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PATH INTERDEPENDENCE AMONG EARLY AND LATE BLOOMERS IN A DYNAMIC HECKSCHER-OHLIN MODEL

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Abstract: The closed economy neoclassical growth model predicts convergence to a capital stock level that is independent of its initial level, suggesting that discrepancies in percapita income among the world's economies should largely disappear in the long-run. This paper shows that international trade among countries differing only in their level of initial capital is sufficient to generate long-run income differences across countries. The long-run level of capital of the country most initially endowed with capital is shown to exceed the level of capital otherwise obtained in autarchy while the country least endowed converges to a capital stock lower than would otherwise be obtained in autarchy.

Key words: International trade, Development, Multiple Equilibria

JEL Classification: O41, F43, F11

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1. Introduction

The closed economy neoclassical version of the Ramsey model predicts that the level of capital stock a country converges to is independent of initial stock. Hence, countries that differ only in their initial levels of capital stock should converge to the same steady state and consequently share common levels of income in the long run. This prediction is based on the assumption of autarchy and may not result when countries engage in international trade. It is well known that the large discrepancies in income levels among countries can be attributed to the point in time when countries started a sustained growth regime (Galor, 2005), and country integration into world markets is often identified as an important contributor to a country's ability to sustain growth, although this point of view has its skeptics (Rodriguez and Rodrik, 2001). Others have attributed discrepancies in income levels among countries to a number of factors such as policies, savings rates and technology (Acemoglu and Ventura, 2002). Whereas the literature on endogenous growth has highlighted the importance of initial conditions on long-run capital and consumption levels (e.g. Lucas (1988) and Caballé and Santos (1993)), the theory of international trade suggests (Stolper and Samuelson (1941)) that if factors of production are not equally distributed among individuals international trade will influence a country's distribution of income. Since factors of production are not equally distributed across countries one might ask: will international trade alleviate or exacerbate income differences across countries in the long run? Others have posed a similar question. For example, Atkeson and Kehoe (2000) pose the question: How does the timing of a country's development relative to that of the rest of the world affect the path of a country's development?

The literature using growth models to provide insights into growth convergence is fairly extensive. Our paper is related to the work of Chen (1992) in that we focus on two countries that are not in long-run equilibrium using a Heckscher-Ohlin like structure. He considers a two country two good dynamic Heckscher-Ohlin model. He includes a leisure-work decision choice in the utility function and finds that if consumers have relatively small discount factors, long-run income levels across countries will differ, depending on initial conditions. Ventura (1997) focuses on conditional convergence among countries using a two-sector growth model in which intermediate products are traded internationally. He shows that convergence of output per capita is affected by the elasticity of substitution between capital and labor. Acemoglu and Ventura (2002) consider a world economy consisting of a continuum of small countries which internationally trade intermediate inputs. Capital is employed in the production of intermediate products and in turn intermediates are used to produce two non-internationally traded commodities, investment and consumption goods. Countries differ in technologies, savings and economic policies. To deal with specialization they employ the Armington specification. They demonstrate that rich countries are those that have low discount factors, create incentives to invest and have better technologies. Mountford (1998) employs an overlapping generations model with two countries and shows they converge to different income levels if consumer's time rate of discount varies across countries.

Overall, our work more closely parallels that of Atkeson and Kehoe. They focus on a single country in the presence of the rest of the world that has converged to its long-run equilibrium. They find the timing of a country's development relative to the rest of the world affects the path of the country's development. A country with identical technologies and inter-temporal preferences as the rest of the world, but a country that begins the development process with a capital-labor ratio lower than that of the rest of the world –a late-bloomer–ends up with a permanently lower level of income than do early-blooming countries. By not focusing on two or more countries that remain in transition to long-run equilibrium, Atkeson and Kehoe's analysis cannot draw inferences regarding differences in countries' transition paths, nor can they infer whether these paths are unique and converge to a unique steady state.

We extend the Atkeson and Kehoe analysis by considering a world economy of two countries, neither of which have converged to their long-run equilibrium. As in Atkeson and Kehoe, the countries produce two commodities, an investment good and a perishable consumption good. Consumers have identical preferences and identical discount factors, and derive satisfaction from consuming the perishable consumption good. Hence, countries only differ in their initial capital endowment. Restricting our analysis to the no specialization case, we show that different initial capital stock endowments are sufficient to generate long-run income differences across countries. We show that in state space, the set of steady states is a ray and demonstrate that, depending on initial conditions, the economies converge to a point on this ray. In other words, different initial endowments of capital can lead to long-run differences in capital, consumption and income across countries. We find that while the steady-state world capital stock is unique, our two otherwise identical countries can converge to different levels of capital stock. We find that the early bloomer (i.e., the country with a higher initial level of capital) converges to a higher long-run level of capital stock than the late bloomer, and the level of capital of the early bloomer is higher than the level obtained in autarchy. The early bloomer enjoys a higher level of consumption and, if the capital producing sector is capital intensive, its pattern of trade is to export the capital intensive good and import the perishable consumption good. The savings rate of the early bloomer can be higher than that of the late bloomer, a difference which can persist in the long-run. These results also suggest that the role played by initial capital stocks in the many growth divergence - convergence studies, is likely more complex, and per-country capital endowments as proportion of world capital may be also relevant for analyzing growth divergence - convergence across countries.

The next section lays out the two country - two sector world economy, and derives the key results. Then, we restrict the parameters of the model so as to yield constant savings rates in each country to make more explicit and reinforce the results obtained in the previous section. To provide insights into rough orders of magnitude, an empirical model is specified and solutions to their respective transition paths are obtained for various levels of initial capital for one country relative to the other. The paths are shown for the case where both countries remain within their cone of diversification. The results show transition paths corresponding to different country levels of initial capital stock and corresponding different steady-states. The results also show that trade can generate long-run differences in capital stocks across countries of more than 20 percent, depending upon differences in initial capital stocks.

2. The 2x2x2 Economy.

The world economy consists of two countries, each of which employs capital and labor using a Cobb-Douglas constant returns to scale technology to produce a capital good Y_{xj} and a perishable-consumption good Y_{cj} that are internationally traded. Each economy consists of a representative individual. Consumers' preferences are identical across countries and are characterized by a constant elasticity of intertemporal substitution $\left(\theta^{-1}\right)$ utility function and discount future utility of consumption at rate $\rho > 0$ (also presumed identical across countries). At the beginning of time each economy j is endowed with an identical amount of labor, L_j, but their endowment of capital, $K_j(0)$, can differ. We do not consider technological change or population growth, and the holding of other country assets is not allowed.

2.1. Consumers' optimization problem.

The instantaneous utility of consumption c_j in the j-th country is given by

$$U(c_j) = \begin{cases} \frac{c_j^{1-\theta}}{1-\theta} \text{ if } \theta \neq 1 \\ \ln c_j \text{ if } \theta = 1 \end{cases} \tag{1}$$

with $\theta > 0$. The consumer of country j (for $j = 1, 2$) chooses paths of consumption $c_j(t)$, and assets $K_j(t)$ given prices to solve

$$\max \int_0^\infty U(c_j(t)) e^{-\rho t} dt \tag{2}$$

subject to

$$\dot{K}_j(t) = \frac{\left(w_{Kj}(t) - \delta p(t)\right) K_j(t) + w_{Lj}(t) L_j - c_j(t)}{p(t)}, \tag{3}$$

where the endowment of labor is normalized to unity ($L_j = 1$). The j-th country's price of capital and labor are denoted by $w_{Kj}(t)$ and $w_{Lj}(t)$, respectively, and δ is the constant and common rate of capital depreciation. Since the consumption and investment goods are traded, the price of the consumption and investment good are equal across countries. Let $p(t)$ denote the price of the investment good in terms of the price of the consumption good which is treated as the numeraire.

The first order and transversality conditions of problem (2) are given by (to avoid notational cluster, we omit expressing variables as a function of time unless needed for clarity)

$$c_j^{-\theta} = \frac{\lambda_j}{p}, \quad \lambda_j \left(\frac{w_{Kj}}{p} - \delta - \rho \right) = -\dot{\lambda}_j, \quad , \lim_{t \to \infty} \lambda_j(t) K_j(t) e^{-\rho t} = 0 \quad \text{for } j = 1, 2 \tag{4}$$

where λ_j is the co-state variable associated with constraint (3). The Euler condition of the consumer of country j is, therefore given by

$$\frac{\dot{c}_j}{c_j} = \frac{1}{\theta} \left(-\frac{\dot{\lambda}_j}{\lambda_j} + \frac{\dot{p}}{p} \right) = \frac{1}{\theta} \left(\frac{w_{Kj}}{p} + \frac{\dot{p}}{p} - \delta - \rho \right) \quad \text{for } j = 1, 2. \tag{5}$$

2.2. Firms

The technologies for producing the investment and consumption good are, respectively

$$Y_{xj} = K_{xj}^{\alpha} L_{xj}^{1-\alpha}, \qquad Y_{cj} = K_{cj}^{\beta} L_{cj}^{1-\beta} \tag{6}$$

with $0<\alpha<1$, $0<\beta<1$, and $\alpha\neq\beta$. K_{ij} and L_{ij} denote the capital and labor services employed in the production of output $i=x,c$ in country $j=1,2$. Subscripts x and c denote the investment good and consumption good, respectively. Profit maximization in each sector implies

$$\begin{aligned} K_{xj} &= \alpha\frac{pY_{xj}}{w_{Kj}}, \qquad L_{xj}=(1-\alpha)\frac{pY_{xj}}{w_{Lj}}, \\ K_{cj} &= \beta\frac{Y_{cj}}{w_{Kj}}, \qquad L_{cj}=(1-\beta)\frac{Y_{cj}}{w_{Lj}} \end{aligned} \tag{7}$$

Using (6) and (7) we obtain the zero profit condition that output price equals marginal cost given by

$$p=\frac{w_{Kj}^{\alpha}w_{Lj}^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \qquad 1=\frac{w_{Kj}^{\beta}w_{Lj}^{1-\beta}}{\beta^{\beta}(1-\beta)^{1-\beta}} \tag{8}$$

Definition.- An equilibrium are paths of quantities $c_j(t)$, $K_j(t)$, $Y_{ij}(t)$, $K_{ij}(t)$, $L_{ij}(t)$ and prices $w_{Kj}(t)$, $w_{Lj}(t)$ and $p(t)$, such that, given prices, $c_j(t)$, $K_j(t)$ solve the optimization problem (2) of the consumer of country j for $j=1,2$. Given prices $K_{ij}(t)$, $L_{ij}(t)$, for $i=x,c$ and $j=1,2$, solve the profit maximization problem of sector i in country j, and the following market clearing conditions are satisfied:

