] The definitions and data sources for the variables used were as follows: y – net change in other cows and heifers one year and over at the end of March in 1000's [4] ; $x_1 = (\cdot5\, P_B + \cdot33\, P_{B-1} + \cdot17\, P_{B-2}) \,/\, (\cdot6(\cdot5\, P_W + \cdot33\, P_{W-1} + \cdot17\, P_{W-2}) + \cdot2\,(\cdot5\, P_L + \cdot33\, P_{L-1} + \cdot17\, P_{L-2}) + \cdot2\,(\cdot5\, P_D + \cdot33\, P_{D-1} + \cdot17\, P_{D-2}))$ where P_B is the price in cents per kg first and second quality bullocks and cows, Homebush [2], P_W is the average price in cents per kg realized for greasy wool at auction [4], P_L is the price in cents per kg for first and second quality export lamb, Homebush [2], and P_D is the average export butter price in cents per kg [3]; x_2 is in million hectares [3], and; x_3 is based on the following weights, +1 for annual rainfall in deciles 9 and 10, 0 for annual rainfall in deciles 4 through 8, −1 for annual rainfall in decile 3, −2 for annual rainfall in decile 2 and −3 for annual rainfall in decile 1 [8].

[13] The covariance matrix from which the standard errors were derived is reported in an appendix.

6.3 UPDATING THE PARAMETER ESTIMATES

For the next or updating period, $n + 1$, we specify a beef cow inventory relationship of the form:

(22) $y_t = x_t \beta_{n+1} + u_t$ for t = 1969–70, 1970–1,

where all terms are as defined for (20) but where elements of β_{n+1} in (22) are not necessarily constrained to be the same as the comparable elements of β_n in (20b). Our objective is to obtain an estimate of β_{n+1} in (22) using the least squares estimator described above to combine new information generated during 1969–71 with the estimates obtained from the sample period 1953–4 through 1968–9 reported in (21).

As before, the new information forthcoming in 1969–71 falls into two categories; information on parameter changes and additional sample observations. For the first category we focus on a change in the β_1 parameter; we assume no changes in the other parameters. In the estimated model the β_1 parameter describes the effect on the inventory of beef cows of a change in the relative expected output price of beef production to the output price of alternative grazing activities and this effect is assumed to be constant and independent of the levels of the price variables. In the discussion of the specified model in section 6.1 it was noted that x_1, the relative expected output price variable for beef production and alternative grazing activities, is a proxy variable for the relative expected net return variable of the underlying product-product economic model. The proxy variable x_1 does not allow for the effects of changing labour costs on the relative attractiveness of beef production to the alternative grazing activities. Labour costs rose in 1969–71 relative to the earlier sample period and available evidence indicates substantially lower labour costs per unit of beef production relative to that incurred in wool, lamb and butter production[14]. In this context it is contended that the proxy variable x_1 underestimates the attractiveness of beef production in more recent times (specifically period $n + 1$ as compared to period n). In short, we expect the β_1 parameter to increase between period n and period $n + 1$, but we are uncertain about the extent of the change.

Specifically we assume $\beta_{1,\, n+1} = a\ \beta_{1,\, n}$ and specify probabilistic information about "a" to reflect our somewhat arbitrary presumption that we are 95 per cent confident that $\beta_{1,\, n+1}$ is zero to twenty per cent greater than $\beta_{1,\, n}$ with an expected increase of 10 per cent[15]. To a large extent these assumptions reflect the ten per cent increase in the Bureau of Agricultural Economics index of price of wages paid between June, 1969 and June, 1971 [3]. Then, in the notation of the parameter change relation (4), the $\bar{A}$ matrix has diagonal elements equal to 1.0, except for $\bar{a}_{22}$ which is 1.1 and all off-diagonal elements are zero, the Ω matrix has all elements zero except the variance term for a_{22} which is set at $(\cdot 05)^2$, the $\bar{d}$ vector is null, and the ø covariance matrix is specified to be null.

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[14] For a discussion of some evidence *see* Davidson [6].

[15] We have approximated the 95 per cent confidence interval as plus or minus two standard deviations.

The second set of information that becomes available in period $n + 1$ is two additional observations, viz.

$$(23)\quad y_{n+1} = \begin{bmatrix} 912 \\ 1\,125 \end{bmatrix}, \text{ and } X_{n+1} = \begin{bmatrix} 1{\cdot}0 & {\cdot}731 & 1{\cdot}866 & -{\cdot}21 \\ 1{\cdot}0 & {\cdot}839 & 1{\cdot}588 & -{\cdot}06 \end{bmatrix}.$$

As above we will consider estimates of the parameter vector β_{n+1} in (22) obtained under four different information states. Case 1—the non-updated estimate—utilizes only the 1953–69, or period n, sample information. Case 2—updated estimate with parameter change information—utilizes the 1953–69 sample information and the independent information that β_1 increases between n and $n + 1$; but it ignores the additional 1969–71 sample observations. Case 3—updated estimates with additional sample data—utilizes the two sets of sample observations 1954–69 and 1969–71; but it ignores the independent information about parameter changes. In essence for this case the least squares estimator is applied to the augmented sample under the assumption of constant parameters. Case 4—updated estimates with parameter change information and with additional sample observations—utilizes both categories of new information generated in 1969–71 and the 1953–69 sample data. For cases 2, 3, and 4 the least squares estimator described in section 4 is employed.