Market clearing for labor and capital in each country requires

$$L_{cj}(t)+L_{xj}(t)=L_j=1 \tag{9}$$

$$K_{cj}(t)+K_{xj}(t)=K_j(t) \tag{10}$$

The international market clearing for the consumption good is given by

$$Y_{c1}(t)+Y_{c2}(t)=c_1(t)+c_2(t)=C(t) \tag{11}$$

where $C(t)$ denotes world consumption. Market clearing for the investment good equals

$$Y_{x1}(t)+Y_{x2}(t)=x_1(t)+x_2(t) \tag{12}$$

where $x_j(t)$ equals

$$x_j(t)=\dot{K}_j(t)-\delta K_j(t) \tag{13}$$

2.3. *Solution*

In the absence of specialization, factor price equalization occurs and thus $w_{Kj} = w_K$ and $w_{Lj} = w_L$. Equation (8) defines w_K and w_L in terms of the price of the investment good. More specifically

$$w_L = \left(\frac{B^{\frac{1}{\beta}}}{A^{\frac{1}{\alpha}} p^{\frac{1}{\alpha}}} \right)^{\frac{\alpha\beta}{\alpha-\beta}}, \qquad w_K = \left(\frac{B^{\frac{1}{1-\beta}}}{A^{\frac{1}{1-\alpha}} p^{\frac{1}{1-\alpha}}} \right)^{\frac{(1-\alpha)(1-\beta)}{\beta-\alpha}} \tag{14}$$

where $A = \alpha^{\alpha} (1-\alpha)^{1-\alpha}$ and $B = \beta^{\beta} (1-\beta)^{1-\beta}$. As in the Heckscher-Ohlin model of international trade if the investment (consumption) good uses capital intensively in its production and the price of the investment good increases (declines) then the rental price of capital increases and the labor wage rate declines.

Using the market clearing condition for labor and capital, we obtain the j-th country's supply of investment and consumption goods, respectively, given by

$$Y_{xj} = \frac{\beta w_L L_j - (1-\beta) w_K K_j}{p(\beta - \alpha)}, \qquad Y_{cj} = \frac{(1-\alpha) w_K K_j - \alpha w_L L_j}{\beta - \alpha} \tag{15}$$

Market clearing in the consumption good implies

$$C = \frac{(1-\alpha) w_K (K_1 + K_2) - 2\alpha w_L (L_1 + L_2)}{\beta - \alpha} \tag{16}$$

which implicitly defines the price of the investment good p. Equation (16) and

$$\dot{c}_1 = \frac{1}{\theta} \left(\frac{w_K}{p} + \frac{\dot{p}}{p} - \delta - \rho \right) c_1, \tag{17}$$

$$\dot{c}_2 = \frac{1}{\theta} \left(\frac{w_K}{p} + \frac{\dot{p}}{p} - \delta - \rho \right) c_2, \tag{18}$$

$$\dot{K}_1 = \frac{(w_K - \delta p) K_1 + w_L - c_1}{p}, \tag{19}$$

$$\dot{K}_2 = \frac{(w_K - \delta p) K_2 + w_L - c_2}{p}, \tag{20}$$

form a system of four differential equations and one static equation in five variables c_1, c_2, K_1, K_2 and p which together determine an equilibrium solution. Note that w_K, w_L can be substituted out employing (14).

2.4. Steady state

If a steady-state exists within each country's cone of diversification, the Euler condition of both countries implies

$$\frac{1}{\theta}\left(\frac{w_K^*}{p^*}-\delta-\rho\right)=0 \quad \Rightarrow \quad \frac{w_K^*}{p^*}\equiv r^*=\delta+\rho \tag{21}$$

where $*$ denotes steady state values. We denote the ratio $\frac{w_K^*}{p^*}$ as the rental rate of capital r^*. Using (8), and setting $p^*=\frac{w_K^*}{r^*}$, the steady state rental price of capital and labor wage rate are, respectively, given by

$$w_K^*=B\left(\frac{\delta+\rho}{A}\right)^{\frac{1-\beta}{1-\alpha}}, \qquad w_L^*=B\left(\frac{A}{\delta+\rho}\right)^{\frac{\beta}{1-\alpha}} \tag{22}$$

The price of the investment good at the steady state is

$$p^*=\frac{(w_K^*)^{\alpha}(w_L^*)^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}=\frac{B}{A^{\frac{1-\beta}{1-\alpha}}}(\delta+\rho)^{\frac{\alpha-\beta}{1-\alpha}} \tag{23}$$

Setting (19) and (20) equal to zero, solving for c_1 and c_2, and substituting c_1 and c_2 into (16) gives the steady state levels of capital in country one and two as the combinations of capital stocks K_1^* and K_2^* satisfying the following equation

$$K_2^*=\frac{\beta w_L^* L}{((1-\alpha)r^*-(\beta-\alpha)\rho)p^*}-K_1^* \tag{24}$$

where L is world labor supply $(L=L_1+L_2=2)$. If these economies were closed, they converge to the same steady state. In the presence of trade, values for K_1^* and K_2^* satisfying equation (24) determine a steady state solution.

The steady state level of consumption in each country equals total factor returns,

$$c_1^* = \rho p^* K_1^* + w_L^*, \tag{25}$$

$$c_2^* = \rho p^* K_2^* + w_L^* \tag{26}$$

Clearly, if $K_1^* \neq K_2^*$, then consumption across countries differ in the steady state.

The steady-state levels of production in each country are given by

$$Y_{xj}^{*}=\frac{\beta w_L^{*}-(1-\beta)\,w_K^{*}K_j^{*}}{p\,(\beta-\alpha)},\qquad Y_{cj}^{*}=\frac{(1-\alpha)\,w_K^{*}K_j^{*}-\alpha w_L^{*}}{\beta-\alpha}\text{ for }j=1,2 \tag{27}$$

Note that the steady state equilibria is a infinite connected set satisfying equation (24). Next we show that ray (24) is a self attracting manifold.

3. Convergence

To facilitate analysis of the model's convergence properties, the system is reduced to three differential equations. Using standard techniques, we show that the economies can converge to different steady-states depending on initial conditions.

3.1. Transforming and reducing the system

A difficulty one encounters when analyzing the asymptomatic properties of system (16) − (20) is that equations (17) and (18) contain the term $\frac{\dot{p}}{p}$. We, however, do not directly have information about this term. To analyze the asymptotic properties of the model it is useful to use the co-state variables and its differential equations obtained from (4) as follows[1]

$$\dot{\lambda}_1 = \left(-\frac{w_K}{p}+\delta+\rho\right)\lambda_1, \tag{28}$$

$$\dot{\lambda}_2 = \left(-\frac{w_K}{p}+\delta+\rho\right)\lambda_2, \tag{29}$$

$$\dot{K}_1 = \frac{(w_K-\delta p)\,K_1+w_L L_1}{p}-\frac{1}{p}\left(\frac{p}{\lambda_1}\right)^{\frac{1}{\theta}}, \tag{30}$$

$$\dot{K}_2 = \frac{(w_K-\delta p)\,K_2+w_L L_2}{p}-\frac{1}{p}\left(\frac{p}{\lambda_2}\right)^{\frac{1}{\theta}} \tag{31}$$

where p is implicitly defined by (16) and w_K and w_L can be substituted out using (14). Since the steady state is a manifold of dimension one at least one of the eigenvalues of the Jacobian matrix of system (28) − (31) must be zero. Indeed it is straightforward to demonstrate that the Jacobian matrix of (28) − (31) has a zero eigenvalue.

The convergence properties of the model are most easily derived by reducing the dimensionality of the dynamic system (28) − (31). Since the countries' Euler conditions are identical, the ratio of

[1] Note from (4) that $c_j=\left(\frac{p}{\lambda_j}\right)^{\frac{1}{\theta}}$.

their consumption levels c_1/c_2 is constant throughout transition to the steady state. Let this ratio equal some constant $\mu > 0$. Using $c_2 = \left(\frac{p}{\lambda_2}\right)^{\frac{1}{\theta}}$, equation (4) implies

$$c_1 + c_2 = (\mu + 1)\, c_2 = (\mu + 1) \left(\frac{p}{\lambda_2}\right)^{\frac{1}{\theta}}. \tag{32}$$

Therefore, equation (16) can be rewritten as

$$G(p, \lambda_2, K_1, K_2) = \left(\frac{\lambda_2}{p}\right)^{\frac{1}{\theta}} \frac{(1-\alpha)\, w_K (K_1 + K_2) - \alpha w_L (L_1 + L_2)}{(\beta - \alpha)(\mu + 1)} = 1 \tag{33}$$

Suppose $(p^o, \lambda_2^o, K_1^o, K_2^o)$ satisfies $G(p^o, \lambda_2^o, K_1^o, K_2^o) = 1$. Presuming differentiability, the Implicit Function Theorem implies the existence of a function $P(\lambda_2, K_1, K_2) = p$ defined on an open ball B about $(\lambda_2^o, K_1^o, K_2^o)$ such that

$$G(P(\lambda_2, K_1, K_2), \lambda_2, K_1, K_2) = 1 \text{ for all } (\lambda_2, K_1, K_2) \in B, \tag{34}$$

and

$$\frac{\partial P(\lambda_2^o, K_1^o, K_2^o)}{\partial \chi} = -\frac{G_\chi (p^o, \lambda_2^o, K_1^o, K_2^o)}{G_p (p^o, \lambda_2^o, K_1^o, K_2^o)} \quad \text{for} \quad \chi = \lambda_2, K_1, K_2 \tag{35}$$

where $G_\chi (p^o, \lambda_2^o, K_1^o, K_2^o)$ denotes the partial derivative of G with respect to χ evaluated at $(p^o, \lambda_2^o, K_1^o, K_2^o)$ satisfying $G(p^o, \lambda_2^o, K_1^o, K_2^o) = 1$. Using $\frac{c_1}{c_2} = \mu$ and $c_2 = \left(\frac{p}{\lambda_2}\right)^{\frac{1}{\theta}}$ we can therefore reduce the equilibrium conditions to a system of three differential equations in three *dynamic* variables λ_2, K_1, and K_2 as follows:

$$\begin{pmatrix} \dot{\lambda}_2 \\ \dot{K}_1 \\ \dot{K}_2 \end{pmatrix} = \begin{pmatrix} (-w_K/P(\lambda_2, K_1, K_2) + \delta + \rho)\, \lambda_2 \\ ((w_K - \delta p) K_1 + w_L L_1)/P(\lambda_2, K_1, K_2) - \mu \left(P(\lambda_2, K_1, K_2)^{1-\theta}/\lambda_2\right)^{\frac{1}{\theta}} \\ ((w_K - \delta p) K_2 + w_L L_2)/P(\lambda_2, K_1, K_2) - \left(P(\lambda_2, K_1, K_2)^{1-\theta}/\lambda_2\right)^{\frac{1}{\theta}} \end{pmatrix} \tag{36}$$

where $P(\lambda_2, K_1, K_2)$ is implicitly defined by (33) and as before w_K, w_L can be replaced by using (14).

3.2. Convergence properties

From Li et al. (2003), we know that the system (36), a system of a lower dimension, preserves the dynamic properties of (28) – (31). Note that any combination of the state variables $(K_1,\ K_2)$ satisfying (24) constitutes a steady-state equilibrium. We now demonstrate that the ray (24) is saddle path stable. This implies that different initial conditions of the state-variables $(K_1(0),\ K_2(0))$ may lead to a different steady state.

These results lead to two claims.

Claim 1. *Different initial conditions of the state variables* K_1 *and* K_2 ***can*** *asymptotically lead to different steady state values of* (K_1^*, K_2^*) *whose sum satisfies a unique world value* $K^* = K_1^* + K_2^*$.

Claim 2. *At each steady state* $(K_1^*,\ K_2^*)$ *there is a neighborhood containing a one-dimensional manifold convergent to a point on the ray of steady states at* $(K_1^*,\ K_2^*)$ *such that for all initial conditions* $(K_1(0),\ K_2(0))$ *of this one-dimensional manifold, the equilibrium path converges exponentially to* $(K_1^*,\ K_2^*)$.

Proposition 1. (Convergence) *The Jacobian matrix of* (36) *evaluated at a steady state has a negative eigenvalue and two positive eigenvalues.*

Proof. See appendix A■

Since the set of steady states is a ray of dimension one (24) and the Jacobian matrix of (36) has a negative eigenvalue and two positive eigenvalues, from Li et al. (2003) it follows that claims 1 and 2 hold. Provided initial conditions are in the neighborhood of a steady states it also follows that for each steady state $(K_1^*,\ K_2^*)$ there is a unique path with different initial conditions converging to $(K_1^*,\ K_2^*)$. Note that a steady state is not stable in the sense that if we disturb this steady state by providing more capital to one of the economies the countries converge to a different steady state.

Corollary. Countries that start with different initial endowments of capital end up at different income levels.

Proposition 2 *The gap between the consumption shares of the two countries at any point in time* t *is determined by the difference in initial capital endowments.*

Proof. Since consumption grows at equal rates across countries, the share of per country consumption in aggregate consumption, $C(t) = c_1(t) + c_2(t)$, is constant,

$$\phi_j = \frac{c_j(t)}{C(t)} \tag{37}$$

Integrating each country's budget constraint and employing the transversality condition we obtain

$$\phi_1 - \phi_2 = \frac{p(0)(K_1(0) - K_2(0))}{C(0)^{\theta} \lim\limits_{t\to\infty} \int_0^t C(\tau)^{1-\theta} e^{-\rho\tau} d\tau} \blacksquare \tag{38}$$

If the utility function is logarithmic, $U(c_j) = \ln c_j$, the difference in consumption across countries at time zero equals

$$c_1(0) - c_2(0) = \rho p(0)(K_1(0) - K_2(0)) \tag{39}$$

Another implication of equation (39) is that the more impatient countries are, (ρ large), the larger the consumption of the country with the largest initial stock of capital. Thus, if $K_1(0) \neq K_2(0)$ consumption across countries will differ transitionally and at the steady state. The country that starts with the largest capital endowment will for ever enjoy larger consumption levels. The consumption ratio of country one and two is given by

$$\frac{\phi_1}{\phi_2} = \mu = \frac{K_1(0)C(0)^{-\theta} p(0) + \int_0^{\infty} C(\tau)^{-\theta} w_L(\tau) e^{-\rho\tau} d\tau}{K_2(0)C(0)^{-\theta} p(0) + \int_0^{\infty} C(\tau)^{-\theta} w_L(\tau) e^{-\rho\tau} d\tau} \tag{40}$$

Corollary. With international trade, the country that starts with the smallest initial capital endowment will have a steady-state consumption level smaller than its steady-state autarchy consumption level.

Proposition 3 *The gap between the level of* countries' *capital stock in the steady state is determined by the difference in initial capital endowments.*

Proof. Substituting (25) – (26) into (38) we obtain

$$K_1^* - K_2^* = \frac{p(0)}{\rho p^*} \frac{C^*}{C(0)^{\theta}} \left(\frac{K_1(0) - K_2(0)}{\int_0^{\infty} C(\tau)^{1-\theta} e^{-\rho\tau} d\tau} \right) \tag{41}$$

where C^* denotes world consumption at the steady state and equals

$$C^* = \left(\frac{(1-\alpha)\,\delta + \rho}{(1-\alpha)\,\delta + (1-\beta)\,\rho} \right) w_L^* L \blacksquare \tag{42}$$

It follows that the country with the largest initial endowment of capital converges to a steady-state capital stock that is larger than the other country. When compared to the steady-state level of capital in autarchy, the country that started with the largest initial endowment of capital will surpass the autarchy level of capital, while the other country will converge to a lower capital stock level that the level that obtains in the case of autarchy.

The differences in country capital stocks imply differences in the pattern of production. From (15) it is straightforward to verify that if production of the investment good is relatively capital intensive, $\alpha > \beta$, then the country with the largest initial stock of capital produces larger amounts of the investment good relative to the other country, while the other produces a larger amount of the consumption good.

Next we look at an analytical solution to the model under a restriction in parameter values[2]. In particular, this restriction in parameter values for the closed economy two-sector Ramsey model leads to the two-sector model by Uzawa (1963) of constant savings rates (as in the Solow model (1956)).

3.3. Analytical solution

Consider a particular solution of the model presented in Section 2.

Proposition 4. *If*

$$\rho = \delta\,(\alpha + \beta\,(\theta - 1)) - \delta, \qquad \alpha + \beta\,(\theta - 1) > 1 \tag{43}$$

then the world has a constant savings rate out of world income equal to $s = \frac{1}{\theta}$.[3]

Proof. See Appendix B.

Proposition 5. *Let* $K(0)$ *denote the world's initial capital endowment (*$K(0) = K_2(0) + K_1(0)$*), and let* $\kappa_j(0) = K_j(0)/K(0)$ *. For* (43) *, the share of each country's consumption in total*

[2] $\rho = \delta\,(\alpha + \beta\,(\theta - 1)) - \delta$

[3] Note that $\alpha + \beta\,(\theta - 1) > 1$ requires $\theta > 1$.

consumption is given by

$$\begin{aligned}\phi_j &= \left(\beta - \frac{1-\alpha}{\theta-1}\right)\left(\kappa_j(0) - \frac{L_j}{L}\right) + \frac{L_j}{L} \qquad (44)\\ &= \left(\frac{\rho}{\delta}\frac{1}{\theta-1}\right)\left(\kappa_j(0) - \frac{1}{2}\right) + \frac{1}{2}\end{aligned}$$

Proof. See Appendix C.■

Since $\theta > 1$ must hold under (43), the term $\left(\frac{\rho}{\delta}\frac{1}{\theta-1}\right)$ is positive. Because we have assumed that each country has the same endowment of labor, $\frac{L_j}{L} = \frac{1}{2}$, it follows from (44) that the country with the largest proportion of initial capital ($\kappa_j(0) > 1/2$) will have a share in aggregate consumption of at least 1/2. This country then benefits proportionally more than the other country from the larger initial capital endowment, and the relative benefit depends upon the magnitude of $\left(\beta - \frac{1-\alpha}{\theta-1}\right) = \frac{\rho}{\delta}\frac{1}{\theta-1} > 0$.

The restriction (43) also implies other regularities in the evolution a country's share of capital stock and savings out of income.

Proposition 6. *Given* (43), *the ratio* $\kappa_j(t) = K_j(t)/K(t)$ *is constant for all* t *and*

$$\kappa_j = \frac{K_j(t)}{K(t)} = \frac{K_j(0)}{K(0)} \qquad (45)$$

Proof. See Appendix C.■

The restriction (43) implies a constant world saving rate, but not necessary identical saving rates across countries. Let $s_j(t)$ denote the savings rate out of income for the j-th country.

Proposition 7. *Given* (43), *the* $j-th$ *country has a constant savings rate, and moreover, if* $K_1(0) \neq K_2(0)$ *then*

$$s_1 \neq s_2. \qquad (46)$$

and the rate s_j *is given by*

$$s_j = \frac{\phi_j \frac{1-\eta}{1-\beta} - \frac{1}{L}}{\phi_j \frac{1-\eta}{1-\beta}(\alpha + \beta(\theta-1)) - \frac{1}{L}} \quad \text{for } j = 1, 2 \qquad (47)$$

where

$$\eta = \frac{(1-\alpha)}{(1-\alpha)+(1-\beta)(\theta-1)} \tag{48}$$

is the share of world labor employed in the production of the investment good.[4]

Proof. See Appendix C.■

Consequently, (47) indicates that the country with the largest ϕ_j (larger initial capital stock) will also have a larger saving rate.[5] This result is in contrast to the literature showing that different savings rates are the result of differences in discount factors across countries (Mountford, 1998, Acemoglu and Ventura, 2002).

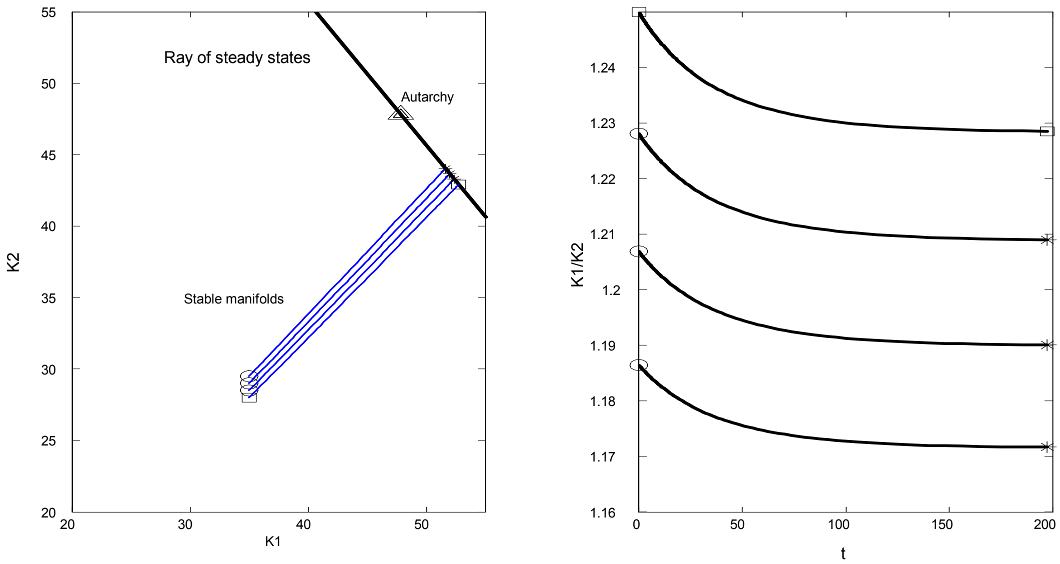
4. Some simulations

For illustrative purposes, we obtain numerical solutions to the model assuming the following parameter values:

$$\delta = 02, \quad \rho = .02, \quad \theta = 2, \quad \beta = .3, \quad \alpha = .45 \tag{49}$$

This solution is not the solution generating constant savings rates saving rates.