Estimates of the parameter vector β_{n+1} in (22) and of the standard errors for these estimates obtained for the four information states described above are reported in table 1[16]. Inclusion of the additional sample observations in the parameter estimates, cases 3 and 4, resulted in

TABLE 1

Estimates of Regression Parameters of 1969–71 *Relation for Inventory of Australian Beef Cows for Different Information States*

Information State[a]	Explanatory variables			
	Constant	X_1	X_2	X_3
Case 1	− 481·3	1,493·9	67·24	330·9
	(204·9)[b]	(373·9)	(99·97)	(92·8)
Case 2	− 481·3	1,643·3	67·24	330·9
	(204·9)	(418·5)	(99·97)	(92·8)
Case 3	− 645·2	1,780·9	116·12	328·3
	(156·6)	(290·9)	(90·9)	(92·0)
Case 4	− 562·6	1,813·1	88·75	330·6
	(151·8)	(292·7)	(92·00)	(92·4)

[a] As defined in text.

[b] Details of the complete covariance matrix are given in appendix.

[16] The covariance matrices from which the standard errors were derived are reported in an appendix.

estimates of $\beta_{1,\ n+1}$ which were greater than those obtained by the estimates which ignored the new sample observations, cases 1 and 2. These results are consistent with our argument that the parameter on the relative output price variable has increased in recent years.

Table 2 reports mean forecasts of changes in the inventory of Australian beef cows for 1969–70, 1970–1 and 1971–2 using the four estimated functions collated in table 1. For this example, and clearly one sample provides a limited basis for assessing forecasting performance, the parameter estimates incorporating both sources of additional information, case 4, provide the most satisfactory forecasts while the parameter estimates which do not incorporate this additional information, case 1, provide the least satisfactory forecasts.

TABLE 2

Forecasts Annual Changes in the Inventory of Australian Beef Cows for 1969–70, 1970–1 *and* 1971–2 *Using the Different Estimated Relations of Table* 1

Forecast function	Forecast period					
	1969–70		1970–71		1971–72	
	Mean forecast	Forecast error	Mean forecast	Forecast error	Mean forecast	Forecast error
Case 1 ..	668	244	859	266	899	603
Case 2 ..	776	136	984	141	1,039	463
Case 3 ..	803	109	1,012	113	996	506
Case 4 ..	860	52	1,080	45	1,098	404

7 CONCLUSION

In this paper we have argued that it is desirable to incorporate forthcoming information in the parameter estimates of time series models to be used in studies of the current situation. A least squares estimator was derived for incorporating independent information about parameter changes and additional sample observations in the updated parameter estimates. Recursive formulae for the sequential updating of parameter estimates are reported in equations (6), (7), (14), (15) and (16). The formulae involve a minimal computational burden.

The bias properties of the updated estimates are directly related to the independent prior information of expected changes, if any, in the parameters of the specified regression model over time. In view of our discussion of section 2 that the typical case with time series econometric models will be one of non-constant parameters, the conventional procedure of assuming constant parameters will result in biased updated estimates. Such biases can be minimized by incorporating quasi-unbiased mean estimates of parameter change in the updated estimates, by

explicitly recognizing one's state of imperfect knowledge about these parameter changes, and by incorporating forthcoming sample observations in the updated parameter estimates. Incorporation of additional sample observations in the parameter estimates increases the precision of the updated estimates.

We have applied the least squares estimator to update estimates of the parameters of an equation describing annual changes in the inventory of Australian beef cows. Judging by the criteria of mean square forecasting error the updated parameter estimates were more successful than forecasts obtained from the model with non-updated parameter estimates.

Throughout the paper we have restricted the discussion to a single regression equation. Just as the least squares estimator has been extended to the estimation of parameters in multiple equation models, including simultaneous equation models, so may the updating estimator reported in this paper.

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APPENDIX: PARAMETER COVARIANCE MATRICES

Case 1. *Non-updated estimate*

	Constant	X_1	X_2	X_3
Constant ..	41,995	..	..	..
X_1	− 63,916	139,830	..	..
X_2	− 10,864	1,507	9,993	..
X_3	1,641	7,335	− 3,889	8,606

Case 2. *Updated estimate with parameter change information*

	Constant	X_1	X_2	X_3
Constant ..	41,995	..	..	..
X_1	− 70,307	175,118	..	..
X_2	− 10,864	1,659	9,993	..
X_3	1,641	8,068	− 3,889	8,606

Case 3. *Updated estimate with sample observations*

	Constant	X_1	X_2	X_3
Constant ..	24,518	..	..	..
X_1	− 33,416	84,400	..	..
X_2	− 5,755	− 6,950	8,257	..
X_3	1,398	7,375	− 3,642	8,468

Case 4. *Updated estimate with parameter change information and with sample information*

	Constant	X_1	X_2	X_3
Constant ..	23,033	..	..	..
X_1	− 29,246	84,520	..	..
X_2	− 6,079	−3,733	8,468	..
X_3	1,739	7,368	− 3,733	8,544