Figure 1



[4]See appendix B.

[5] $\frac{\partial s_j}{\partial \phi_j} = \frac{(\alpha+\beta(\theta-1)-1)\frac{1-\eta}{1-\beta}}{\left(\phi_j \frac{1-\eta}{1-\beta}(\alpha+\beta(\theta-1))-\frac{1}{L}\right)^2}\frac{1}{L}$

In Figure 1, (left diagram) displays a phase diagram of the country capital stocks for the case of four different initial capital stock endowments of country two (the y axis), holding the initial endowment of country one (the x axis) constant. The initial conditions are shown by o. The symbol □ is used to identify the base simulation. Steady-state values are denoted by *. For comparison purposes, the triangle (△) shows the autarchy level of steady-state capital stock. We have also plotted the ray of steady states.

Figure 1confirms the proposition that the economies converge to different steady states depending on initial condition. In particular, the country that stars with the largest capital stock also converges to a capital stock that is larger than the other country. The diagram on the right plots the ratio of the evolution of capital stock in country one to the stock of capital in country two. Countries remain within their cone of diversification so that specializations does not take place (see Figure 2). The right diagram in Figure 1 shows that the model with trade can generate long-run differences in capital stocks across countries of more than 20 percent.

Figure 2 Capital and Production

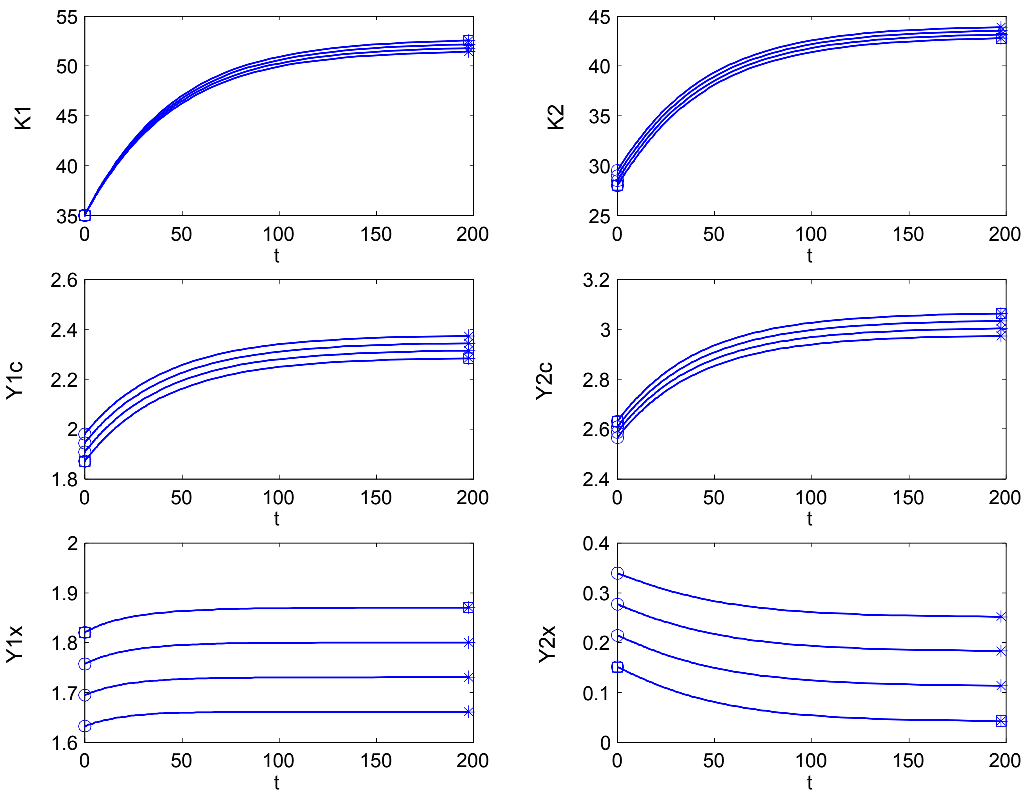
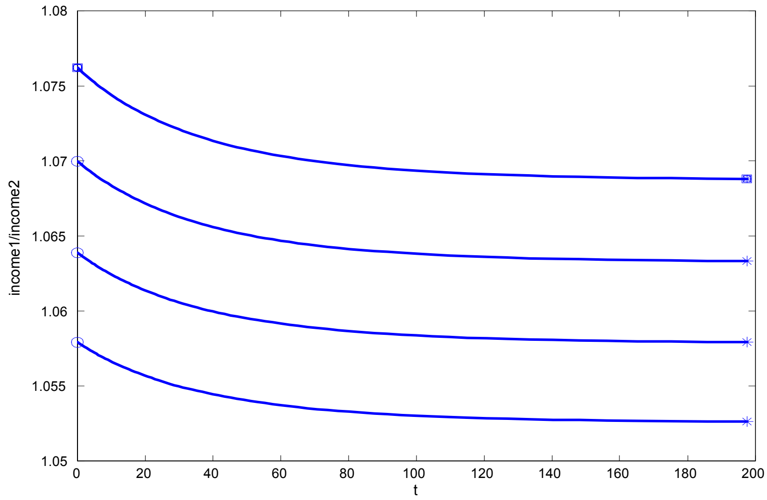


Figure 3 plots income ratios for the four simulations performed. Note that long-run income differences range from about 5 to 7 percent.

Table 3 Income ratio



5. Conclusions

Large discrepancies in income levels among countries can be attributed, in part, to when they started the growth process. Those starting the process later, the late-bloomers, are typically characterized as having lower capital to output ratios than the early-bloomers. The closed economy neoclassical version of the Ramsey model predicts that a county's long run equilibrium is independent of its initial capital. If countries are integrated, according to our findings, they may not converge to similar capital and percapita income levels in the long run. We consider a world of two open and competitive economies that produce an investment good and a perishable good, and only differ in the level of their initial capital stock. The early-bloomer (the country whose initial capital stock exceeds that of the other country) is shown to converge not only to a higher level of capital than the late-bloomer, but also to an amount that exceeds its autarchy level. In particular, we show that in state space the set of steady states is a ray and demonstrate that, depending on initial conditions, the economies converge to a point on this ray. In other words, different initial endowments of capital can lead to long-run differences in capital, consumption and income across countries. These and corollary results are shown for the general case, and more specifically for the case where the model's parameters are restricted to allow a constant savings rate. The last section of the paper further demonstrates these results with an empirical example.

We leave for future research the study of whether or not allowing for international borrowing and lending leads to equalization of income cross countries.

6. Appendix A

The Jacobian matrix of (36) equals

$$J^* = \begin{pmatrix} \Lambda_1 \frac{\partial p\left(\lambda_2^*, K_1^*, K_2^*\right)}{\partial \lambda_2} & \Lambda_1 \frac{\partial p\left(\lambda_2^*, K_1^*, K_2^*\right)}{\partial K_1} & \Lambda_1 \frac{\partial p\left(\lambda_2^*, K_1^*, K_2^*\right)}{\partial K_2} \\ H_1 \frac{\partial p\left(\lambda_2^*, K_1^*, K_2^*\right)}{\partial \lambda_2} + \frac{1}{\theta} \frac{1}{p^*} \frac{c_1^*}{\lambda_2^*} & H_1 \frac{\partial p\left(\lambda_2^*, K_1^*, K_2^*\right)}{\partial K_1} + \rho & H_1 \frac{\partial p\left(\lambda_2^*, K_1^*, K_2^*\right)}{\partial K_2} \\ H_2 \frac{\partial p\left(\lambda_2^*, K_1^*, K_2^*\right)}{\partial \lambda_2} + \frac{1}{\theta} \frac{1}{p^*} \frac{c_2}{\lambda_2} & H_2 \frac{\partial p\left(\lambda_2^*, K_1^*, K_2^*\right)}{\partial K_1} & H_2 \frac{\partial p\left(\lambda_2^*, K_1^*, K_2^*\right)}{\partial K_2} + \rho \end{pmatrix} \tag{50}$$

where

$\Lambda_1 = \frac{1-\alpha}{\beta-\alpha} \frac{\lambda_2^*}{p^*} \frac{w_K^*}{p^*}$,

$H_1 = -\frac{1}{(p^*)^2} \left(\frac{(1-\alpha) w_K^* K_1^* - \alpha w_L^* L_1}{(\beta-\alpha)} + \frac{1-\theta}{\theta} c_1^* \right) = -\frac{1}{(p^*)^2} \left(Y_{c1}^* + \frac{1-\theta}{\theta} c_1^* \right)$

$H_2 = -\frac{1}{(p^*)^2} \left(\frac{(1-\alpha) w_K^* K_2^* - \alpha w_L^* L_2}{(\beta-\alpha)} + \frac{1-\theta}{\theta} c_2^* \right) = -\frac{1}{(p^*)^2} \left(Y_{c2}^* + \frac{1-\theta}{\theta} c_2^* \right)$

and the derivatives of p are given by (35). Note that since $\frac{\partial p\left(\lambda_2^*, K_1^*, K_2^*\right)}{\partial K_1} = \frac{\partial p\left(\lambda_2^*, K_1^*, K_2^*\right)}{\partial K_2}$ the Jacobian matrix (50) can be written as

$$J^* = \begin{pmatrix} J_{11}^* & J_{12}^* & J_{12}^* \\ J_{21}^* & J_{23}^* + \rho & J_{23}^* \\ J_{31}^* & J_{32}^* & J_{32}^* + \rho \end{pmatrix} \tag{51}$$

where J_{ih}^* is the row i and column h element of matrix (50). One can readily verify that any matrix as the one indicated in (51) has an eigenvalue equal to ρ. Let ε be an eigenvalue of matrix (51), the characteristic equation of (51) equals

$$\begin{aligned} & \left(J_{11}^* - \varepsilon\right) \left[\left(J_{23}^* + \rho - \varepsilon\right) \left(J_{32}^* + \rho - \varepsilon\right) - J_{23}^* J_{32}^* \right] \qquad (52) \\ & - \underbrace{J_{12}^* J_{21}^* (J_{32}^*}_{a} + \rho - \varepsilon) + \underbrace{J_{12}^* J_{23}^* J_{31}^*}_{b} + \underbrace{J_{12}^* J_{21}^* J_{32}^*}_{a'} - \underbrace{J_{12}^* J_{31}^* (J_{23}^*}_{b'} + \rho - \varepsilon) \\ = \; & \left(J_{11}^* - \varepsilon\right) \left[\left(J_{23}^* + \rho - \varepsilon\right) \left(J_{32}^* + \rho - \varepsilon\right) - J_{23}^* J_{32}^* \right] - J_{12}^* J_{21}^* \left(\rho - \varepsilon\right) - J_{12}^* J_{31}^* \left(\rho - \varepsilon\right). \end{aligned}$$

The terms a, and $a\prime$ and b and $b\prime$, respectively cancel each other. By inspection it is easy to see that if $\varepsilon = \rho$ then (52) equals zero and therefore $\varepsilon = \rho$ is one of the eigenvalues of matrix (51) which is positive.

Further, note that

$$H_1 + H_2 = -\frac{\left(Y_{1c}^* + \frac{1-\theta}{\theta} c_1^*\right)}{(p^*)^2} - \frac{\left(Y_{2c}^* + \frac{1-\theta}{\theta} c_2^*\right)}{(p^*)^2} = -\frac{C^*}{\theta (p^*)^2} \tag{53}$$

where $C^* = c_1^* + c_2^*$ and

$$\frac{\partial G}{\partial K_2} = G_{K_2}(p^*, \lambda_2^*, K_1^*, K_2^*) = \left(\frac{\lambda_2^*}{p^*}\right)^{\frac{1}{\theta}} \frac{(1-\alpha) w_K^*}{(\beta - \alpha)(\mu + 1)} = \frac{(1-\alpha) w_K^*}{(\beta - \alpha) C^*} \tag{54}$$

and

$$\frac{\partial G}{\partial \lambda_2} = G_{\lambda_2}(p^*, \lambda_2^*, K_1^*, K_2^*) = \frac{1}{\theta}\frac{1}{\lambda_2^*} \tag{55}$$

Using (53) – (55) we obtain

$$-(H_1 + H_2) G_{K_2} = \frac{C^*}{\theta (p^*)^2} G_{K_2} = \frac{1}{\theta}\frac{w_K^*}{(p^*)^2}\frac{1-\alpha}{\beta - \alpha} = G_{\lambda_2}\Lambda_1$$

$$\Rightarrow \quad \Lambda_1 G_{\lambda_2} = \frac{C^*}{\theta (p^*)^2} G_{K_2} \tag{56}$$

Using (53) – (56) the Jacobian matrix (50) can be rewritten as

$$J^* = \begin{pmatrix} \frac{C^*}{\theta (p^*)^2}\frac{\partial p}{\partial K_2} & \Lambda_1 \frac{\partial p}{\partial K_2} & \Lambda_1 \frac{\partial p}{\partial K_2} \\ \frac{C^*}{\Lambda_1 \theta (p^*)^2}\left(H_1 \frac{\partial p}{\partial K_2} + G_{K_2}\frac{c_1^*}{p^*}\right) & H_1 \frac{\partial p}{\partial K_2} + \rho & H_1 \frac{\partial p}{\partial K_2} \\ \frac{C^*}{\Lambda_1 \theta (p^*)^2}\left(H_2 \frac{\partial p}{\partial K_2} + G_{K_2}\frac{c_2^*}{p^*}\right) & H_2 \frac{\partial p}{\partial K_2} & H_2 \frac{\partial p}{\partial K_2} + \rho \end{pmatrix} \tag{57}$$

Next we find the determinant of the Jacobian matrix (we first operate on matrix (51) to avoid notational cluster)

$$\begin{aligned}
\det J^* &= J_{11}^*\left[\left(J_{23}^*+\rho\right)\left(J_{32}^*+\rho\right)-J_{23}^*J_{32}^*\right]-\underbrace{J_{12}^*J_{21}^*\left(J_{32}\right.}_{a}+\rho)+\underbrace{J_{12}^*J_{23}^*J_{31}^*}_{b} \qquad (58)\\
&\quad +\underbrace{J_{12}^*J_{21}^*J_{32}^*}_{a'}-\underbrace{J_{12}^*J_{31}^*\left(J_{23}^*\right.}_{b'}+\rho)\\
&= J_{11}^*\left[\left(J_{23}^*+\rho\right)\left(J_{32}^*+\rho\right)-J_{23}^*J_{32}^*\right]-J_{12}^*J_{21}^*\rho-J_{12}^*J_{31}^*\rho\\
&= \rho\left(J_{11}^*\left(J_{23}^*+J_{32}^*+\rho\right)-J_{12}^*\left(J_{21}^*+J_{31}^*\right)\right)
\end{aligned}$$

Making appropriate substitutions from (57) and simplifying using (53) we obtain

$$\det J^* = \frac{\partial p}{\partial K_2}\frac{\rho C^*}{\theta\left(p^*\right)^2}\left(\rho-\frac{G_{K_2}}{p^*}C^*\right) \qquad (59)$$

Substituting for G_{K_2} from (54) we get

$$\begin{aligned}
\det J^* &= \frac{\partial p}{\partial K_2}\frac{\rho C^*}{\theta\left(p^*\right)^2}\left(\rho-\frac{(1-\alpha)}{(\beta-\alpha)}\underbrace{\frac{w_K^*}{p^*}}_{r^*}\right) && (60)\\
&= \frac{\partial p}{\partial K_2}\frac{\rho}{\theta}\frac{C^*}{\left(p^*\right)^2}\left(\rho-\frac{(1-\alpha)}{(\beta-\alpha)}(\delta+\rho)\right) && (61)\\
&= -\frac{G_{K_2}}{G_p}\frac{\rho}{\theta}\frac{C^*}{\left(p^*\right)^2}\left(\frac{\rho\left(\beta-\alpha\right)-(1-\alpha)(\delta+\rho)}{(\beta-\alpha)}\right) && (62)\\
&= \frac{1}{G_p}\left(\frac{1-\alpha}{(\beta-\alpha)^2}\right)\frac{\rho}{\theta}\frac{w_K^*}{\left(p^*\right)^2}\left(\delta\left(1-\alpha\right)+\rho\left(1-\beta\right)\right) && (63)
\end{aligned}$$

Therefore, the sign of $\det J^*$ depends on the sign of G_p which equals

$$\frac{\partial G}{\partial p} = -\frac{1}{\theta}\frac{1}{p^*}-\frac{1}{C^*}\underbrace{\left(\frac{(1-\alpha)(1-\beta)}{(\beta-\alpha)^2}\frac{w_K^*}{p^*}\left(K_1^*+K_2^*\right)+\frac{\alpha\beta}{(\beta-\alpha)^2}\frac{w_L^*}{p^*}\left(L_1+L_2\right)\right)}_{>0} \qquad (64)$$

which is trivially negative. Since the determinant of a matrix equals the product of its eigenvalues, and since $\det J^* < 0$ and by (52) one of the eigenvalues of J^* equals $\rho > 0$, then it must be the case that the other two eigenvalues have opposite signs.■

7. Appendix B

To show that the model present in section 2 with the restrictions (43) generates constant savings rates we first solve the closed economy Uzawa (1963) model of constant savings rates and then demonstrate that the model presented in section two and the model with constant savings rates are equivalent under the given restriction in parameter values.

Uzawa-Solow model closed economy.

We now solve the two commodity closed economy model were consumption and savings are a constant fraction of total income as in Uzawa (1963). Consider an economy with production functions as the ones specified in (6) . To maximize profits firms set (we have dropped the country subscript as we are, at the moment, solving for the case of a closed economy)

$$w_K^{Uz} = \alpha p^{Uz} \frac{Y_x^{Uz}}{K_x^{Uz}} = \beta \frac{Y_c^{Uz}}{K_c^{Uz}}, \qquad w_L^{Uz} = (1-\alpha)\, p^{Uz} \frac{Y_x^{Uz}}{L_x^{Uz}} = (1-\beta)\, \frac{Y_c^{Uz}}{L_c^{Uz}} \tag{65}$$

We use the superscript Uz to distinguish this model from the model presented in Section 2. Define income Y^{Uz} as labor income plus capital, which equals

$$Y^{Uz} = w_L^{Uz} L^{Uz} + w_K^{Uz} K^{Uz} \tag{66}$$

where L^{Uz} is the labor endowment of the economy, and K^{Uz} is total capital stock of the economy. Following Uzawa where a constant fraction of income is saved and a constant fraction of income is used for consumption, we have

$$C^{Uz} = (1-s)\, Y^{Uz}, \qquad sav^{Uz} = sY^{Uz} \tag{67}$$

where C^{Uz} denotes consumption, sav^{Uz} stand for savings and s is a constant satisfying $0 < s < 1$. We now set sav^{Uz} to be equal to the value of investment so that $sav^{Uz} = p^{Uz} Y_x^{Uz}$ and use the market clearing condition for the consumption good so that $C^{Uz} = Y_c^{Uz}$, (65) becomes

$$w_K^{Uz} = \frac{\alpha s Y^{Uz}}{K_x^{Uz}} = \frac{\beta\,(1-s)\, Y^{Uz}}{K_c^{Uz}}, \qquad w_L^{Uz} = \frac{(1-\alpha)\, sY^{Uz}}{L_x^{Uz}} = \frac{(1-\beta)\,(1-s)\, Y^{Uz}}{L_c^{Uz}} \tag{68}$$

Let $0 < \nu < 1$ be the (possible variable) fraction of total capital used in the production of the investment good so that $K_x^{Uz} = \nu K^{Uz}$, and let η be the fraction of total labor employed in the production of the investment good, (68) becomes

$$w_K^{Uz} = \frac{\alpha}{\nu}\frac{p^{Uz}Y_x^{Uz}}{K^{Uz}} = \alpha\left(\frac{s}{\nu}\right)\frac{Y^{Uz}}{K^{Uz}} = \beta\left(\frac{1-s}{1-\nu}\right)\frac{Y^{Uz}}{K^{Uz}}, \tag{69}$$

$$w_L^{Uz} = (1-\alpha)\left(\frac{s}{\eta}\right)\frac{Y^{Uz}}{L^{Uz}} = (1-\beta)\left(\frac{1-s}{1-\eta}\right)\frac{Y^{Uz}}{L^{Uz}} \tag{70}$$

Since s is constant (69) implies that v is also a constant and equals

$$\alpha\left(\frac{s}{\nu}\right) = \beta\left(\frac{1-s}{1-\nu}\right) \qquad \Rightarrow \qquad \nu = \frac{\alpha s}{\alpha s + \beta(1-s)} \tag{71}$$

Note that the fraction of capital employed in the production of the investment good ν is larger the larger the capital intensity of the investment good is[6]. Similarly, η is also a constant equal to

$$(1-\alpha)\left(\frac{s}{\eta}\right) = (1-\beta)\left(\frac{1-s}{1-\eta}\right) \qquad \Rightarrow \qquad \eta = \frac{(1-\alpha)s}{(1-\alpha)s + (1-\beta)(1-s)}. \tag{72}$$

Since equation (71) indicates that the amount of capital K_x^{Uz} used in the production of the investment good is a constant fraction of total capital. Using the equation of motion of capital, similar to the one specified in (13) we get

$$\dot{K}^{Uz} = Y_x^{Uz} - \delta K^{Uz} = \left(\nu K^{Uz}\right)^{\alpha}(\eta L)^{1-\alpha} - \delta K^{Uz} \tag{73}$$

Note that (73) is a Bernoulli differential equation that can be solved analytically to obtain the path of capital K^{Uz} as follows

$$K^{Uz}(t) = \left(\left(K(0)^{1-\alpha} - \frac{\nu^{\alpha}(\eta L)^{1-\alpha}}{\delta}\right)e^{-(1-\alpha)\delta t} + \frac{\nu^{\alpha}(\eta L)^{1-\alpha}}{\delta}\right)^{\frac{1}{1-\alpha}} \tag{74}$$

Taking the limit of (74) as t goes to infinity one obtains the result that capital converges in the long run to

$$K^{Uz*} = \left(\frac{\nu^{\alpha}}{\delta}\right)^{\frac{1}{1-\alpha}}\eta L \tag{75}$$

[6] $\frac{\partial \nu}{\partial \alpha}\frac{\alpha}{\nu} = v\frac{\beta}{\alpha}\frac{1-s}{s}$

Using the market clearing condition for consumption, we obtain the consumption path

$$C^{Uz}(t) = Y_C^{Uz}(t) = \left((1-\nu) K^{Uz}(t)\right)^{\beta} \left((1-\eta) L\right)^{1-\beta} \tag{76}$$

and therefore consumption growths at the rate

$$\frac{\dot{C}^{Uz}}{C^{Uz}} = \beta \frac{\dot{K}^{Uz}}{K^{Uz}} \tag{77}$$

We now solve for the price of the investment good and labor wage rate in transition. Since $K_x^{Uz} = \nu K^{Uz}$ then

$$\frac{Y_x^{Uz}}{K_x^{Uz}} = \frac{\left(\nu K^{Uz}\right)^{\alpha} (\eta L)^{1-\alpha}}{\nu K^{Uz}} \tag{78}$$

From the optimality conditions, it follows that

$$\alpha p^{Uz} Y_x^{Uz} / K_x^{Uz} = w_K^{Uz} \Rightarrow \alpha p^{Uz} \left(\nu K^{Uz}\right)^{\alpha} (\eta L)^{1-\alpha} / \left(\nu K^{Uz}\right) = w_K^{Uz} \tag{79}$$

Employing (14) we obtain

$$w_K^{Uz} = \alpha p^{Uz} \frac{\left(\nu K^{Uz}\right)^{\alpha} (\eta L)^{1-\alpha}}{\nu K^{Uz}} \tag{80}$$

Therefore, the price of the investment and the labor wage rate from (14) equal

$$\begin{aligned} p^{Uz}(t) &= B \left(\frac{\alpha^{\alpha-\beta}}{A^{1-\beta}}\right)^{\frac{1}{1-\alpha}} \left(\frac{\eta L}{\nu K^{Uz}(t)}\right)^{\alpha-\beta}, \\ w_L^{Uz}(t) &= \left(\frac{B^{\frac{1}{\beta}}}{A^{\frac{1}{\alpha}} p^{Uz}(t)^{\frac{1}{\alpha}}}\right)^{\frac{\alpha\beta}{\alpha-\beta}} = B \left(\frac{A}{\alpha}\right)^{\frac{\beta}{1-\alpha}} \left(\frac{\nu K^{Uz}(t)}{\eta L}\right)^{\beta} \end{aligned} \tag{81}$$

Ramsey-Uzawa-Solow equivalence.

We now show that the Uzawa-Solow two-sector model and the two-sector model with endogenous savings rate are equivalent when

$$\rho = \delta\left(\alpha + \beta(\theta - 1)\right) - \delta, \tag{82}$$

$$v = \frac{\alpha}{\alpha + \beta(\theta - 1)}^{7} \tag{83}$$

and the restriction

$$\alpha + \beta(\theta - 1) > 1 \tag{84}$$

holds where we require $\theta > 1$.

We denote with the superscript Uz the derivations that come from the Uzawa-Solow type model. From (69), income Y^{Uz} equals $\frac{\nu}{s}\frac{w_K^{Uz}}{\alpha}K^{Uz}$. Therefore $\frac{\dot{Y}^{Uz}}{Y^{Uz}} = \frac{\dot{w}_K^{Uz}}{w_K^{Uz}} + \frac{\dot{K}^{Uz}}{K^{Uz}}$. Given that $C^{Uz} = (1-s)Y^{Uz}$, the following rates of change are equal

$$\frac{\dot{C}^{Uz}}{C^{Uz}} = \frac{\dot{Y}^{Uz}}{Y^{Uz}} = \frac{\dot{w}_K^{Uz}}{w_K^{Uz}} + \frac{\dot{K}^{Uz}}{K^{Uz}} \tag{85}$$

Since profits are maximized under the Uzawa-Solow setting, equation (14) also holds and, therefore

$$\frac{\dot{w}_K^{Uz}}{w_K^{Uz}} = -\left(\frac{1-\beta}{\beta-\alpha}\right)\frac{\dot{p}^{Uz}}{p^{Uz}}. \tag{86}$$

Substituting (86) into (85) we obtain

$$\frac{\dot{C}^{Uz}}{C^{Uz}} = -\left(\frac{1-\beta}{\beta-\alpha}\right)\frac{\dot{p}^{Uz}}{p^{Uz}} + \frac{\dot{K}^{Uz}}{K^{Uz}}. \tag{87}$$

Using (85) and (77)we get

$$\left(\frac{\dot{w}_K^{Uz}}{w_K^{Uz}}\right) = \frac{\dot{C}^{Uz}}{C^{Uz}} - \frac{\dot{K}^{Uz}}{K^{Uz}} = (\beta - 1)\frac{\dot{K}^{Uz}}{K^{Uz}} \tag{88}$$

equalizing (86) and (88) we get

$$\frac{\dot{p}^{Uz}}{p^{Uz}} = (\beta - \alpha)\frac{\dot{K}^{Uz}}{K^{Uz}} \tag{89}$$

Using (87), (89) and adding and subtracting $\frac{1}{\theta}\frac{\dot{p}^{Uz}}{p^{Uz}}$ we get

$$\begin{aligned}\frac{\dot{C}^{Uz}}{C^{Uz}} &= \frac{1}{\theta}\frac{\dot{p}^{Uz}}{p^{Uz}} - \frac{\dot{p}^{Uz}}{p^{Uz}}\left(\frac{1}{\theta} + \left(\frac{1-\beta}{\beta-\alpha}\right)\right) + \frac{\dot{K}^{Uz}}{K^{Uz}} \qquad (90) \\ &= \frac{1}{\theta}\frac{\dot{p}^{Uz}}{p^{Uz}} - (\beta-\alpha)\frac{\dot{K}^{Uz}}{K^{Uz}}\left(\frac{1}{\theta} + \left(\frac{1-\beta}{\beta-\alpha}\right)\right) + \frac{\dot{K}^{Uz}}{K^{Uz}} \\ &= \frac{1}{\theta}\frac{\dot{p}^{Uz}}{p^{Uz}} + \left(\frac{\theta-(\beta-\alpha)-\theta(1-\beta)}{\theta}\right)\frac{\dot{K}^{Uz}}{K^{Uz}} \\ &= \frac{1}{\theta}\frac{\dot{p}^{Uz}}{p^{Uz}} + \left(\frac{\alpha+\beta(\theta-1)}{\theta}\right)\frac{\dot{K}^{Uz}}{K^{Uz}} \\ &= \frac{1}{\theta}\frac{\dot{p}^{Uz}}{p^{Uz}} + \left(\frac{\alpha+\beta(\theta-1)}{\theta}\right)\left(\frac{Y_x^{Uz}-\delta K^{Uz}}{K^{Uz}}\right) \text{ using } (73) \qquad (91)\end{aligned}$$

From (69) we obtain $\frac{Y_x^{Uz}}{K^{Uz}} = \frac{w_K^{Uz}}{p^{Uz}}\frac{\nu}{\alpha}$, and substituting this result into (91) yields

$$\begin{aligned}\frac{\dot{C}^{Uz}}{C^{Uz}} &= \frac{1}{\theta}\frac{\dot{p}^{Uz}}{p^{Uz}} + \left(\frac{\alpha+\beta(\theta-1)}{\theta}\right)\left(\frac{w_K^{Uz}}{p^{Uz}}\frac{\nu}{\alpha} - \delta\right) \qquad (92) \\ &= \frac{1}{\theta}\frac{\dot{p}^{Uz}}{p^{Uz}} + \frac{\nu}{\alpha}\left(\frac{\alpha+\beta(\theta-1)}{\theta}\right)\frac{w_K^{Uz}}{p^{Uz}} - \left(\frac{\alpha+\beta(\theta-1)}{\theta}\right)\delta\end{aligned}$$

In case of endogenous savings, (the model presented in section 2) the consumer Euler condition is given by (5)

$$\frac{\dot{C}^R}{C^R} = \frac{1}{\theta}\frac{\dot{p}^R}{p^R} + \frac{1}{\theta}\frac{w_K^R}{p^R} - \frac{\delta+\rho}{\theta} \qquad (93)$$

where the superscript R (Ramsey type model) is use to distinguish variables that come from the model when savings rates are endogenous. Now use

$$\rho = \delta(\alpha+\beta(\theta-1)) - \delta \qquad (94)$$

and

$$v = \frac{\alpha}{\alpha+\beta(\theta-1)} \text{ or } \frac{1}{\theta} = \frac{v}{\alpha}\left(\frac{\alpha+\beta(\theta-1)}{\theta}\right) \qquad (95)$$

to obtain

$$\frac{\dot{C}^R}{C^R} = \frac{1}{\theta}\frac{\dot{p}^R}{p^R} + \frac{1}{\theta}\frac{w_K^R}{p^R} - \left(\frac{(\alpha+\beta(\theta-1))}{\theta}\right)\delta \tag{96}$$

$$= \frac{1}{\theta}\frac{\dot{p}^R}{p^R} + \frac{v}{\alpha}\left(\frac{\alpha+\beta(\theta-1)}{\theta}\right)\frac{w_K^R}{p^R} - \left(\frac{(\alpha+\beta(\theta-1))}{\theta}\right)\delta$$

$$= \frac{\dot{C}^{Uz}}{C^{Uz}}.$$

8. Appendix C

Proof of proposition 5.

Let $\hat{\lambda}_j = \lambda_j e^{\rho t}$, taking the log time derivative of $\hat{\lambda}_j$ and employing (4) we get

$$\frac{\dot{\hat{\lambda}}_j}{\hat{\lambda}_j} = \frac{\dot{\lambda}_j}{\lambda_j} - \rho = -\frac{w_K}{p} + \delta. \tag{97}$$

Integrating the consumers budget and using $c_j^{-\theta} p = \hat{\lambda}_j e^{\rho t}$ (from (4)) and setting $c_j(t) = \phi_j C(t)$ we obtain

$$\hat{\lambda}_j(t) K_j = \hat{\lambda}_j(0) K_j(0) + \phi_j^{-\theta}\int_0^t w_L(\tau) C(\tau)^{-\theta} e^{-\rho\tau} d\tau - \phi_j^{1-\theta}\int_0^t C(\tau)^{1-\theta} e^{-\rho\tau} d\tau \tag{98}$$

To obtain ϕ_j, we first solve for the integrals in equation (98). Let K_x denote the world capital employed in the production of the investment good $(K_x = K_{x1} + K_{x2})$. Note that since the world (aggregate) economy behaves as a closed we can employ the results from the analytical solution for the Uzawa-Solow type model. Using (76) and (81) to obtain

$$w_L C^{-\theta} = H^{-\theta} B \left(\frac{A}{\alpha}\right)^{\frac{\beta}{1-\alpha}} \left(\frac{\nu}{\eta L}\right)^{\beta} K^{\beta(1-\theta)} \tag{99}$$

and

$$C^{1-\theta} = H^{1-\theta} K^{\beta(1-\theta)} \tag{100}$$

where $H = \left((1-\nu)^{\beta}((1-\eta)L)^{1-\beta}\right)$. Substituting (99) and (100) into (98) we get

$$\hat{\lambda}_j(t)K_j = \hat{\lambda}_j(0)K_j(0) + \left(\frac{\phi_j^{-\theta}}{H^{\theta}}B\left(\frac{A}{\alpha}\right)^{\frac{\beta}{1-\alpha}}\left(\frac{\nu}{\eta L}\right)^{\beta} - \frac{\phi_j^{1-\theta}}{H^{\theta-1}}\right)\int_0^t K(\tau)^{\beta(1-\theta)}e^{-\rho\tau}d\tau \tag{101}$$

using (74) and (43) and integrating $\int_0^t K(\tau)^{\beta(1-\theta)}e^{-\rho\tau}d\tau$ we get

$$\int_0^t K(\tau)^{\beta(1-\theta)}e^{-\rho\tau}d\tau$$

$$= \int_0^t \underbrace{\left(\left(\frac{\left(K(0)^{1-\alpha} - \left(K^{Uz*}\right)^{1-\alpha}\right)}{e^{(1-\alpha)\delta\tau}} + \left(K^{Uz*}\right)^{1-\alpha}\right)^{\frac{1}{1-\alpha}}\right)}_{K(\tau)\text{ from (74)}}^{\beta(1-\theta)} e^{-\rho\tau}dt \tag{102}$$

$$= \int_0^t \left(K(0)^{1-\alpha} - \left(K^{Uz*}\right)^{1-\alpha} + \left(K^{Uz*}\right)^{1-\alpha}e^{(1-\alpha)\delta\tau}\right)^{\frac{\beta(1-\theta)}{1-\alpha}}$$

$$* e^{\overbrace{(-\delta(\alpha+\beta(\theta-1))+\delta)}^{-\rho\text{ from (43)}}\tau - \delta\beta(1-\theta)\tau}d\tau$$

$$= \int_0^t \left(K(0)^{1-\alpha} - \left(K^{Uz*}\right)^{1-\alpha} + \left(K^{Uz*}\right)^{1-\alpha}e^{(1-\alpha)\delta\tau}\right)^{-\frac{\rho}{(1-\alpha)\delta}-1} e^{(1-\alpha)\delta\tau}d\tau$$

$$= -\frac{\left(K(0)^{1-\alpha} - \left(K^{Uz*}\right)^{1-\alpha} + \left(K^{Uz*}\right)^{1-\alpha}e^{(1-\alpha)\delta\tau}\right)^{-\frac{\rho}{(1-\alpha)\delta}}}{\left(K^{Uz*}\right)^{1-\alpha}\rho}\Bigg|_0^t$$

$$= \frac{1}{\left(K^{Uz*}\right)^{1-\alpha}\rho}\left(\frac{1}{K_0^{\frac{\rho}{\delta}}} - \frac{1}{\left(K(t)^{Uz}\right)^{\frac{\rho}{\delta}}e^{\rho t}}\right)\text{ using (74)}. \tag{103}$$

Substituting (103) into (101) yields

$$\hat{\lambda}_j(t) K_j = \hat{\lambda}_j(0) K_j(0) + \left(B \left(\frac{A}{\alpha}\right)^{\frac{\beta}{1-\alpha}} \left(\frac{\nu}{\eta L}\right)^{\beta} - \phi_j H \right) \frac{\phi_j^{-\theta}}{H^{\theta} \rho \left(K^{Uz*}\right)^{1-\alpha}} \left(\frac{1}{K_0^{\frac{\rho}{\delta}}} - \frac{1}{K(t)^{\frac{\rho}{\delta}} e^{\rho t}} \right). \tag{104}$$

Using $c_j(t)^{-\theta} p(t) e^{-\rho t} = \hat{\lambda}_j(t)$ from (4) and setting $c_j(t) = \phi_j C(t)$ we get

$$\begin{aligned} \hat{\lambda}_j(t) K_j &= \phi_j^{-\theta} C(t)^{-\theta} p(t) e^{-\rho t} K_j(t) \\ &= \phi_j^{-\theta} C(0)^{-\theta} p(0) K_j(0) + \\ &\quad + \left(B \left(\frac{A}{\alpha}\right)^{\frac{\beta}{1-\alpha}} \left(\frac{\nu}{\eta L}\right)^{\beta} - \phi_j H \right) \frac{\phi_j^{-\theta}}{H^{\theta} \rho \left(K^{Uz*}\right)^{1-\alpha}} \left(\frac{1}{K_0^{\frac{\rho}{\delta}}} - \frac{1}{K(t)^{\frac{\rho}{\delta}} e^{\rho t}} \right). \end{aligned} \tag{105}$$

Taking the limit as t approaches infinity, and using the transversality condition, we obtain

$$0 = C(0)^{-\theta} p(0) K_j(0) + \left(B \left(\frac{A}{\alpha}\right)^{\frac{\beta}{1-\alpha}} \left(\frac{\nu}{\eta L}\right)^{\beta} - \phi_j H \right) \frac{1}{H^{\theta} \rho \left(K^{Uz*}\right)^{1-\alpha} K(0)^{\frac{\rho}{\delta}}}. \tag{106}$$

Using the analytical solution for C and p (from (76) and (81)), setting

$$C(0)^{-\theta} p(0) = \frac{K(0)^{\beta(1-\theta)-\alpha}}{H^{\theta}} B \left(\frac{\alpha^{\alpha-\beta}}{A^{1-\beta}}\right)^{\frac{1}{1-\alpha}} \left(\frac{\eta L}{\nu}\right)^{\alpha-\beta}$$

in (106), using $\frac{\rho}{\delta} = (\alpha + \beta(\theta - 1)) - 1$ and solving for ϕ_j we get

$$\begin{aligned} \phi_j &= \frac{\nu^{\beta}}{\eta^{\beta} L^{\beta}} \frac{B}{H} \left(\frac{A}{\alpha}\right)^{\frac{\beta}{1-\alpha}} \rho \left(1 + \rho \left(K^{Uz*}\right)^{1-\alpha} K_j(0) \left(K(0)\right)^{\frac{\rho}{\delta} + \beta(1-\theta) - \alpha} \left(\frac{\alpha^{\alpha}}{A}\right)^{\frac{1}{1-\alpha}} \left(\frac{\eta^{\alpha} L^{\alpha}}{\nu^{\alpha}}\right) \right) \\ &= \frac{\nu^{\beta}}{\eta^{\beta}} \frac{B}{(1-\nu)^{\beta} (1-\eta)^{1-\beta}} \left(\frac{A}{\alpha}\right)^{\frac{\beta}{1-\alpha}} \left(\frac{\rho}{\delta} \frac{\eta}{1-\alpha} \frac{K_j(0)}{K(0)} + \frac{1}{L} \right) \end{aligned} \tag{107}$$

Adding ϕ_j for $j = 1, 2$, leads to the result

$$\phi_1 + \phi_2 = 1 = \frac{\nu^{\beta}}{\eta^{\beta}} \frac{B}{(1-\nu)^{\beta} (1-\eta)^{1-\beta}} \left(\frac{A}{\alpha}\right)^{\frac{\beta}{1-\alpha}} \left(\frac{\rho}{\delta} \frac{\eta}{1-\alpha} + 1 \right) \tag{108}$$

which implies

$$\frac{\nu^{\beta}}{\eta^{\beta}}\frac{B}{(1-\nu)^{\beta}(1-\eta)^{1-\beta}}\left(\frac{A}{\alpha}\right)^{\frac{\beta}{1-\alpha}} = \frac{\delta(1-\alpha)}{\rho\eta+\delta(1-\alpha)} \tag{109}$$

From (43) we obtain

$$-\frac{\rho}{\delta}\frac{1}{(\theta-1)} = \frac{1-\alpha}{\theta-1}-\beta, \tag{110}$$

Using $s=\frac{1}{\theta}$, η as defined in (72) , η, therefore, equals

$$\eta=\frac{(1-\alpha)}{(1-\alpha)+(1-\beta)(\theta-1)} \tag{111}$$

substituting for η into $\frac{\delta(1-\alpha)}{\rho\eta+\delta(1-\alpha)}$ we get

$$\frac{\delta(1-\alpha)}{\rho\eta+\delta(1-\alpha)}=\frac{(1-\alpha)+(1-\beta)(\theta-1)}{(\theta-1)} \tag{112}$$

thus (107) becomes

$$\phi_j = \underbrace{\frac{\delta(1-\alpha)}{\rho\eta+\delta(1-\alpha)}}_{\text{using (108)}}\left(\frac{\rho}{\delta}\frac{\eta}{1-\alpha}\frac{K_j(0)}{K(0)}+\frac{L_j}{L}\right) \tag{113}$$

$$= \underbrace{\frac{(1-\alpha)+(1-\beta)(\theta-1)}{(\theta-1)}}_{\text{using (112)}}\left(\frac{\rho}{(1-\alpha)\delta}\underbrace{\frac{(1-\alpha)}{(1-\alpha)+(1-\beta)(\theta-1)}}_{\text{using (111)}}\frac{K_j(0)}{K(0)}+\frac{L_j}{L}\right)$$

$$= \left(\frac{\rho}{\delta}\frac{1}{(\theta-1)}\frac{K_j(0)}{K(0)}+\frac{(1-\alpha)+(1-\beta)(\theta-1)}{(\theta-1)}\frac{L_j}{L}\right)$$

$$= \frac{\rho}{\delta}\frac{1}{(\theta-1)}\left(\frac{K_j(0)}{K(0)}-\frac{L_j}{L}\right)+\frac{L_j}{L} \blacksquare$$

Proof of proposition 6. Using (105) one can also verify that

$$-\hat{\lambda}_j(t)K_j(t) = \int_t^{\infty}\left(w_L(\tau)c_j(\tau)^{-\theta}-c_j(\tau)^{1-\theta}\right)e^{-\rho\tau}d\tau \tag{114}$$

$$= \left(\frac{\phi_j^{-\theta}B\left(\frac{A}{\alpha}\right)^{\frac{\beta}{1-\alpha}}\left(\frac{\nu}{\eta L}\right)^{\beta}}{H^{\theta}}-\frac{\phi_j^{1-\theta}}{H^{-1+\theta}}\right)\int_t^{\infty}K(\tau)^{\beta(1-\theta)}e^{-\rho\tau}d\tau$$

or

$$p(t)\,C(t)^{-\theta}\,e^{-\rho t}K_j(t) = \left(\frac{\phi_j H - B\left(\frac{A}{\alpha}\right)^{\frac{\beta}{1-\alpha}}\left(\frac{\nu}{\eta L}\right)^{\beta}}{\left(K^{Uz*}\right)^{1-\alpha}\rho K(t)^{\frac{\rho}{\delta}}\,e^{\rho t}}\right)\frac{1}{H^{\theta}} \tag{115}$$

Simplification leads to

$$\kappa_j = \frac{K_j(t)}{K(t)} = \frac{\delta}{\rho}\frac{1-\alpha}{\eta}\left(\phi_j\left(\frac{1-\eta}{1-\beta}\right)-\frac{1}{L}\right) \tag{116}$$

Therefore, the ratio $\frac{K_j(t)}{K(t)}$ is constant for all t. Solving for ϕ_j we obtain

$$\phi_j = \left(\frac{\rho}{\delta}\frac{\eta}{1-\alpha}\kappa_j+\frac{1}{L}\right)\frac{1-\beta}{1-\eta} \tag{117}$$

$$= \left(\frac{\rho}{\delta}\frac{1}{\theta-1}\right)\left(\kappa_j(0)-\frac{1}{2}\right)+\frac{1}{2} \tag{118}$$

Therefore, solving for κ_j yields

$$\kappa_j = \frac{\delta}{\rho}\frac{1-\alpha}{\eta}\left(\frac{1-\eta}{1-\beta}\left(\left(\frac{\rho}{\delta}\frac{1}{\theta-1}\right)\left(\kappa_j(0)-\frac{1}{2}\right)+\frac{1}{2}\right)-\frac{1}{2}\right) \tag{119}$$

$$= \frac{1-\eta}{\eta}\frac{1-\alpha}{1-\beta}\frac{1}{\theta-1}\kappa_j(0)+\left(\left(\frac{1-\eta}{1-\beta}-1\right)\frac{\delta}{\rho}-\frac{1-\eta}{1-\beta}\frac{1}{\theta-1}\right)\frac{1-\alpha}{\eta}\frac{1}{2} \tag{120}$$

$$= \kappa_j(0) \tag{121}$$

Note that

$$\left(\frac{1-\eta}{1-\beta}-1\right)\frac{\delta}{\rho} = \frac{1-\eta}{1-\beta}\left(1-\frac{1-\beta}{1-\underbrace{\eta}_{\eta=\frac{(1-\alpha)s}{(1-\alpha)s+(1-\beta)(1-s)}}}\right)\frac{\delta}{\rho} \tag{122}$$

$$= \frac{1-\eta}{1-\beta}\left(\frac{(1-s)-(1-\alpha)s-(1-\beta)(1-s)}{(1-s)}\right)\frac{\delta}{\rho}$$

$$= \frac{1-\eta}{1-\beta}\left(\frac{-(1-\alpha)s+\beta(1-s)}{(1-s)}\right)\frac{\delta}{\rho} \text{ using } s=\frac{1}{\theta}$$

$$= \frac{1-\eta}{1-\beta}\left(\frac{-(1-\alpha)+\beta(\theta-1)}{\theta-1}\right)\frac{\delta}{\rho}$$

$$= \frac{1-\eta}{1-\beta}\left(\frac{1}{\theta-1}\right) \text{ using (43)} \blacksquare$$

Proof of proposition 7. Per country income is given by

$$Y_j = w_L L_j + w_K K_j$$

$$= B\left(\frac{\nu K}{\eta L}\right)^{\beta}\left(\frac{A}{\alpha}\right)^{\frac{\beta}{1-\alpha}}\left(\left(\frac{\alpha}{A}\right)^{\frac{1}{1-\alpha}}\frac{\eta}{\nu}\frac{K_j}{K}L+1\right)$$

$$= B\left(\frac{\nu K}{\eta L}\right)^{\beta}\left(\frac{1-\alpha}{\alpha}\right)^{\beta}\left(\left(\frac{\alpha}{1-\alpha}\right)\frac{\eta}{\nu}\kappa_j L+1\right)$$

Since the ratio $\frac{K_j(t)}{K(t)}$ is constant, this implies that income in each country grows at the same rate as aggregate (world) capital which, in turn, implies

$$\frac{\dot{Y}_j}{Y_j} = \beta\frac{\dot{K}}{K} = \frac{\dot{C}}{C} = \frac{\dot{c}_j}{c_j} \tag{123}$$

Since $c_j = \phi_j C$, the ratio $\frac{c_j}{Y_j} \equiv (1-s_j)$ is constant, that is, the consumption expenditure is a constant fraction of income. Using $\frac{1-\nu}{\beta} = \frac{1-\alpha}{\alpha}\frac{\nu}{\eta}\frac{1-\eta}{1-\beta}$ and $H = \left((1-\nu)^{\beta}\left((1-\eta)L\right)^{1-\beta}\right)$ it can be shown that the consumption share of country j equals

$$1-s_j = \frac{\phi_i HK(t)^{\beta}}{B\left(\frac{\nu K(t)}{\eta L}\right)^{\beta}\left(\frac{1-\alpha}{\alpha}\right)^{\beta}\left(\frac{\alpha}{1-\alpha}\frac{\eta}{\nu}\kappa_j L+1\right)} \tag{124}$$

$$= \frac{1-\eta}{1-\beta}\frac{1-\alpha}{\alpha}\frac{\nu}{\eta}\left(\frac{\phi_i}{\kappa_j+\frac{1}{L}\frac{1-\alpha}{\alpha}\frac{\nu}{\eta}}\right)$$

$$= \frac{1-\nu}{\beta}\left(\frac{\phi_i}{\kappa_j+\frac{1}{L}\frac{1-\alpha}{\alpha}\frac{\nu}{\eta}}\right)$$

$$= \frac{1-\nu}{\beta}\frac{\eta}{1-\alpha}\left(\frac{\phi_i}{\phi_j\frac{\delta}{\rho}\frac{1-\eta}{1-\beta}+\frac{1}{L}\left(\frac{\nu}{\alpha}-\frac{\delta}{\rho}\right)}\right)$$

which implies

$$s_j = \frac{\phi_j\frac{\delta}{\rho}\frac{1-\eta}{1-\beta}-\frac{1-\nu}{\beta}\frac{\eta}{1-\alpha}\phi_i+\frac{1}{L}\left(\frac{\nu}{\alpha}-\frac{\delta}{\rho}\right)}{\phi_j\frac{\delta}{\rho}\frac{1-\eta}{1-\beta}+\frac{1}{L}\left(\frac{\nu}{\alpha}-\frac{\delta}{\rho}\right)} \tag{125}$$

$$= \frac{\phi_j\frac{1-\eta}{1-\beta}-\frac{1}{L}}{\phi_j\frac{1-\eta}{1-\beta}\frac{\delta}{\rho}\frac{1}{\left(\frac{\delta}{\rho}-\frac{\nu}{\alpha}\right)}-\frac{1}{L}}$$

Note that

$$\left(\frac{\delta}{\rho}-\frac{\nu}{\alpha}\right) = \frac{\delta}{\rho}-\frac{1}{\alpha+\beta(\theta-1)} \tag{126}$$

$$= \frac{1}{(\alpha+\beta(\theta-1)-1)(\alpha+\beta(\theta-1))}, \text{ and}$$

$$\frac{\delta}{\rho}\frac{1}{\left(\frac{\delta}{\rho}-\frac{\nu}{\alpha}\right)} = (\alpha+\beta(\theta-1))$$

Therefore;

$$s_j = \frac{\phi_j\frac{1-\eta}{1-\beta}-\frac{1}{L}}{\phi_j\frac{1-\eta}{1-\beta}(\alpha+\beta(\theta-1))-\frac{1}{L}} \blacksquare \tag{127}$$

Thus if (43) holds, saving's rates are constant but also different across countries as long as $K_1(0) \neq$

$K_2(0)$.

Since $\alpha + \beta(\theta - 1) > 1$ s_j is increasing in ϕ_j and therefore, increasing in $K_j(0)$

$$\frac{\partial s_j}{\partial \phi_j} = \frac{(\alpha + \beta(\theta - 1) - 1)\frac{1-\eta}{1-\beta}}{\left(\phi_j \frac{1-\eta}{1-\beta}(\alpha + \beta(\theta - 1)) - \frac{1}{L}\right)^2} \frac{1}{L}$$

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